EXACT SOLUTIONS FOR THE SINGULARLY PERTURBED RICCATI EQUATION AND EXACT WKB ANALYSIS

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Abstract. The singularly perturbed Riccati equation is the first-order nonlinear ordinary differential equation $\hbar \partial_x f = af^2 + bf + c$ in the complex domain where $\hbar$ is a small complex parameter. We prove an existence and uniqueness theorem for exact solutions with prescribed asymptotics as $\hbar \to 0$ in a half-plane. These exact solutions are constructed using the Borel–Laplace method; that is, they are Borel summations of the formal divergent $\hbar$-power series solutions. As an application, we prove existence and uniqueness of exact WKB solutions for the complex one-dimensional Schrödinger equation with a rational potential.

§1. Introduction

The purpose of this article is to analyze the *singularly perturbed Riccati equation*

$$\hbar \partial_x f = af^2 + bf + c, \tag{1}$$

where x is a complex variable and $\hbar$ is a small complex perturbation parameter, and where the coefficients a, b, c are holomorphic functions of $(x, \hbar)$ which admit asymptotic expansions as $\hbar \to 0$. The main problem we pose here is to construct *canonical exact solutions*, that is, solutions that are holomorphic in both variables and that are uniquely determined by their prescribed asymptotics as $\hbar \to 0$. This is a quintessential problem in singular perturbation theory.

1.1 Motivation

Existence and uniqueness theory for first-order ODEs is obviously a very well-developed subject which can also be analyzed in the presence of a parameter like $\hbar$. However, it gives no information about the asymptotic behavior of solutions as $\hbar \to 0$. Attempting to solve an equation like (1) by expanding it in powers of $\hbar$ generically leads to divergent power series solutions.

Of course, the subject of Riccati equations is vast with an exceptionally long history, appearing in a very wide variety of contexts (see, e.g., [15]). Our motivation has two primary sources.

One is the exact WKB analysis of Schrödinger equations in the complex domain [3, 4, 7, 8, 16, 20]. This very powerful approximation technique was popularized in the early days of quantum mechanics and goes back to as early as Liouville. However, the natural question of existence of *exact* solutions with prescribed asymptotic behavior as $\hbar \to 0$ (often called *exact WKB solutions*) has remained open in general. (Though in the course of finishing a

draft of this paper, we became aware of the recent work of Nemes [10].) Our main result can be used to give a positive answer to this question in a large class of problems (generalizing in particular the recent results of Nemes). This is briefly described in a special case in §6.3, and a full description is given in [12].

Another interesting problem serving as motivation for this paper is encountered in the analysis of singularly perturbed differential systems and more generally meromorphic connections on holomorphic vector bundles over Riemann surfaces. Given a singularly perturbed differential system with a singular point, the question is that of constructing a filtration by growth rates on the vector space of local solutions which is holomorphically varying in $\hbar$ and has a well-defined limit as $\hbar \to 0$. For a large class of systems, the main result in this article can be used to construct such filtrations and furthermore show that they converge to the eigendecomposition of the unperturbed system as $\hbar \to 0$ as well as to the eigendecomposition of the principal part of the system as x tends to the singular point (see [13]).

1.2 Setting and overview of main results

Take a domain $\mathsf{X} \subset \mathbb{C}_x$, a sector $\mathsf{S} \subset \mathbb{C}_\hbar$ at the origin, and consider a Riccati equation (1) whose coefficients a, b, c are holomorphic functions of $(x, \hbar) \in \mathsf{X} \times \mathsf{S}$ which admit locally uniform asymptotic expansions $\hat{a}, \hat{b}, \hat{c}$ as $\hbar \to 0$ in S. More details are presented in §2, but for the purposes of this introduction, let us focus on the most ubiquitous scenario where a, b, c are in fact polynomials in $\hbar$. The leading-order part in $\hbar$ of the Riccati equation (1) is the quadratic equation $a_0 f_0^2 + b_0 f_0 + c_0 = 0$, which generically has two distinct local holomorphic solutions $f_0^\pm$ away from *turning points* (i.e., the zeros of the discriminant $D_0 := b_0^2 - 4a_0 c_0$).

Let $\mathsf{U} \subset \mathsf{X}$ be a domain free of turning points that supports a univalued square-root branch $\sqrt{D_0}$. Then it is well known (see Theorem 3.1) that (1) has precisely two formal solutions $\hat{f}_\pm$ on U which are uniquely determined by the leading-order solutions $f_0^\pm$ via a recursion on the coefficients $f_k^\pm$. The main goal of this paper is to promote—in a *canonical* way—the formal solutions $\hat{f}_\pm$ to *exact solutions* $f_\pm$ (formally defined in §2), that is, holomorphic solutions defined on $\mathsf{U}_0 \times \mathsf{S}_0$ where $\mathsf{U}_0 \subset \mathsf{U}$ and $\mathsf{S}_0 \subset \mathsf{S}$ is some sectorial domain such that $f_\pm \sim \hat{f}_\pm$ as $\hbar \to 0$ in S_0.

Although existence of exact solutions is a classical fact in the theory of singularly perturbed differential equations (see, e.g., [21, Theorem 26.1]), they are inherently nonunique due to the problem of missing exponential corrections in asymptotic expansions. Part of the issue is that classical techniques in general give no control on the size of the opening of the sectorial domain S_0 (see, e.g., the remark in [21, p. 144], immediately following Theorem 26.1). In particular, it is impossible in general to *identify* a given exact solution with its asymptotic formal solution.

In this paper, we develop a general procedure applicable to a large class of problems to obtain *canonical* exact solutions which indeed can be identified in a precise sense with their corresponding asymptotic formal solutions. In order to achieve this, the opening angle $|\mathsf{A}|$ of S must be at least π, the most fundamental case being $|\mathsf{A}| = \pi$. For the purposes of this introduction, let us assume that $\mathsf{A} = (-\frac{\pi}{2}, +\frac{\pi}{2})$.

Fix a basepoint $x_0 \in \mathsf{X}$ that is not a turning point, choose a local square-root branch $\sqrt{D_0}$ near x_0, and consider the *Liouville transformation*

$$z = \Phi(x) := \int_{x_0}^x \sqrt{D_0(t)} \mathrm{d}t. \tag{2}$$

Suppose that x_0 has a neighborhood $\mathsf{W} \subset \mathsf{X}$ which is mapped by Φ to a horizontal strip $\mathsf{H} = \left\{ z \mid -r < \mathrm{Im}(z) < r \right\}$ of some width $r > 0$. Suppose furthermore that the $\hbar$-polynomial coefficients a_k, b_k, c_k of a, b, c are bounded on W by $\sqrt{D_0}$. Then, under these assumptions, the main results of this paper can be summarized as follows.

THEOREM 1.1. *The Riccati equation (1) has a pair of canonical exact solutions $f_\pm$ near $x_0 \in \mathsf{X}$ which are asymptotic to the formal solutions $\hat{f}_\pm$ as $\hbar \to 0$ in the right half-plane. Namely, there is a neighborhood $\mathsf{U}_0 \subset \mathsf{X}$ of x_0 and a sectorial subdomain $\mathsf{S}_0 \subset \mathsf{S}$ with the same opening A such that the Riccati equation (1) has a unique pair of holomorphic solutions $f_\pm$ on $\mathsf{U}_0 \times \mathsf{S}_0$ which are Gevrey asymptotic to $\hat{f}_\pm$ as $\hbar \to 0$ along the closed arc $\bar{\mathsf{A}}$ uniformly for all $x \in \mathsf{U}_0$:*

$$f_\pm \simeq \hat{f}_\pm \qquad as\ \hbar \to 0\ along\ \bar{\mathsf{A}},\ unif.\ \forall x \in \mathsf{U}_0. \tag{3}$$

Moreover, $f_\pm$ is the uniform Borel resummation of the formal solution $\hat{f}_\pm$:

$$f_\pm = \mathcal{S}\big[\hat{f}_\pm\big]. \tag{4}$$

This is a special case of Theorems 5.1 and 5.2, which are the two main results of this paper.

1.3 Discussion and method

We construct the canonical exact solutions $f_\pm$ by employing relatively basic and classical techniques from complex analysis which form the basis for the more modern and sophisticated theory of resurgent asymptotic analysis. Namely, we use the *Borel–Laplace method*, also known as the theory of *Borel–Laplace summability*. We stress that the Borel–Laplace method *is nothing other than the theory of Laplace transforms, written in slightly different variables*, echoing the words of Sokal [17]. As such, we have tried to keep our presentation very hands-on and self-contained, so the knowledge of basic complex analysis should be sufficient to follow.

An additional significant benefit of our approach is that we obtain uniqueness of the solution in the same sector where the initial data are specified. This feature does not hold for other less explicit approaches, such as, e.g., [21, Theorem 26.1] where an existence theorem is proved only on a smaller subsector and there is no hope of uniqueness.

Finally, we want to take the opportunity to acknowledge the unpublished work of Koike and Schäfke on the Borel summability of WKB solutions of Schrödinger equations with polynomial potentials. See [19, §3.1] for a brief account of their work. Their ideas (which were kindly explained to the author in a private communication from Kohei Iwaki) provided the initial inspiration for the more general strategy of the proof pursued in this article.

§2. Singularly perturbed Riccati equations

2.1 Background assumptions

Throughout the paper, we fix a complex plane $\mathbb{C}_x$ with coordinate x and another complex plane $\mathbb{C}_\hbar$ with coordinate $\hbar$. Let X be a domain in $\mathbb{C}_x$ or indeed a coordinate chart on a Riemann surface. Let $\mathsf{S} \subset \mathbb{C}_\hbar$ be a sectorial domain at the origin with opening arc A. We assume that $0 < |\mathsf{A}| \le 2\pi$.

Consider the Riccati equation

$$\hbar \partial_x f = af^2 + bf + c \tag{5}$$

whose coefficients a, b, c are holomorphic functions of $(x, \hbar) \in \mathsf{X} \times \mathsf{S}$ admitting locally uniform asymptotic expansions with holomorphic coefficients as $\hbar \to 0$ along A:

$$a(x, \hbar) \sim \hat{a}(x, \hbar) := \sum_{n=0}^{\infty} a_n(x)\hbar^n,$$

$$b(x, \hbar) \sim \hat{b}(x, \hbar) := \sum_{n=0}^{\infty} b_n(x)\hbar^n, \qquad \text{as } \hbar \to 0 \text{ along } \mathsf{A}, \text{ loc.unif. } \forall x \in \mathsf{X}, \qquad (6)$$

$$c(x, \hbar) \sim \hat{c}(x, \hbar) := \sum_{n=0}^{\infty} c_n(x)\hbar^n.$$

The main problem we pose in this article is to find canonical *exact solutions* of the Riccati equation (5) in the following sense.

DEFINITION 2.1. Fix any phase $\theta \in \mathsf{A}$. A *local θ-exact solution* of the Riccati equation (5) near a point $x_0 \in \mathsf{X}$ is a holomorphic solution $f = f(x, \hbar)$, defined on a domain $\mathsf{U}_0 \times \mathsf{S}_0$ where $\mathsf{U}_0 \subset \mathsf{X}$ is a neighborhood of x_0 and $\mathsf{S}_0 \subset \mathsf{S}$ is a sectorial subdomain with opening $\mathsf{A}_0 \subset \mathsf{A}$ containing θ, such that f admits an asymptotic expansion with holomorphic coefficients as $\hbar \to 0$ along A_0 uniformly for all $x \in \mathsf{U}_0$.

A *θ-exact solution* on a domain $\mathsf{U} \subset \mathsf{X}$ is a holomorphic solution $f = f(x, \hbar)$ which is a local θ-exact solution near every point in U. That is, f is a holomorphic solution defined on a domain $\mathbb{U} \subset \mathsf{U} \times \mathsf{S}$ with the following property: for every $x_0 \in \mathsf{X}$, there is a domain neighborhood $\mathsf{U}_0 \subset \mathsf{U}$ of x_0 and a sectorial domain $\mathsf{S}_0 \subset \mathsf{S}$ with opening $\mathsf{A}_0 \subset \mathsf{A}$ containing θ such that f admits an asymptotic expansion with holomorphic coefficients as $\hbar \to 0$ along A_0 uniformly for all $x \in \mathsf{U}_0$.

2.2 Examples

The following is a list, included here for illustrative purposes only, containing a few explicit examples of Riccati equations to which the main results in this paper can be applied.

The most typical situation is one where the coefficients a, b, c of the Riccati equation (5) are polynomials in $\hbar$ with coefficients which are rational functions of x. In this case, X is the complement of the poles in $\mathbb{C}_x$, and the sectorial domain S can be taken to be the whole open right half-plane $\{\mathrm{Re}(\hbar) > 0\}$. The simplest example is

(1) $\quad \hbar\partial_x f = f^2 - x$.

This Riccati equation is examined in great detail in §6.2. It arises in the exact WKB analysis of the Airy equation $\hbar^2\partial_x^2\psi(x, \hbar) = x\psi(x, \hbar)$ (see [12]). In this case, $\mathsf{X} = \mathbb{C}_x$ and the sectorial domain S is the open right half-plane $\{\mathrm{Re}(\hbar) > 0\}$. If U is any of the three sectorial domains in $\mathbb{C}_x$ given by $\{0 < \arg(x) < +4\pi/3\}$, or $\{+2\pi/3 < \arg(x) < +2\pi\}$, or $\{-2\pi/3 < \arg(x) < +2\pi/3\}$, then on each of these domains, the main existence and uniqueness result in this paper produces a pair of canonical exact solutions. More generally,

(2) $\quad \hbar\partial_x f = f^2 + q(x)$,

where $q(x)$ is any polynomial or a rational function with poles of order 2 or higher. In this case, S can again be arranged to be the right half-plane, and U is a sectorial domain near a pole of $q(x)$ of order 2 or higher.

Many Riccati equations arise from the WKB analysis of classical second-order differential equation. For example, the following Riccati equation appears in the WKB analysis of the Gauss hypergeometric equation:

(3) $\hbar \partial_x f = f^2 + \frac{\gamma - (\alpha + \beta + 1)x}{x(x-1)} f + \frac{\alpha\beta}{x(x-1)}$ for any $\alpha, \beta, \gamma \in \mathbb{C}^*$.

Riccati equations also arise in the analysis of singularly perturbed second-order systems. For example, the Riccati equation

(4) $\hbar \partial_x f = \hbar f^2 + (1-x)f + x\hbar$

arises in the analysis of the system $\hbar \partial_x \psi + \begin{bmatrix} 1 & -\hbar \\ x\hbar & x \end{bmatrix} \psi = 0$. See [13].

Our methods also apply to the following nontrivial deformation of example (1):

(5) $\hbar \partial_x f = f^2 - x + xE(x,\hbar)$ where $E(x,\hbar) := \int_0^{+\infty} \frac{e^{-\xi/\hbar}}{x+\xi} d\xi$.

The sectorial domain S in this case is the open right half-plane. The function E is holomorphic in $\hbar \in \mathsf{S}$, and it admits a locally uniform asymptotic expansion as $\hbar \to 0$ in the right half-plane. Notice, however, that E is not holomorphic at $\hbar = 0$, and it also has nonisolated singularities along the negative real axis in $\mathbb{C}_x$. Nevertheless, if U is the domain given by $\{-2\pi/3 < \arg(x) < +2\pi/3\}$ or by $\{0 < \arg(x) < +\pi\}$ or $\{+\pi < \arg(x) < +2\pi\}$, then our method yields canonical exact solutions on U.

§3. Formal perturbation theory

In this section, we analyze the Riccati equation from a purely formal perspective whereby we ignore all analytic considerations in the $\hbar$-variable.

Thus, we consider the *formal Riccati equation*

$$\hbar \partial_x \hat{f} = \hat{a}\hat{f}^2 + \hat{b}\hat{f} + \hat{c}, \tag{7}$$

where $\hat{a}, \hat{b}, \hat{c}$ are arbitrary formal power series in $\hbar$ with holomorphic coefficients on some domain X in $\mathbb{C}_x$. By definition, a *formal solution* of (7) on a domain $\mathsf{U} \subset \mathsf{X}$ is any formal power series with holomorphic coefficients $\hat{f} = \hat{f}(x,\hbar)$ that satisfies the formal equation (7).

3.1 Leading-order solutions

Consider the *leading-order equation* corresponding to (7):

$$a_0 f_0^2 + b_0 f_0 + c_0 = 0. \tag{8}$$

It is a quadratic equation in the unknown variable f_0, and we refer to its solutions as *leading-order solutions* of the Riccati equation. Generically, they are locally holomorphic, but may have poles and branch-point singularities.

The discriminant of (8),

$$D_0 := b_0^2 - 4a_0 c_0, \tag{9}$$

which we call the *leading-order discriminant* of the Riccati equation, is a holomorphic function on X. We always assume that D_0 is not identically zero. The zeros of D_0 are called *turning points* of the Riccati equation, and all other points in X are called *regular points*. Locally, away from turning points, there is at least one holomorphic leading-order solution. For reference, we state the following elementary lemma.

LEMMA 3.1 (Holomorphic leading-order solutions). *Let $\mathsf{U} \subset \mathsf{X}$ be any domain free of turning points such that a univalued square-root branch $\sqrt{D_0}$ of D_0 can be chosen on U. Then the leading-order equation (8) has at least one holomorphic solution on U. In addition, if a_0 is nonvanishing on U, then (8) has two holomorphic solutions. Moreover, any holomorphic solution is bounded on U whenever the coefficients a_0, b_0, c_0 are bounded by $\sqrt{D_0}$ on U.*

We will always label the leading-order solutions as follows:

$$f_0^\pm := \frac{-b_0 \pm \sqrt{D_0}}{2a_0} \qquad \text{if } a_0 \not\equiv 0, \tag{10a}$$

$$f_0^+ := -c_0/b_0 \qquad \text{if } a_0 \equiv 0. \tag{10b}$$

This choice of labels yields the following relations:

$$\pm\sqrt{D_0} = 2a_0 f_0^\pm + b_0 \qquad \text{and} \qquad \sqrt{D_0} = a_0\big(f_0^+ - f_0^-\big). \tag{11}$$

Thus, if a_0 is nonvanishing, then both $f_0^\pm$ from (10a) are holomorphic functions on U. If a_0 has zeros in U, then f_0^+ from (10a) remains holomorphic on U, but f_0^- has poles where a_0 has zeros. If $a_0 \equiv 0$, then f_0^+ from (10b) is a holomorphic function on U.

3.2 Existence and uniqueness of formal solutions

The following elementary theorem says that a formal Riccati equation (7) always has at least one local solution away from turning points, and it is uniquely specified in the leading-order.

THEOREM 3.1 (Formal existence and uniqueness theorem). *Consider the formal Riccati equation (7). Assume that its leading-order discriminant D_0 is not identically zero. Let $\mathsf{U} \subset \mathsf{X}$ be a domain free of turning points that supports a univalued square-root branch $\sqrt{D_0}$.*

(1) *If $a_0 \equiv 0$, then (7) has a unique formal solution $\hat{f}_+$ on U. Its leading-order term is f_0^+ from (10b).*
(2) *If $a_0 \not\equiv 0$ and nonvanishing on U, then (7) has exactly two distinct formal solutions $\hat{f}_\pm$ on U. Their leading-order terms $f_0^\pm$ are given by (10a).*
(3) *If $a_0 \not\equiv 0$ but has zeros in U, then (7) has a unique formal solution $\hat{f}_+$ on U. Its leading-order term f_0^+ is the unique holomorphic leading-order solution on U given by (10a).*

Moreover, the coefficients $f_k^\pm$ of $\hat{f}_\pm$ for $k \geq 1$ are given by the following recursive formula:

$$f_k^\pm = \pm\frac{1}{\sqrt{D_0}}\partial_x f_{k-1}^\pm \mp \frac{1}{\sqrt{D_0}}\left(\sum_{k_1+k_2+k_3=k}^{k_2,k_3\neq k} a_{k_1} f_{k_2}^\pm f_{k_3}^\pm + \sum_{k_1+k_2=k}^{k_2\neq k} b_{k_1} f_{k_2}^\pm + c_k\right). \tag{12}$$

Proof. We expand the formal Riccati equation (7) order-by-order in $\hbar$:

$$\hbar^0 \;\Big|\; 0 = a_0 f_0^2 + b_0 f_0 + c_0; \tag{13}$$

$$\hbar^1 \;\Big|\; \partial_x f_0 = (2a_0 f_0 + b_0)f_1 + a_1 f_0^2 + b_1 f_0 + c_1; \tag{14}$$

$$\hbar^2 \;\Big|\; \partial_x f_1 = (2a_0 f_0 + b_0)f_2 + a_0 f_1^2 + 2a_1 f_0 f_1 + a_2 f_0^2 + b_1 f_1 + b_2 f_0 + c_2; \tag{15}$$

$$\vdots$$

$$\hbar^k \; \Big| \; \partial_x f_{k-1} = (2a_0 f_0 + b_0) f_k + \sum_{\substack{k_1+k_2+k_3=k}}^{k_2,k_3 \neq k} a_{k_1} f_{k_2} f_{k_3} + \sum_{\substack{k_1+k_2=k}}^{k_2 \neq k} b_{k_1} f_{k_2} + c_k. \qquad (16)$$

$$\vdots$$

Observe that these are no longer differential equations because the derivative term at each order depends only on the solutions from previous orders. If we fix a leading-order solution $f_0^\pm$, then the expression $(2a_0 f_0^\pm + b_0)$, appearing as a factor in front of $f_k^\pm$ in each equation (16), is simply $\pm\sqrt{D_0}$. From the assumption that $D_0 \not\equiv 0$, it follows that at each order in $\hbar$, we can uniquely solve for $f_k^\pm$. This establishes the formula (12), from which the other statements readily follow. □

REMARK 3.1. In Theorem 3.1(3), the Riccati equation (7) also has a *singular* formal solution $\hat{f}_-$ on U whose leading-order term is the singular leading-order solution f_0^- on U. The singularities of the coefficients of $\hat{f}_-$ are poles occurring at the zeros of a_0. We will examine in more detail the singularities of formal (and exact) solutions in a forthcoming paper.

REMARK 3.2 (Formal discriminant). Since in the generic situation the Riccati equation has precisely two formal solutions $\hat{f}_+, \hat{f}_-$, we can introduce a notion of discriminant for the Riccati equation analogous to the discriminant of a quadratic equation by simply mimicking the formula.

Thus, let $\mathsf{U} \subset \mathsf{X}$ be a domain free of turning points that supports a univalued square-root branch $\sqrt{D_0}$, and suppose that a_0 is nonvanishing on U. We define the *formal discriminant* of the Riccati equation (7) by the following formula:

$$\hat{D} := \hat{a}^2 (\hat{f}_+ - \hat{f}_-)(\hat{f}_- - \hat{f}_+). \qquad (17)$$

It is a formal power series with holomorphic coefficients on U, and its leading-order term is precisely the leading-order discriminant D_0. This quantity plays an important role in addressing global questions in the WKB analysis that will be studied elsewhere.

3.3 Gevrey regularity of formal solutions

In this subsection, we prove the following general result about the regularity of formal solutions, which generalizes Proposition A.1.1 in [1, p. 19] (see also [20, p. 252]), where it is assumed that $\hat{a} = -1, \hat{b} = 0$, and $\hat{c}$ is an entire holomorphic function of x only (i.e., $\hat{c}(x,\hbar) = c_0(x)$).

PROPOSITION 3.1 (Local Gevrey regularity of formal solutions). *Consider a formal Riccati equation (7) on X with leading-order discriminant $D_0 \not\equiv 0$. Let $\mathsf{U} \subset \mathsf{X}$ be any domain free of turning points that supports a univalued square-root branch $\sqrt{D_0}$, and let $\hat{f}$ be a formal solution on U. If the coefficients $\hat{a}, \hat{b}, \hat{c}$ are locally uniformly Gevrey series on U, then so is $\hat{f}$. In particular, the formal Borel transform $\hat{\phi}(x,\xi) := \hat{\mathfrak{B}}[\hat{f}](x,\xi)$ of $\hat{f}$ is a locally uniformly convergent power series in ξ.*

Concretely, Proposition 3.1 says that if the coefficients a_k, b_k, c_k of the power series $\hat{a}, \hat{b}, \hat{c}$ grow at most like $k!$, then the coefficients f_k of any formal solution $\hat{f}$ likewise grow at most like $k!$. This is made precise in the following corollary.

COROLLARY 3.1 (At most factorial growth). *Consider a formal Riccati equation (7) on X with leading-order discriminant $D_0 \not\equiv 0$. Let $\mathsf{U} \subset \mathsf{X}$ be any domain free of turning points that supports a univalued square-root branch $\sqrt{D_0}$, and let $\hat{f}$ be a formal solution on U.*

Take any pair of nested compactly contained subsets $\mathsf{U}_0 \Subset \mathsf{U}_1 \Subset \mathsf{U}$, *and suppose that there are real constants* $A, B > 0$ *such that*

$$\left|a_k(x)\right|, \left|b_k(x)\right|, \left|c_k(x)\right| \leq AB^k k! \qquad (\forall k \geq 0, \forall x \in \mathsf{U}_1). \tag{18}$$

Then there are real constants $C, M > 0$ *such that*

$$\left|f_k(x)\right| \leq CM^k k! \qquad (\forall k \geq 0, \forall x \in \mathsf{U}_0). \tag{19}$$

Proof of Proposition 3.1. Let $\mathbb{D}_R \subset \mathsf{U}$ be any sufficiently small disk of some radius $R > 0$ on which $\hat{a}, \hat{b}, \hat{c}$ are uniformly Gevrey and $\sqrt{D_0}$ is bounded both above and below by a nonzero constant. Thus, there are real constants $A, B > 0$, which give the following uniform bounds:

$$\left|a_k(x)\right|, \left|b_k(x)\right|, \left|c_k(x)\right| \leq AB^k k! \qquad \text{and} \qquad A^{-1} \leq \left|\sqrt{D_0(x)}\right| \leq A \tag{20}$$

for all integers $k \geq 0$ and all $x \in \mathbb{D}_R$. It will be convenient for us to assume without loss of generality that $A \geq 3$ and $R < 1$. We will prove that $\hat{f}$ is a uniformly Gevrey power series on any compactly contained subset of $\mathbb{D}_R$. In fact, we will prove something a little bit stronger as follows. For any $r \in (0, R)$, denote by $\mathbb{D}_r \subset \mathbb{D}_R$ the concentric subdisk of radius r. Then Proposition 3.1 follows from the following claim.

CLAIM 3.1. *There exist real constants* $C, M > 0$ *such that, for any* $r \in (0, R)$,

$$\left|f_k(x)\right| \leq CM^k \delta^{-k} k! \tag{21}$$

for all integers $k \geq 0$ *and uniformly for all* $x \in \mathbb{D}_r$, *where* $\delta := R - r$. *(The constants* C, M *are independent of* r, x, k, *but may depend on* R, A, B.) *In particular, for any* $r \in (0, R)$, *the power series* $\hat{f}$ *is Gevrey uniformly for all* $x \in \mathbb{D}_r$.

Proof. First, it is easy to find a constant $C > 0$ (independent of r) such that

$$\left|f_0(x)\right| \leq C \tag{22}$$

uniformly for all $x \in \mathbb{D}_R$ (see Lemma 3.1). Without loss of generality, assume that

$$C \geq A \geq 3. \tag{23}$$

Then the bound (21) will be demonstrated in two main steps. First, we will recursively construct a sequence $(M_k)_{k=0}^{\infty}$ of positive real numbers such that, for all $k \geq 0$ and all $r \in (0, R)$, we have the following uniform bound for all $x \in \mathbb{D}_r$:

$$\left|f_k(x)\right| \leq CM_k \delta^{-k} k!. \tag{24}$$

Then we will show that there is a constant $M > 0$ (independent of r) such that $M_k \leq M^k$ for all k. $\square$

CONSTRUCTION OF $(M_k)_{k=0}^{\infty}$. The bound (24) for $k = 0$ is just the bound (22) if we put $M_0 := 1$. Now, we use induction on k and formula (12). Assume that we have already constructed positive real numbers $M_0, \ldots, M_{k-1}$ such that, for all $i = 0, \ldots, k-1$, all $r \in (0, R)$, and all $x \in \mathbb{D}_r$, we have the bound

$$\left|f_i(x)\right| \leq CM_i \delta^{-i} i!. \tag{25}$$

In order to derive an estimate for f_k, we first need to estimate the derivative term $\partial_x f_{k-1}$, for which we use Cauchy estimates as follows.

SUB-CLAIM. For all $r \in (0, R)$ and all $x \in \mathbb{D}_r$,

$$\left| \partial_x f_{k-1}(x) \right| \leq C^2 M_{k-1} \delta^{-k} k!. \tag{26}$$

Proof of Sub-Claim. For every $r \in (0, R)$, define

$$\delta_k := \delta \frac{k}{k+1} \qquad \text{and} \qquad r_k := R - \delta_k.$$

Inequality (25) holds in particular for $i = k - 1$, $r = r_k$. So for all $x \in \mathbb{D}_{r_k}$, we find

$$\left| f_{k-1}(x) \right| \leq C M_{k-1} \delta_k^{1-k} (k-1)! \leq C^2 M_{k-1} \delta^{-k} k! \frac{\delta}{k+1}.$$

Here, we have used the estimate $(1 + 1/k)^{k-1} \leq e \leq C$. Finally, notice that for every $x \in \mathbb{D}_r$, the closed disk around x of radius $r_k - r = \delta - \delta_k = \frac{\delta}{k+1}$ is contained inside the disk $\mathbb{D}_{r_k}$. Therefore, Cauchy estimates imply (26). $\square$

Using (20), (23), (25), (26), and the fact that $\delta < 1$, we can now estimate f_k:

$$|f_k| \leq C \left(\left| \partial_x f_{k-1} \right| + \sum_{\substack{k_1+k_2+k_3=k \\ }}^{k_2, k_3 \neq k} |a_{k_1}| \cdot |f_{k_2}| \cdot |f_{k_3}| + \sum_{\substack{k_1+k_2=k \\ }}^{k_2 \neq k} |b_{k_1}| \cdot |f_{k_2}| + |c_k| \right)$$

$$\leq C \left(C^2 M_{k-1} \delta^{-k} k! + \delta^{-k} C^3 k! \sum_{\substack{k_1+k_2+k_3=k \\ }}^{k_2, k_3 \neq k} B^{k_1} M_{k_2} M_{k_3} \right.$$

$$\left. + \delta^{-k} C^2 k! \sum_{\substack{k_1+k_2=k \\ }}^{k_2 \neq k} B^{k_1} M_{k_2} + C B^k k! \right)$$

$$\leq C^4 \left(M_{k-1} + \sum_{\substack{k_1+k_2+k_3=k \\ }}^{k_2, k_3 \neq k} B^{k_1} M_{k_2} M_{k_3} + \sum_{\substack{k_1+k_2=k \\ }}^{k_2 \neq k} B^{k_1} M_{k_2} + B^k \right) \delta^{-k} k!.$$

We can therefore define, for $k \geq 1$,

$$M_k := C^3 \left(M_{k-1} + \sum_{\substack{k_1+k_2+k_3=k \\ }}^{k_2, k_3 \neq k} B^{k_1} M_{k_2} M_{k_3} + \sum_{\substack{k_1+k_2=k \\ }}^{k_2 \neq k} B^{k_1} M_{k_2} + B^k \right). \tag{27}$$

CONSTRUCTION OF M. To see that $M_k \leq M^k$ for some $M > 0$, we argue as follows. Consider the following power series in an abstract variable t:

$$\hat{p}(t) := \sum_{k=0}^{\infty} M_k t^k \qquad \text{and} \qquad q(t) := \sum_{k=1}^{\infty} B^k t^k \quad \in \quad \mathbb{C}[[t]].$$

Note that $\hat{p}(0) = M_0 = 1$ and $q(0) = 0$, and notice that $q(t)$ is convergent. We will show that $\hat{p}(t)$ is also convergent. The key is the observation that they satisfy the following algebraic equation:

$$\hat{p}(t) - 1 = C^3 \left(t \hat{p}(t) + q(t) \hat{p}(t)^2 + \left(\hat{p}(t) - 1 \right)^2 + q(t) \hat{p}(t) + q(t) \right). \tag{28}$$

This equation was found by trial and error, and it is straightforward to verify directly by substituting the power series $\hat{p}(t), q(t)$ and comparing the coefficients of t^k using the defining formula (27) for M_k. Namely, the terms M_{k-1} and B^k in (27) correspond, respectively, to

the terms $t\hat{p}(t)$ and $q(t)$ in (28), whereas the first and second sums correspond, respectively, to $q(t)\hat{p}(t)^2 + \big(\hat{p}(t) - 1\big)^2$ and $q(t)\hat{p}(t)$.

Now, consider the following holomorphic function in two complex variables (t, p):

$$F(t, p) := -p + 1 + C^3 \Big(tp + q(t)p^2 + (p-1)^2 + q(t)p + q(t) \Big).$$

It has the following properties:

$$F(0, 1) = 0 \qquad \text{and} \qquad \left. \frac{\partial F}{\partial p} \right|_{(t, p) = (0, 1)} = -1 \neq 0.$$

By the Holomorphic Implicit Function Theorem, there exists a unique holomorphic function $p(t)$ near $t = 0$ such that $p(0) = 1$ and $F\big(t, p(t)\big)\big) = 0$. Thus, $\hat{p}(t)$ must be its convergent Taylor series expansion at $t = 0$ and its coefficients grow at most exponentially: that is, there is a constant $M > 0$ such that $M_k \leq M^k$. This completes the proof of the Claim and hence of Proposition 3.1.

§4. WKB geometry

In this intermediate section, we introduce a coordinate transformation which plays a central role in the construction of exact solutions in §5. It is used to determine regions in $\mathbb{C}_x$ where the Borel–Laplace method can be applied to the Riccati equation.

The material of this section can essentially be found in [18, §§9–11] (see also [2, §3.4]). These references use the language of foliations given by quadratic differentials on Riemann surfaces. The relevant quadratic differential is $D_0(x)\mathrm{d}x^2$. The reader may be more familiar with the set of critical leaves of this foliation, which is encountered in the literature under various names including *Stokes curves*, *Stokes graph*, *spectral network*, *geodesics*, and *critical trajectories* [3, 5, 6, 8, 11].

To keep the discussion a little more elementary, we state the relevant definitions and facts by appealing directly to explicit formulas using the *Liouville transformation* (defined below) commonly used in the WKB analysis of Schrödinger equations.

4.1 The Liouville transformation

Throughout this section, we remain in the background setting of §2.1. Recall the leading-order discriminant $D_0 = b_0^2 - 4a_0c_0$, which is a holomorphic function on X, assumed not identically zero. Fix a phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, a basepoint $x_0 \in \mathsf{X}$, and a univalued square-root branch $\sqrt{D_0}$ near x_0 (i.e., either in a disk or a sectorial neighborhood of x_0). Consider the following local coordinate transformation near x_0, called the *Liouville transformation*:

$$z = \Phi(x) := \int_{x_0}^x \sqrt{D_0(t)}\mathrm{d}t. \tag{29}$$

Let $\mathsf{U} \subset \mathsf{X}$ be any domain which is free of turning points, supports a univalued square-root branch $\sqrt{D_0}$ (e.g., U is simply connected), and contains x_0 in the interior or on the boundary. Then the Liouville transformation defines a (possibly multivalued) local biholomorphism $\Phi : \mathsf{U} \longrightarrow \mathbb{C}_z$. Notice that turning points are precisely the locations in X where Φ fails to be conformal.

REMARK 4.1. The basepoint of integration x_0 can in principle be chosen even on the boundary of X or at infinity in $\mathbb{C}_x$ provided that the integral is well defined. Liouville

transformations such as (29) are encountered in the analysis of the Schrödinger equation $\hbar^2 \partial_x^2 \psi - q(x)\psi = 0$ as described, for example, in Olver's textbook [14, §6.1]. However, note that our formula (29) in the special case of the Schrödinger equation reads

$$\Phi(x) = \int_{x_0}^x \sqrt{D_0(t)}\mathrm{d}t = 2\int_{x_0}^x \sqrt{q(t)}\mathrm{d}t, \tag{30}$$

which differs from formula (1.05) in [14, §6.1] by a factor of 2.

4.2 WKB trajectories

Let $x_0 \in \mathsf{X}$ be a regular point, and consider the Liouville transformation (29). A *WKB θ-trajectory* through x_0 is the real one-dimensional smooth curve Γ_θ on X locally determined by the following equation:

$$\Gamma_\theta = \Gamma_\theta(x_0) \quad : \quad \mathrm{Im}\big(e^{-i\theta}\Phi(x)\big) = 0. \tag{31}$$

A *WKB θ-trajectory $\pm$-ray* (or simply a *WKB ray* if the context is clear) emanating from x_0 is the component $\Gamma_\theta^\pm$ of Γ_θ given, respectively, by

$$\Gamma_\theta^\pm = \Gamma_\theta^\pm(x_0) \quad : \quad \pm\mathrm{Re}\big(e^{-i\theta}\Phi(x)\big) \geq 0; \tag{32}$$

WKB trajectories are regarded by definition as being maximal under inclusion. Explicitly, (31) and (32) read

$$\mathrm{Im}\left(e^{-i\theta}\int_{x_0}^x \sqrt{D_0(t)}\mathrm{d}t\right) = 0 \quad\text{and}\quad \pm\mathrm{Re}\left(e^{-i\theta}\int_{x_0}^x \sqrt{D_0(t)}\mathrm{d}t\right) \geq 0. \tag{33}$$

The Liouville transformation Φ with basepoint x_0 maps the WKB θ-trajectory $\Gamma_\theta(x_0)$ to a possibly infinite straight line segment $(\tau_- e^{i\theta}, \tau_+ e^{i\theta}) \subset e^{i\theta}\mathbb{R} \subset \mathbb{C}_z$ containing the origin $0 = \Phi(x_0)$, that is, with $\tau_- < 0 < \tau_+$. Maximality means that this line segment is the largest possible image. The image of the WKB θ-trajectory $\pm$-ray emanating from x_0 is then, respectively, the line segment $[0, \tau_+ e^{i\theta})$ or $(\tau_- e^{i\theta}, 0]$.

All other nearby WKB θ-trajectories can be locally described by an equation of the form $\mathrm{Im}\big(e^{-i\theta}\Phi(x)\big) = c$ for some $c \in \mathbb{R}$. That is, if $\mathsf{U}_0 \subset \mathsf{X}$ is a simply connected neighborhood of x_0 free of turning points, then any WKB θ-trajectory Γ_θ' intersecting U_0 is locally given by this equation with $c = \mathrm{Im}\, e^{-i\theta}\Phi(x_0')$ for some $x_0' \in \mathsf{U}_0$. Its image in $\mathbb{C}_z$ under Φ is an interval on the parallel line containing $z_0' := \Phi(x_0')$:

$$z_0' + e^{i\theta}\mathbb{R} := \big\{z = z_0' + \xi \mid \xi \in e^{i\theta}\mathbb{R}\big\}.$$

Our primary focus is *infinite WKB rays* $\Gamma_\theta^\pm$, defined as having $|\tau_\pm| = \infty$, respectively. An *infinite WKB trajectory* is one with at least one infinite ray. A *generic WKB trajectory* is one with both rays being infinite.

An infinite WKB trajectory may be a *closed WKB trajectory* if it is a simple closed curve in the complement of the turning points. A closed WKB θ-trajectory has the property that there is a nonzero time $\omega \in \mathbb{R}$ such that $\Phi^{-1}(e^{i\theta}\omega) = \Phi^{-1}(0)$ (see [18, §9.2]). This only happens when the Liouville transformation is analytically continued along the trajectory to a multivalued function. We refer to the smallest possible positive such $\omega \in \mathbb{R}_+$ as the *WKB trajectory period*. It follows from general considerations (see [18, §9]) that if the WKB θ-trajectory through x_0 is a closed trajectory, then all nearby WKB θ-trajectories are also closed with the same period.

A nonclosed infinite WKB ray may tend to a single point, limit to a dense subset of X, or escape X altogether. Formally, the *limit* of an infinite WKB ray $\Gamma_\theta^\pm$ by definition, respectively, is the limit set

$$\overline{\Phi^{-1}\Big([\tau e^{i\theta}, +\infty \cdot e^{i\theta})\Big)} \quad \text{as} \quad \tau \to +\infty \quad \text{or} \quad \overline{\Phi^{-1}\Big((-\infty \cdot e^{i\theta}, \tau e^{i\theta}]\Big)} \quad \text{as} \quad \tau \to -\infty.$$

Obviously, this definition is independent of the chosen basepoint x_0 along the trajectory. If the limit is a single point $x_\infty \in \mathbb{C}_x$, then this point (sometimes called an *infinite critical point*) is necessarily a pole of D_0 of order $m \geq 2$ (see [18, §10.2]). Given $\alpha \in \{+, -\}$, it also follows from general considerations that if the WKB θ-trajectory α-ray emanating from x_0 tends to an infinite critical point, then x_0 has a disk neighborhood U_0 such that every WKB θ-trajectory α-ray emanating from U_0 tends to the same infinite critical point.

Finite WKB rays—those with finite τ_+ or τ_-—are inadmissible for our construction of exact solutions in §5. As τ approaches τ_+ or τ_-, respectively, such a WKB trajectory either tends to a turning point or escapes to the boundary of X in finite time. If it tends to a single point on the boundary of X, this point is either a turning point or a simple pole of the discriminant D_0 (see [18, §10.2]). For this reason, turning points and simple poles are sometimes collectively referred to as *finite critical points*. A *singular WKB ray* is one that approaches a finite critical point. They are important in the global analysis of exact solutions, which will be discussed in detail elsewhere.

A *WKB θ-strip domain* containing x_0 is any domain neighborhood of x_0 which is swept out by generic WKB θ-trajectories. It necessarily has the form

$$\mathsf{W}_\theta = \mathsf{W}_\theta(x_1, r) := \Phi^{-1}\big(\mathsf{H}_\theta\big) \subset \mathbb{C}_x,$$
$$\text{where} \quad \mathsf{H}_\theta = \mathsf{H}_\theta(z_1, r) := \Big\{ z \ \Big| \ \text{dist}(z, z_1 + e^{i\theta}\mathbb{R}) < r \Big\} \subset \mathbb{C}_z, \tag{34}$$

for some $r > 0$ and some $x_1 = \Phi^{-1}(z_1) \in X$. Similarly, a *WKB $(\theta, \pm)$-half-strip domain* containing x_0 is any domain neighborhood of x_0 which is swept out by infinite WKB θ-trajectory $\pm$-rays. It necessarily has the form

$$\mathsf{W}_\theta^\pm = \mathsf{W}_\theta^\pm(x_1, r) := \Phi^{-1}\big(\mathsf{H}_\theta^\pm\big) \subset \mathbb{C}_x,$$
$$\text{where} \quad \mathsf{H}_\theta^\pm = \mathsf{H}_\theta^\pm(z_1, r) := \Big\{ z \ \Big| \ \text{dist}(z, z_1 + e^{i\theta}\mathbb{R}_\pm) < r \Big\} \subset \mathbb{C}_z. \tag{35}$$

Note that we obviously have $\mathsf{W}_\theta = \mathsf{W}_\theta^- \cup \mathsf{W}_\theta^+$. The intersection $\mathsf{W}_\theta^- \cap \mathsf{W}_\theta^+ = \Phi^{-1}\big(\{|z - z_1| < r\}\big)$ may be called a *WKB disk* around x_1, and it is clearly independent of θ.

If the WKB θ-trajectory through x_0 is not closed, then WKB θ-strip W_θ is a simply connected domain conformally equivalent to the infinite strip H_θ via the Liouville transformation $\Phi : \mathsf{W}_\theta \xrightarrow{\sim} \mathsf{H}_\theta$.

On the other hand, if the WKB θ-trajectory through x_0 is closed, then W_θ is swept out by closed WKB θ-trajectories, so W_θ has the topology of an annulus. In this case, W_θ is sometimes called a *WKB θ-ring domain*. The Liouville transformation Φ is a multivalued holomorphic function on W_θ, but the inverse $\Phi^{-1} : \mathsf{H}_\theta \to \mathsf{W}_\theta$ is still necessarily a local biholomorphism.

§5. Exact perturbation theory

We can now state and prove our main results. Throughout this section, we remain in the background setting of §2.1. Namely, X is a domain in $\mathbb{C}_x$ and $\mathsf{S} \subset \mathbb{C}_\hbar$ is a sectorial domain

with opening A. In addition, we assume that $|\mathsf{A}| = \pi$ so that $\mathsf{A} = \mathsf{A}_\theta := (\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})$ for some $\theta \in \mathbb{R}$.

5.1 Existence and uniqueness of local exact solutions

The main result of this paper is the following theorem.

THEOREM 5.1 (Main exact existence and uniqueness theorem). *Consider the Riccati equation*

$$\hbar \partial_x f = af^2 + bf + c \tag{36}$$

whose coefficients a, b, c are holomorphic functions of $(x, \hbar) \in \mathsf{X} \times \mathsf{S}$ admitting locally uniform asymptotic expansions $\hat{a}, \hat{b}, \hat{c}$ as $\hbar \to 0$ along A. Assume that $D_0 = b_0^2 - 4a_0c_0 \not\equiv 0$. Fix a regular point $x_0 \in \mathsf{X}$, a square-root branch $\sqrt{D_0}$ near x_0, and a sign $\alpha \in \{+, -\}$. In addition, we assume the following hypotheses:

(1) *There is a WKB (θ, α)-half-strip domain $\mathsf{W} = \mathsf{W}_\theta^\alpha \subset \mathsf{X}$ containing x_0; assume in addition that a_0 is nonvanishing on W if $\alpha = -$.*
(2) *The asymptotic expansions of the coefficients a, b, c are valid with Gevrey bounds as $\hbar \to 0$ along the closed arc $\bar{\mathsf{A}}_\theta = [\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}]$, with respect to the asymptotic scale $\sqrt{D_0}$, uniformly for all $x \in \mathsf{W}$:*

$$a \simeq \hat{a}, \quad b \simeq \hat{b}, \quad c \simeq \hat{c} \qquad \text{as } \hbar \to 0 \text{ along } \bar{\mathsf{A}}_\theta, \text{ wrt } \sqrt{D_0}, \text{ unif. } \forall x \in \mathsf{W}. \tag{37}$$

Then the Riccati equation has a canonical local exact solution f_α^θ near x_0 which is asymptotic to the formal solution $\hat{f}_\alpha$ as $\hbar \to 0$ in the direction θ. Namely, for any compactly contained domain $\mathsf{U}_0 \Subset \mathsf{W}$, there is a sectorial domain $\mathsf{S}_0 \subset \mathsf{S}$ with the same opening A_θ such that the Riccati equation has a unique holomorphic solution f_α^θ on $\mathsf{U}_0 \times \mathsf{S}_0$ which is Gevrey asymptotic to $\hat{f}_\alpha$ as $\hbar \to 0$ along the closed arc $\bar{\mathsf{A}}_\theta$ uniformly for all $x \in \mathsf{U}_0$:

$$f_\alpha^\theta \simeq \hat{f}_\alpha \qquad \text{as } \hbar \to 0 \text{ along } \bar{\mathsf{A}}_\theta, \text{ unif. } \forall x \in \mathsf{U}_0. \tag{38}$$

We will prove this theorem in §5.1.2. First, let us make some remarks.

REMARK 5.1. (1) For the reader's convenience, we recall here that hypothesis (2) in Theorem 5.1 explicitly means that there are real constants $A, B > 0$ such that for all $n \geq 0$, all $x \in \mathsf{W}$, and all sufficiently small $\hbar \in \mathsf{S}$,

$$\left| a(x, \hbar) - \sum_{k=0}^{n-1} a_k(x) \hbar^k \right| \leq \left| \sqrt{D_0(x)} \right| AB^n n! |\hbar|^n, \tag{39}$$

and similarly for b and c. Likewise, the asymptotic condition (38) reads explicitly as follows: there are real constants $C, M > 0$ such that for all $n \geq 0$, all $x \in \mathsf{U}_0$, and all sufficiently small $\hbar \in \mathsf{S}_0$,

$$\left| f_\alpha^\theta(x, \hbar) - \sum_{k=0}^{n-1} f_k^\alpha(x) \hbar^k \right| \leq CM^n n! |\hbar|^n. \tag{40}$$

(2) If all the hypotheses of Theorem 5.1 are satisfied for both signs $\alpha = +, -$, then we obviously obtain a pair of distinct canonical exact solutions f_+^θ, f_-^θ near x_0. Let us also note that the solution f_α^θ obviously does not depend in any serious way on the chosen basepoint x_0.

(3) In many applications, including the exact WKB analysis of Schrödinger, the coefficients of the Riccati equation (36) do not satisfy hypothesis (2) in Theorem 5.1 on the nose and we have to do an additional transformation in order to apply our theorem. This is discussed in §5.4.

(4) The somewhat abstract general hypotheses of this theorem get significantly simplified in many notable situations which are discussed in §6.

(5) The sectorial domain S_0 in the conclusion of Theorem 5.1 can be chosen to be a Borel disK $\left\{ \hbar \mid \mathrm{Re}(e^{i\theta}/\hbar) > 1/d_0 \right\}$ bisected by the direction θ of sufficiently small diameter $d_0 > 0$.

We have the following immediate corollary of the uniqueness property of canonical exact solutions.

COROLLARY 5.1 (Extension to larger domains). *Let* $\mathsf{U} \subset \mathsf{X}$ *be a domain free of turning points that supports a univalued square-root branch* $\sqrt{D_0}$. *Fix a sign* $\alpha \in \{+, -\}$ *such that the leading-order solution* f_0^α *is holomorphic on* U. *In addition, assume that hypotheses (1)–(3) in Theorem 5.1 are satisfied for every point* $x_0 \in \mathsf{U}$.

Then the Riccati equation (36) has a canonical exact solution f_α^θ *on* U *asymptotic to the formal solution* $\hat{f}_\alpha$ *as* $\hbar \to 0$ *in the direction* θ. *Namely, there is a domain* $\mathbb{U} \subset \mathsf{U} \times \mathsf{S}$ *and a holomorphic solution* f_α^θ *defined on* $\mathbb{U}$ *with the following property: for every point* $x_0 \in \mathsf{U}$, *there is a neighborhood* $\mathsf{U}_0 \subset \mathsf{U}$ *of* x_0 *and a sectorial domain* $\mathsf{S}_0 \subset \mathsf{S}$ *with the same opening* A_θ *such that* $\mathsf{U}_0 \times \mathsf{S}_0 \subset \mathbb{U}$ *and* f_α^θ *is the unique holomorphic solution on* $\mathsf{U}_0 \times \mathsf{S}_0$ *satisfying (38). In particular,* f_α^θ *is the unique solution on* $\mathbb{U}$ *with the following locally uniform Gevrey asymptotics:*

$$f_\alpha^\theta \simeq \hat{f}_\alpha \qquad \text{as } \hbar \to 0 \text{ along } \bar{\mathsf{A}}_\theta, \text{ loc.unif. } \forall x \in \mathsf{U}. \tag{41}$$

In particular, the domain $\mathsf{U} \subset \mathsf{X}$ in Corollary 5.1 can be a union of WKB half-strips. In fact, examining the proof of Theorem 5.1 more closely, it is readily seen that on any WKB half-strip, we can state the asymptotic property of canonical exact solutions more precisely as follows.

PROPOSITION 5.1 (Asymptotics on WKB half-strips). *Assume all the hypotheses of Theorem 5.1. Then the canonical local exact solution* f_α^θ *uniquely extends to an exact solution on* W *with the following locally uniform Gevrey asymptotics:*

$$f_\alpha^\theta \simeq \hat{f}_\alpha \qquad \text{as } \hbar \to 0 \text{ along } \bar{\mathsf{A}}_\theta, \text{ loc.unif. } \forall x \in \mathsf{W}. \tag{42}$$

In fact, even more is true. Let $r > 0$ *be such that* $\mathsf{W} = \mathsf{W}_\theta^\alpha(x_0', r)$. *For any* $r_0 \in (0, r)$, *let* $\mathsf{W}_0 := \mathsf{W}_\theta^\alpha(x_0', r_0)$. *Then there is a sectorial domain* $\mathsf{S}_0 \subset \mathsf{S}$ *with the same opening* A_θ *such that the canonical exact solution* f_α^θ *extends to a holomorphic solution on* $\mathsf{W}_0 \times \mathsf{S}_0$ *with the following uniform Gevrey asymptotic property:*

$$f_\alpha^\theta \simeq \hat{f}_\alpha \qquad \text{as } \hbar \to 0 \text{ along } \bar{\mathsf{A}}_\theta, \text{ unif. } \forall x \in \mathsf{W}_0. \tag{43}$$

Proof. The fact that f_α^θ extends to any W_0 as stated follows immediately from the proof of Theorem 5.1. The uniqueness part of the construction guarantees that all these extensions coincide. □

5.1.1. Riccati equations on horizontal half-strips

The strategy of the proof of Theorem 5.1 is to use the Liouville transformation Φ_α to transform the Riccati equation into one in standard form over a horizontal half-strip in the z-space, and then apply the Borel–Laplace method. First, we give a general description of this standard form of the Riccati equation and prove the corresponding version of the exact existence and uniqueness theorem (Lemma 5.1). Then we will show that any Riccati equation satisfying the assumptions of Theorem 5.1 can be put into this standard form, thereby deducing our claims.

Let $\mathsf{H}_+ \subset \mathbb{C}_z$ be a horizontal half-strip around the positive real axis $\mathbb{R}_+ \subset \mathbb{C}_z$ of some radius $r > 0$, and let $\mathsf{S}_+ \subset \mathbb{C}_\hbar$ a Borel disk bisected by the positive real axis of some diameter $d > 0$:

$$\mathsf{H}_+ := \big\{ z \mid \operatorname{dist}(z, \mathbb{R}_+) < r \big\} \quad \text{and} \quad \mathsf{S}_+ := \big\{ \hbar \mid \operatorname{Re}(1/\hbar) > 1/d \big\}. \tag{44}$$

The opening of S_+ is the semicircular arc $\mathsf{A}_+ := (-\frac{\pi}{2}, +\frac{\pi}{2})$. Consider the following singularly perturbed Riccati equation on $\mathsf{H}_+ \times \mathsf{S}_+$:

$$\hbar \partial_z F = F + \hbar \big(A_2 F^2 + A_1 F + A_0 \big), \tag{45}$$

where A_i are holomorphic functions of $(z, \hbar) \in \mathsf{H}_+ \times \mathsf{S}_+$ which admit uniform Gevrey asymptotic expansions as $\hbar \to 0$ along the closed arc $\bar{\mathsf{A}}_+$:

$$A_i \simeq \hat{A}_i, \qquad \text{as } \hbar \to 0 \text{ along } \bar{\mathsf{A}}_+, \text{ unif. } \forall z \in \mathsf{H}_+. \tag{46}$$

Denote their leading-order parts by $a_i = a_i(z)$. The corresponding leading-order equation is simply $F_0 = 0$. By Theorem 3.1(1), this Riccati equation has a unique formal solution $\hat{F}_+$ on H_+, and its leading order part is $F_0^+ = 0$ and its next-to-leading order part is $F_1^+ = -a_0$.

LEMMA 5.1 (Main Lemma). *For every $r_0 \in (0, r)$, there is $d_0 \in (0, d]$ such that the Riccati equation (45) has a canonical exact solution F_+ defined on*

$$\mathsf{H}_0^+ \times \mathsf{S}_0^+ := \big\{ z \mid \operatorname{dist}(z, \mathbb{R}_+) < r_0 \big\} \times \big\{ \hbar \mid \operatorname{Re}(1/\hbar) > 1/d_0 \big\} \subset \mathsf{H}_+ \times \mathsf{S}_+. \tag{47}$$

Namely, F_+ is the unique holomorphic solution on $\mathsf{H}_0^+ \times \mathsf{S}_0^+$, which admits the formal solution $\hat{F}_+$ as its uniform Gevrey asymptotic expansion along $\bar{\mathsf{A}}_+$:

$$F_+(z, \hbar) \simeq \hat{F}_+(z, \hbar) \quad \text{as } \hbar \to 0 \text{ along } \bar{\mathsf{A}}_+, \text{ unif. } \forall z \in \mathsf{H}_0^+. \tag{48}$$

Moreover, it has the following properties.

(P1) *The formal Borel transform*

$$\hat{\varphi}_+(x, \xi) = \hat{\mathfrak{B}}[\hat{F}_+](x, \xi) := \sum_{n=0}^\infty \varphi_n^+(x) \xi^n \quad \text{where} \quad \varphi_k^+(x) := \tfrac{1}{k!} F_{k+1}^+(x)$$

converges uniformly on H_0^+.

(P2) *For any $\epsilon \in (0, r - r_0)$, let $\Xi_+ := \big\{ \xi \mid \operatorname{dist}(\xi, \mathbb{R}_+) < \epsilon \big\}$. Then the analytic Borel transform*

$$\varphi_+(z, \xi) = \mathfrak{B}_+[F_+](z, \xi) := \frac{1}{2\pi i} \oint F_+(x, \hbar) e^{\xi/\hbar} \frac{d\hbar}{\hbar^2}$$

is uniformly convergent for all $(z, \xi) \in \mathsf{H}_0^+ \times \Xi_+$. Here, the integral "$\oint$" is done along the boundary of any Borel disk $\mathsf{S}_+' \subsetneq \mathsf{S}_+$ of strictly smaller diameter $d' < d$, traversed anticlockwise.

(P3) The Laplace transform $\mathfrak{L}_+\big[\varphi_+\big](z, \hbar)$ is uniformly convergent for all $(z, \hbar) \in \mathsf{H}_0^+ \times \mathsf{S}_0^+$ and satisfies $F_+(z, \hbar) = \mathfrak{L}_+\big[\varphi_+\big](z, \hbar)$.

(P4) Therefore, F_+ is the uniform Borel resummation of its asymptotic power series $\hat{F}_+$; that is, $F_+(z, \hbar) = \mathcal{S}_+\big[\hat{F}_+\big](z, \hbar)$ for all $(z, \hbar) \in \mathsf{H}_0^+ \times \mathsf{S}_0^+$.

(P5) If the coefficients A_0, A_1, A_2 are periodic in z with period $\omega \in \mathbb{C}$, then so is F_+.

Proof. We use the Borel–Laplace method to construct the exact solution F_+. Namely, we first apply the Borel transform to obtain a first-order nonlinear PDE which is easy to rewrite as an integral equation. Then most of the heavy lifting is constrained to solving this integral equation, which we do via the method of successive approximations. The desired solution F_+ is then obtained by applying the Laplace transform.

Uniqueness.

Suppose F_+, F_+' are two such exact solutions defined on $\mathsf{H}_0^+ \times \mathsf{S}_0^+$. Their difference $F_+ - F_+'$ is a holomorphic function on $\mathsf{H}_0^+ \times \mathsf{S}_0^+$ which is uniformly Gevrey asymptotic to 0 as $\hbar \to 0$ along $\bar{\mathsf{A}}_+$. By Nevanlinna's theorem [9, Theorem 5.3.9], there can be only one holomorphic function on S_0^+ (namely, the constant function 0), which is Gevrey asymptotic to 0 as $\hbar \to 0$ along $\bar{\mathsf{A}}_+$. Thus, $F_+ - F_+'$ is must be the zero function.

Step 1: *The analytic Borel transform.*

It follows from Nevanlinna's theorem [9, Theorem 5.3.9] that there is some tubular neighborhood $\Xi_+ := \big\{\xi \big| \mathrm{dist}(\xi, \mathbb{R}_+) < \epsilon\big\}$ for some $\epsilon > 0$ such that the analytic Borel transforms $\alpha_i(z, \xi) := \mathfrak{B}_+[A_i](z, \xi)$ are holomorphic functions on $\mathsf{H}_+ \times \Xi_+ \subset \mathbb{C}_{z\xi}^2$ with uniformly at most exponential growth as $|\xi| \to +\infty$. Moreover, $A_i(z, \hbar) = a_i(z) + \mathfrak{L}_+\big[\alpha_i\big]$ for all $(z, \xi) \in \mathsf{H}_+ \times \Xi_+$.

Dividing (45) through by $\hbar$ and applying the analytic Borel transform $\mathfrak{B}_+$, we obtain the following PDE with convolution product:

$$\partial_z \phi - \partial_\xi \phi = \alpha_0 + a_1 \phi + \alpha_1 * \phi + a_2 \varphi * \phi + \alpha_2 * \phi * \phi, \tag{49}$$

where the unknown variables ϕ and F are related by $\phi = \mathfrak{B}_+[F]$ and $F = \mathfrak{L}_+[\phi]$.

Step 2: *The integral equation.*

The principal part of the PDE (49) has constant coefficients, so it is easy to rewrite it as an equivalent integral equation as follows. Consider the holomorphic change of variables $T : (z, \xi) \longmapsto (w, t) = (z + \xi, \xi)$. For any function $\alpha = \alpha(z, \xi)$ of two variables, introduce the following notation:

$$T^* \alpha(z, \xi) := \alpha\big(T(z, \xi)\big) = \alpha(z + \xi, \xi) \quad \text{and} \quad T_* \alpha(w, t) := \alpha\big(T^{-1}(w, t)\big) = \alpha(w - t, t).$$

Note that $T^* T_* \alpha = \alpha$. Under this change of coordinates, the differential operator $\partial_z - \partial_\xi$ transforms into $-\partial_t$, and so the left-hand side of (49) becomes $-\partial_t\big(T_* \phi\big)$. Integrating from 0 to t, and imposing the initial condition $\phi(z, 0) = \phi_0(z) := a_0(z)$, the left-hand side of the PDE (49) becomes $-T_* \phi$. Applying T^*, we therefore obtain the following integral equation for $\phi = \phi(z, \xi)$:

$$\phi = \phi_0 - T^* \int_0^t T_*\Big(\alpha_0 + a_1 \phi + \alpha_1 * \phi + a_2 \phi * \phi + \alpha_2 * \phi * \phi\Big) \mathrm{d}u. \tag{50}$$

Introduce the following notation: for any function $\alpha = \alpha(z, \xi)$ of two variables,

$$I_+\big[\alpha\big](z,\xi) := -T^* \int_0^t T_* \alpha \mathrm{d}u = -\int_0^\xi \alpha(z+\xi-u,u)\mathrm{d}u = \int_0^\xi \alpha(z+t,\xi-t)\mathrm{d}t, \qquad (51)$$

where the integration path is the straight line segment connecting 0 to ξ. Then the integral equation (50) can be written more succinctly as

$$\phi = \phi_0 + I_+\Big[\alpha_0 + a_1\phi + \alpha_1 * \phi + a_2\phi * \phi + \alpha_2 * \phi * \phi\Big]. \qquad (52)$$

Step 3: *Method of successive approximations.*

To solve (52), we use the method of successive approximations. Consider a sequence of functions $\{\phi_n\}_{n=0}^\infty$ defined recursively by $\phi_0 = a_0$, $\phi_1 := I_+\big[\alpha_0 + a_1\phi_0\big]$, and for $n \geq 2$ by the following formula:

$$\phi_n := I_+\left[a_1\phi_{n-1} + \alpha_1 * \phi_{n-2} + \sum_{\substack{i,j \geq 0 \\ i+j=n-2}} a_2\phi_i * \phi_j + \sum_{\substack{i,j \geq 0 \\ i+j=n-3}} \alpha_2 * \phi_i * \phi_j \right]. \qquad (53)$$

CLAIM 5.1 (Main Claim). *Let ϵ be so small that $\epsilon < r - r_0$. Then the infinite series*

$$\phi_+(z,\xi) := \sum_{n=0}^\infty \phi_n(z,\xi) \qquad (54)$$

defines a holomorphic solution of the integral equation (52) on the domain

$$\mathbf{H}_+ := \big\{ (z,\xi) \in \mathsf{H}_+ \times \Xi_+ \ \big|\ z+\xi \in \mathsf{H}_+ \big\}$$

with at most exponential growth at infinity in ξ; more precisely, it satisfies the following uniform exponential bound: there are real constants $A, K > 0$ such that

$$\big|\phi_+(z,\xi)\big| \leq Ae^{K|\xi|} \qquad\qquad \forall(z,\xi) \in \mathbf{H}_+. \qquad (55)$$

In particular, ϕ_+ is a well-defined holomorphic solution on $\mathsf{H}_0^+ \times \Xi_+ \subset \mathbf{H}_+$ where it satisfies the exponential estimate above.

Assuming this claim, only one step remains in order to complete the proof of Lemma 5.1, which is to take the Laplace transform of ϕ_+.

Step 4: *The Laplace transform.*

Let

$$F_+(z,\hbar) := \mathfrak{L}_+\big[\phi_+\big](z,\hbar) = \int_0^{+\infty} e^{-\xi/\hbar}\phi(z,\xi)\mathrm{d}\xi. \qquad (56)$$

This integral is uniformly convergent for all $z \in \mathsf{H}_0^+$ provided that $\mathrm{Re}(\hbar^{-1}) > C_2$. Thus, if we take $d_0 \in (0,d]$ strictly smaller than $1/C_2$, then formula (56) defines a holomorphic solution of the Riccati equation (45) on the domain $\mathsf{H}_0^+ \times \mathsf{S}_0^+$ where $\mathsf{S}_0^+ := \big\{\hbar \ \big|\ \mathrm{Re}(\hbar^{-1}) > 1/d_0 \big\}$. Furthermore, Nevanlinna's theorem implies that F_+ admits a uniform Gevrey asymptotic expansion on H_0^+ as $\hbar \to 0$ along $\bar{\mathsf{A}}_+$, and this asymptotic expansion is necessarily the formal solution $\hat{F}_+$.

Proof of Claim 5.1.

If we assume for the moment that the series (54) is uniformly convergent on $\mathbf{H}_+$, it is easy to check by direct substitution that it satisfies the integral equation (52). To demonstrate uniform convergence of the series ϕ_+, we first note the following exponential estimates on the coefficients of (52): there are constants $C, L > 0$ such that for each $i = 0, 1, 2$ and for all $(z, \xi) \in \mathbf{H}_+$,

$$|a_i| \leq C \qquad \text{and} \qquad |\alpha_i| \leq Ce^{L|\xi|}. \tag{57}$$

We prove the Main Technical Claim by showing that there are constants $A, M > 0$ such that for all n and for all $(z, \xi) \in \mathbf{H}_+$,

$$\left| \phi_n(z, \xi) \right| \leq AM^n \frac{|\xi|^n}{n!} e^{L|\xi|}. \tag{58}$$

This is enough to deduce the uniform convergence of the infinite series ϕ_+ as well as the exponential estimate (55) by taking $K := M + L$ because

$$\left| \phi_+(z, \xi) \right| \leq \sum_{n=0}^{\infty} |\phi_n| \leq \sum_{n=0}^{\infty} AM^n \frac{|\xi|^n}{n!} e^{L|\xi|} = Ae^{(M+L)|\xi|}.$$

To show (58), we will first recursively construct a sequence of positive real numbers $(M_n)_{n=0}^{\infty}$ such that for all n and for all $(z, \xi) \in \mathbf{H}_+$,

$$\left| \phi_n(z, \xi) \right| \leq M_n \frac{|\xi|^n}{n!} e^{L|\xi|}. \tag{59}$$

We will then show that there are $A, M > 0$ such that $M_n \leq AM^n$ for all n.

CONSTRUCTION OF M_0, M_1. We can take $M_0 := C$ because $\left| \phi_0(z, \xi) \right| = \left| a_0(z) \right| \leq C$. We can take $M_1 := C(1 + C)$ because Lemma A.1 gives the estimate

$$\left| \phi_1(z, \xi) \right| \leq \int_0^{\xi} |\alpha_0| |\mathrm{d}u| + \int_0^{\xi} |a_1| |\phi_0| |\mathrm{d}u| \leq C(1 + C) \int_0^{|\xi|} e^{Lr} \mathrm{d}r \leq C(1 + C)|\xi| e^{L|\xi|}.$$

CONSTRUCTION OF M_n FOR $n \geq 2$. We assume that the estimate (59) holds for $\phi_0, \ldots, \phi_{n-1}$ and derive an estimate for ϕ_n. Using Lemmas A.1–A.3, we obtain the following bounds on the terms in the recursive formula (53):

$$\left| I_+ \left[a_1 \phi_{n-1} \right] \right| \leq CM_{n-1} \frac{|\xi|^n}{n!} e^{L|\xi|},$$

$$\left| I_+ \left[\alpha_1 * \phi_{n-2} \right] \right| \leq CM_{n-2} \frac{|\xi|^n}{n!} e^{L|\xi|},$$

$$\left| I_+ \left[a_2 \phi_i * \phi_j \right] \right| \leq CM_i M_j \frac{|\xi|^n}{n!} e^{L|\xi|} \quad \text{if } i + j = n - 2,$$

$$\left| I_+ \left[\alpha_2 * \phi_i * \phi_j \right] \right| \leq CM_i M_j \frac{|\xi|^n}{n!} e^{L|\xi|} \quad \text{if } i + j = n - 3.$$

Using these estimates in (53), we find

$$
\begin{aligned}
|\phi_n| \le {}& \left| I_+\big[a_1\phi_{n-1}\big] \right| + \left| I_+\big[\alpha_1 * \phi_{n-2}\big] \right| \\
& + \sum_{\substack{i,j\ge 0 \\ i+j=n-2}} \left| I_+\big[a_2\phi_i * \phi_j\big] \right| + \sum_{\substack{i,j\ge 0 \\ i+j=n-3}} \left| I_+\big[\alpha_2 * \phi_i * \phi_j\big] \right| \\
\le {}& C\left(M_{n-1} + M_{n-2} + \sum_{\substack{i,j\ge 0 \\ i+j=n-2}} M_i M_j + \sum_{\substack{i,j\ge 0 \\ i+j=n-3}} M_i M_j \right) \frac{|\xi|^n}{n!} e^{L|\xi|}.
\end{aligned}
$$

We can therefore define, for all $n \ge 2$,

$$
M_n := C\left(M_{n-1} + M_{n-2} + \sum_{\substack{i,j\ge 0 \\ i+j=n-2}} M_i M_j + \sum_{\substack{i,j\ge 0 \\ i+j=n-3}} M_i M_j \right). \tag{60}
$$

BOUNDS ON M_n. Consider the following power series in an abstract variable t:

$$
\hat{p}(t) := \sum_{n=0}^{\infty} M_n t^n \in \mathbb{C}[[t]].
$$

We will show that $\hat{p}(t)$ is in fact a convergent power series. First, we observe that $\hat{p}(0) = M_0 = C$ and that $\hat{p}(t)$ satisfies the following algebraic equation:

$$
\hat{p} = C\Big(1 + t + \hat{p}t + \hat{p}t^2 + \hat{p}^2 t^2 + \hat{p}^2 t^3 \Big), \tag{61}
$$

which can be seen by expanding and comparing the coefficients using the defining formula (60) for M_n. Consider the holomorphic function $G = G(p,t)$ of two variables, defined by

$$
G(p,t) := -p + C\Big(1 + t + pt + pt^2 + p^2 t^2 + p^2 t^3 \Big).
$$

It has the following properties:

$$
G(C,0) = 0, \qquad \left.\frac{\partial G}{\partial p}\right|_{(p,t)=(C,0)} = -1 \ne 0.
$$

Thus, by the Holomorphic Implicit Function Theorem, there exists a function $p(t)$, holomorphic at $t = 0$, satisfying $p(0) = C$ and $G\big(p(t),t\big) = 0$ for all t sufficiently close to 0. Since $\hat{p}(0) = C$ and $G\big(\hat{p}(t),t\big) = 0$, the power series $\hat{p}(t)$ must be the Taylor expansion of $p(t)$ at $t = 0$. As a result, $\hat{p}(t)$ is in fact a convergent power series, which means its coefficients grow at most exponentially: there are constants $A, M > 0$ such that $M_n \le AM^n$ for all n. This completes the proof of Claim 5.1 and therefore of Lemma 5.1. $\square$

5.1.2. Proof of Theorem 5.1

We can now finish the proof of the main result in this paper.

Proof of Theorem 5.1. We immediately restrict our attention to a Borel disk in the $\hbar$-plane of some diameter $d > 0$ bisected by the direction θ; that is, without loss of generality, assume that $\mathsf{S} = \big\{ \hbar \mid \mathrm{Re}(e^{i\theta}/\hbar) > 1/d \big\}$. Note that the rotation $\hbar \mapsto e^{-i\theta}\hbar$ sends S to the Borel disk $\mathsf{S}_+ = \big\{ \hbar \mid \mathrm{Re}(1/\hbar) > 1/d \big\}$ from (44).

Next, let $r > 0$ be such that $\mathsf{W}:=\mathsf{W}_\theta^\alpha = W_\theta^\alpha(x_0; r) = \Phi^{-1}(\mathsf{H})$ is the WKB half-strip from the hypothesis, where $\mathsf{H}:=\mathsf{H}_\theta^\alpha = \mathsf{H}_\theta^\alpha(0, r) = \{z \mid \text{dist}(z, e^{i\theta}\mathbb{R}_\alpha) < r\}$. Recall that $\Phi^{-1}: \mathsf{H} \to \mathsf{W}$ is a local biholomorphism. Put $\mathsf{H}_+:=\{z \mid \text{dist}(z, \mathbb{R}_+) < r\}$ and $\Phi_\alpha^\theta:=\varepsilon_\alpha e^{-i\theta}\Phi$, so that $\mathsf{W} = (\Phi_\alpha^\theta)^{-1}(\mathsf{H}_+)$. Furthermore, Φ_α^θ transforms the differential operator $\frac{\varepsilon_\alpha}{\sqrt{D_0}}\partial_x$ into $e^{-i\theta}\partial_z$.

Consider now holomorphic functions of $(x, \hbar) \in \mathsf{W} \times \mathsf{S}$ denoted by $a_*, a_{**}, b_*, b_{**}, c_*, c_{**}$, obtained from a, b, c by removing the leading and the next-to-leading order terms, respectively; that is, they are defined by the following relations:

$$
\begin{aligned}
a &= a_0 + \hbar a_* \quad \text{and} \quad a_* = a_1 + \hbar a_{**}, \\
b &= b_0 + \hbar b_* \quad \text{and} \quad b_* = b_1 + \hbar b_{**}, \\
c &= c_0 + \hbar c_* \quad \text{and} \quad c_* = c_1 + \hbar c_{**}.
\end{aligned}
\tag{62}
$$

Recall that the leading and the next-to-leading orders f_0^α, f_1^α of the formal solution $\hat{f}_\alpha$ are holomorphic functions on W that satisfy the following identities:

$$
\begin{aligned}
a_0(f_0^\alpha)^2 + b_0 f_0^\alpha + c_0 &= 0 \qquad \text{and} \qquad \varepsilon_\alpha\sqrt{D_0} = 2a_0 f_0^\alpha + b_0, \\
\partial_x f_0^\alpha &= \varepsilon_\alpha\sqrt{D_0}f_1^\alpha + a_1(f_0^\alpha)^2 + b_1 f_0^\alpha + c_1,
\end{aligned}
\tag{63}
$$

where $\varepsilon_\pm = \pm 1$. Using these expressions, a straightforward calculation shows that the change of the unknown variable $f \mapsto \tilde{f}$ given by $f = f_0^\alpha + \hbar(f_1^\alpha + \tilde{f})$ transforms the Riccati equation (36) into the following Riccati equation on $\mathsf{W} \times \mathsf{S}$:

$$
\varepsilon_\alpha\frac{\hbar}{\sqrt{D_0}}\partial_x\tilde{f} - \tilde{f} = \hbar(\tilde{a}\tilde{f}^2 + \tilde{b}\tilde{f} + \tilde{c}),
\tag{64}
$$

where

$$
\begin{aligned}
\tilde{a}&:=\frac{\varepsilon_\alpha}{\sqrt{D_0}}a, \qquad \tilde{b}:=\frac{\varepsilon_\alpha}{\sqrt{D_0}}\left(b_* + 2af_1^\alpha + 2a_*f_0^\alpha\right), \\
\tilde{c}&:=\frac{\varepsilon_\alpha}{\sqrt{D_0}}\left(-\partial_x f_1^\alpha + a(f_1^\alpha)^2 + \left(2a_*f_0^\alpha + b_*\right)f_1^\alpha + \left(a_{**}(f_0^\alpha)^2 + b_{**}f_0^\alpha + c_{**}\right)\right).
\end{aligned}
\tag{65}
$$

Finally, transforming equation (64) by Φ_α^θ and applying a rotation $\hbar \mapsto e^{-i\theta}\hbar$, we obtain a Riccati equation on $\mathsf{H}_+ \times \mathsf{S}_+$ of the form (45) where the coefficients A_0, A_1, A_2 are given by

$$
A_2(z, \hbar):=\tilde{a}(x(z), e^{i\theta}\hbar), \quad A_1(z, \hbar):=\tilde{b}(x(z), e^{i\theta}\hbar), \quad A_0(z, \hbar):=\tilde{c}(x(z), e^{i\theta}\hbar),
\tag{66}
$$

where $x(z) = (\Phi_\alpha^\theta)^{-1}(z)$ and the unknown variables $\tilde{f}$ and F are related by

$$
F(z, \hbar) = \tilde{f}(x(z), e^{i\theta}\hbar).
\tag{67}
$$

Theorem 5.1 now follows from Lemma 5.1. □

5.2 Borel summability of formal solutions

In this subsection, we translate Theorem 5.1 and its method of proof into the language of Borel–Laplace theory. Namely, it follows directly from our construction that the canonical exact solutions are the Borel resummation of the corresponding formal solutions. Let us make this statement precise and explicit. The following theorem is a direct consequence of the proof of Theorem 5.1.

THEOREM 5.2 (Borel summability of formal solutions). *Assume all the hypotheses of Theorem 5.1. Then the local formal solution $\hat{f}_\alpha$ is Borel summable in the direction θ*

uniformly near x_0. Namely, the canonical local exact solution f_α^θ is the uniform Borel resummation of $\hat{f}_\alpha$ in the direction θ: for all $\hbar \in S_0$ and uniformly for all $x \in U_0$,

$$f_\alpha^\theta(x, \hbar) = f_0^\alpha(x) + \mathcal{S}_\theta[\hat{f}_\alpha](x, \hbar). \tag{68}$$

A lot of information is packed into Theorem 5.2. Let us unpack it into the following series of explicit statements, all of which are deduced immediately from the proof of Theorem 5.1.

LEMMA 5.2. *Assume all the hypotheses of Theorem 5.1, and let f_α^θ be the canonical exact solution defined on $U_0 \times S_0$.*

(1) *The formal Borel transform $\hat{\varphi}_\alpha$ of $\hat{f}_\alpha$, given by*

$$\hat{\varphi}_\alpha(x, \xi) = \hat{\mathfrak{B}}[\hat{f}_\alpha](x, \xi) = \sum_{n=0}^{\infty} \varphi_n^\alpha(x) \xi^n \quad \text{where} \quad \varphi_k^\alpha(x) := \tfrac{1}{k!} f_{k+1}^\alpha(x), \tag{69}$$

is a uniformly convergent power series in ξ.

(2) *In particular, the power series coefficients of the formal solution $\hat{f}_\alpha$ grow at most factorially in k: there are real constants $C, M > 0$ such that*

$$\left| f_k^\alpha(x) \right| \leq C M^k k! \qquad\qquad (\forall k \geq 0, \forall x \in U_0). \tag{70}$$

(3) *There is some $\epsilon > 0$ such that the analytic Borel transform φ_α^θ of f_α^θ in the direction θ, given by*

$$\varphi_\alpha^\theta(x, \xi) = \mathfrak{B}_\theta[f_\alpha^\theta](x, \xi) = \frac{1}{2\pi i} \oint_\theta f_\alpha^\theta(x, \hbar) e^{\xi/\hbar} \frac{d\hbar}{\hbar^2}, \tag{71}$$

is uniformly convergent for all $(x, \xi) \in U_0 \times \Xi_\theta$ where

$$\Xi_\theta := \left\{ \xi \in \mathbb{C}_\xi \mid \operatorname{dist}(\xi, e^{i\theta}\mathbb{R}_+) < \epsilon \right\}.$$

Here, the integral "$\oint_\theta$" is done anticlockwise along the boundary of any Borel disk $S_\theta = \left\{ \hbar \mid \operatorname{Re}(e^{i\theta}/\hbar) > 1/d \right\} \subsetneq S$ of diameter d so small that its boundary (without the origin) is contained in S. Furthermore, φ_α^θ defines the analytic continuation of the formal Borel transform $\hat{\varphi}_\alpha$ along the ray $e^{i\theta}\mathbb{R}_+ \subset \mathbb{C}_\xi$. In particular, there are no singularities in the Borel plane $\mathbb{C}_\xi$ along the ray $e^{i\theta}\mathbb{R}_+$.

(4) *The Laplace transform of φ_α^θ in the direction θ, given by*

$$\mathfrak{L}_\theta[\varphi_\alpha^\theta](x, \hbar) = \int_{e^{i\theta}\mathbb{R}_+} e^{-\xi/\hbar} \varphi_\alpha^\theta(x, \xi) d\xi,$$

is uniformly convergent for all $(x, \hbar) \in U_0 \times S_0$ and satisfies the following identity:

$$f_\alpha^\theta(x, \hbar) = f_0^\alpha(x) + \mathfrak{L}_\theta[\varphi_\alpha^\theta](x, \hbar). \tag{72}$$

The fact that identity (68) holds uniformly for all $x \in U_0$ means in particular that operations such as differentiation and integration with respect to x can be exchanged with the operation of Borel resummation. Thus, we have the following corollary.

COROLLARY 5.2. *Assume all the hypotheses of Theorem 5.1, and let f_α^θ be the canonical exact solution defined on $U_0 \times S_0$. Then the formal power series on U_0 given by the derivative*

$\partial_x \hat{f}_\alpha$ and the integral $\int_{x_0'}^x \hat{f}_\alpha$ from any basepoint $x_0' \in \mathsf{U}_0$ are uniformly Borel summable on U_0, and the following identities hold uniformly for all $x \in \mathsf{U}_0$:

$$\partial_x f_\alpha^\theta(x, \hbar) = \partial_x f_0^\alpha(x) + \mathcal{S}_\theta \big[\, \partial_x \hat{f}_\alpha \,\big](x, \hbar), \tag{73}$$

$$\int_{x_0'}^x f_\alpha^\theta(t, \hbar) \mathrm{d}t = \int_{x_0'}^x f_0^\alpha(t) \mathrm{d}t + \mathcal{S}_\theta \left[\int_{x_0'}^x \hat{f}_\alpha \mathrm{d}t \right](x, \hbar). \tag{74}$$

Thanks to the tighter control on the asymptotics of canonical exact solutions on WKB half-strips, all of the above statements extend uniformly over strictly smaller WKB halfstrips. Explicitly, we have the following corollary.

PROPOSITION 5.2 (Uniform Borel summability on WKB half-strips). *Assume all the hypotheses of Theorem 5.1. Let $r > 0$ be such that $\mathsf{W} = \mathsf{W}_\theta^\alpha(x_0, r)$. For any $r_0 \in (0, r)$, let $\mathsf{W}_0 := \mathsf{W}_\theta^\alpha(x_0, r_0)$. Then the formal solution $\hat{f}_\alpha$ is Borel summable in the direction θ uniformly for all $x \in \mathsf{W}_0$. Furthermore, the formal power series given by the derivative $\partial_x \hat{f}_\alpha$ and the integral $\int_{x_0'}^x \hat{f}_\alpha$ from any basepoint $x_0' \in \mathsf{U}_0$ are uniformly Borel summable on W_0, and identities (73) and (74) hold uniformly for all $x \in \mathsf{W}_0$.*

5.3 Explicit recursion for the Borel transform

The analytic Borel transform φ_α^θ can be given a reasonably explicit presentation as follows. Define an integral operator I acting on holomorphic functions $\varphi = \varphi(x, \xi)$ by the following formula, wherever it makes sense:

$$I\big[\varphi\big](x, \xi) := \int_0^\xi \varphi(x_t, \xi - t) \mathrm{d}t \qquad \text{where} \qquad x_t := \Phi^{-1}\big(\Phi(x) + t\big). \tag{75}$$

and the integration contour is the straight line segment from 0 to $\xi \in \mathbb{C}$. In particular, for any $x \in \mathsf{U}_0$ and any sufficiently small $\xi \in e^{i\theta}\mathbb{R}_\alpha$, the path $\{x_t\}_{t=0}^\xi$ is a segment of the WKB (θ, α)-ray emanating from x.

Recall functions $\tilde{a}, \tilde{b}, \tilde{c}$ defined by the identities (65). Their leading-order parts in $\hbar$ are, respectively,

$$\tilde{a}_0 = \frac{\varepsilon_\alpha}{\sqrt{D_0}} a_0, \qquad \tilde{b}_0 = \frac{\varepsilon_\alpha}{\sqrt{D_0}} \Big(b_1 + 2a_0 f_1^\alpha + 2a_1 f_0^\alpha\Big), \qquad \tilde{c}_0 = f_2^\alpha, \tag{76}$$

where last identity was obtained by comparing with (15). Finally, denote their analytic Borel transforms in direction θ as follows:

$$\begin{aligned}
\beta_0 &:= \mathfrak{B}_\theta[\tilde{c}], & & & \tilde{a} &= \tilde{a}_0 + \mathfrak{L}_\theta[\beta_2], \\
\beta_1 &:= \mathfrak{B}_\theta[\tilde{b}], & &\text{so that} & \tilde{b} &= \tilde{b}_0 + \mathfrak{L}_\theta[\beta_1], \\
\beta_2 &:= \mathfrak{B}_\theta[\tilde{a}], & & & \tilde{c} &= \tilde{c}_0 + \mathfrak{L}_\theta[\beta_0].
\end{aligned}$$

PROPOSITION 5.3 (Recursive formula for the Borel transform). *Assume all the hypotheses of Theorem 5.1, and let $r > 0$ be such that $\mathsf{W} = \mathsf{W}_\theta^\alpha(x_0; r)$. For any $r_0 \in (0, r)$ and any $\epsilon \in (0, r - r_0)$, let $\mathsf{W}_0 := \mathsf{W}_\theta^\alpha(x_0; r_0)$ and $\Xi_\theta := \big\{\xi \mid \mathrm{dist}(\xi, e^{i\theta}\mathbb{R}_+) < \epsilon\big\}$. Then the analytic Borel transform φ_α^θ can be expressed more explicitly as follows: uniformly for all $(x, \xi) \in \mathsf{W}_0 \times \Xi_\theta$,*

$$\varphi_\alpha^\theta(x, \xi) = f_1^\alpha(x) + \int_0^\xi \sigma_\alpha^\theta(x, t) \mathrm{d}t \qquad \text{with} \qquad \sigma_\alpha^\theta(x, \xi) := \sum_{k=0}^\infty \sigma_k(x, \xi), \tag{77}$$

where $\sigma_0:=\tilde{c}_0 = f_2^\alpha$, $\sigma_1:=I\big[\beta_0 + \tilde{b}_0\sigma_0\big]$, and for $k \geq 2$,

$$\sigma_k:=I\left[\tilde{b}_0\sigma_{k-1} + \beta_1 * \sigma_{k-2} + \sum_{\substack{i,j\geq 0 \\ i+j=k-2}} \tilde{a}_0\sigma_i * \sigma_j + \sum_{\substack{i,j\geq 0 \\ i+j=k-3}} \beta_2 * \sigma_i * \sigma_j\right]. \tag{78}$$

Proof. The proof is straightforward and amounts to transforming some of the main constructions in the proof of Lemma 5.1 using the inverse Liouville transformation Φ back to the x-variable. Let $\Phi_\alpha^\theta:=\varepsilon_\alpha e^{-i\theta}\Phi$, and let H_+ be the image of W under Φ_α^θ. Since Φ^{-1} is a local biholomorphism $\mathsf{H}_+ \to \mathsf{W}$, for a fixed $x \in \mathsf{W}$ and every $t \in [0,\xi]$, there is a unique point $x_t:=\Phi^{-1}\big(\Phi(x)+t\big) \in \mathsf{W}$ such that $\Phi(x)+t = \Phi(x_t)$. Note in particular that, since Φ may be multivalued on W, the point x_t does not depend on the choice of branch of $\Phi(x)$. Thus, the integral operator I from (75) is defined by using Φ to transform the integral operator I_+ from (51) defined in the proof of Lemma 5.1. Likewise, the sequence $\{\phi_k(z,\xi)\}_{k=0}^\infty$ defined in the proof of Lemma 5.1 by the recursive formula (53) transforms under Φ to give the sequence $\sigma_k(\Phi^{-1}(z),\xi):=\phi_k\big(z,e^{-i\theta}\xi\big)$. □

5.4 Monic Riccati equations

In many situations, such as those arising in the context of the exact WKB analysis of second-order ODEs, the coefficients of the Riccati equation satisfy hypothesis (2) in Theorem 5.1 only after an additional transformation.

EXAMPLE 5.1. For example, consider the Riccati equation $\hbar\partial_x f = f^2 - x$, which is encountered in the WKB analysis of the deformed Airy differential equation. The coefficients are $a = 1, b = 0, c = -x$, and the leading-order discriminant D_0 is $4x$. In this case, a WKB half-strip W is necessarily an unbounded domain, and the asymptotic condition (37) reduces to requiring that $c = -x$ is bounded on W by $\sqrt{D_0} = 2\sqrt{x}$, which is not the case. Therefore, Theorem 5.1 cannot be applied to this Riccati equation directly.

However, this can be remedied by making a change of the unknown variable $f \mapsto g$ given by $f = \sqrt{D_0}g$ for $x \in \mathsf{W}$. It transforms the Riccati equation $\hbar\partial_x f = f^2 - x$ into

$$\hbar\partial_x g = 2\sqrt{x}g^2 - \tfrac{1}{2}\hbar x^{-1}g + \tfrac{1}{2}\sqrt{x}. \tag{79}$$

Notice that the leading-order discriminant of this Riccati equation remains $D_0 = 4x$ and that its coefficients are now bounded at infinity by $\sqrt{D_0} = 2\sqrt{x}$. Therefore Theorem 5.1 can be applied to (79).

More generally, this transformation is necessary when dealing with *monic* Riccati equations, that is, whenever the coefficient A of the Riccati equation is identically 1. This is always the case in the exact WKB analysis of second-order ODEs [12]. Spelled out, we have the following version of our results.

THEOREM 5.3 (Exact existence and uniqueness for monic equations). *Consider the following monic Riccati equation:*

$$\hbar\partial_x f = f^2 + pf + q, \tag{80}$$

where p,q are holomorphic functions of $(x,\hbar) \in \mathsf{X} \times \mathsf{S}$ which admit locally uniform asymptotic expansions $\hat{p}, \hat{q}$ as $\hbar \to 0$ along A. Assume that $D_0 = p_0^2 - 4q_0 \not\equiv 0$. Fix a regular point $x_0 \in \mathsf{X}$,

a square-root branch $\sqrt{D_0}$ near x_0, and a sign $\alpha \in \{+,-\}$. In addition, we assume the following hypotheses:

(1) *There is a WKB (θ, α)-half-strip domain $\mathsf{W} = \mathsf{W}_\theta^\alpha \subset \mathsf{X}$ containing x_0.*
(2) *The asymptotic expansions of the coefficients p, q are valid with Gevrey bounds as $\hbar \to 0$ along the closed arc $\bar{\mathsf{A}}_\theta = [\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}]$, with respect to the asymptotic scales $\sqrt{D_0}$ and D_0, respectively, uniformly for all $x \in \mathsf{W}$:*

$$p \simeq \hat{p} \qquad \text{as } \hbar \to 0 \text{ along } \bar{\mathsf{A}}_\theta, \text{ wrt } \sqrt{D_0}, \text{ unif. } \forall x \in \mathsf{W}, \tag{81}$$

$$q \simeq \hat{q} \qquad \text{as } \hbar \to 0 \text{ along } \bar{\mathsf{A}}_\theta, \text{ wrt } D_0, \text{ unif. } \forall x \in \mathsf{W}. \tag{82}$$

(3) *The logarithmic derivative $\partial_x \log D_0$ is bounded by D_0 on W.*

Then all the conclusions of Theorem 5.1, as well as Theorem 5.2, Lemma 5.2, and Corollary 5.2 hold verbatim. Furthermore, all the conclusions of Proposition 5.1 hold verbatim with the only exception that the asymptotic statement (43) must be replaced with the following asymptotic statement with respect to the asymptotic scale $\sqrt{D_0}$:

$$f_\alpha^\theta \simeq \hat{f}_\alpha \qquad \text{as } \hbar \to 0 \text{ along } \bar{\mathsf{A}}_\theta, \text{ wrt } \sqrt{D_0}, \text{ unif. } \forall x \in \mathsf{W}_0. \tag{83}$$

Proof. The change of the unknown variable $f \mapsto g$ given by $f = \sqrt{D_0}g$ transforms (80) into

$$\hbar \partial_x g = \sqrt{D_0} g^2 + \left(p - \hbar \partial_x \log \sqrt{D_0}\right) g + \frac{1}{\sqrt{D_0}} q. \tag{84}$$

This transformation is well defined for all $x \in \mathsf{W}$ because W necessarily supports the univalued square-root branch $\sqrt{D_0}$. Notice that the leading-order discriminant of this Riccati equation remains D_0. Note that $\partial_x \log \sqrt{D_0}$ is bounded by $\sqrt{D_0}$ if and only if $\partial_x \log D_0$ is bounded by D_0, as provided by hypothesis (3). Now, it is obvious that the hypotheses of Theorem 5.3 imply that the Riccati equation (84) satisfies all the hypotheses of Theorem 5.1. It yields the canonical local exact solution g_α^θ near x_0, and therefore the canonical local exact solution $f_\alpha^\theta = \sqrt{D_0} g_\alpha^\theta$ near x_0. □

Of course, a general Riccati equation (36) can always be put into the monic form (5.3) via the change of the unknown variable $f \mapsto g = af$, which yields

$$\hbar \partial_x g = g^2 + (b + \hbar \partial_x \log a) g + ac. \tag{85}$$

If a is nowhere-vanishing, then this transformation makes sense globally on X.

Likewise, Corollary 5.1 is also true for the monic Riccati equation (5.3), though with slightly simplified hypotheses as follows.

COROLLARY 5.3 (Extension to larger domains). *Consider the Riccati equation (80) with $D_0 \not\equiv 0$. Fix a sign $\alpha \in \{+,-\}$, and let $\mathsf{U} \subset \mathsf{X}$ be a domain free of turning points that supports a univalued square-root branch $\sqrt{D_0}$. In addition, assume that hypotheses (1)–(3) in Theorem 5.3 is satisfied for every point $x_0 \in \mathsf{U}$. Then the conclusions of Corollary 5.1 hold verbatim.*

However, Propositions 5.2 and 5.3 are no longer true for the canonical exact solutions of the monic Riccati equation (5.3). In this case, one needs either to factorize $\sqrt{D_0}$ out of f_α^θ and apply these propositions to the regularized Riccati equation (84), or identify and remove the *principal part* of f_α^θ in the limit along the WKB rays. The latter procedure can

be explicitly formalized when the WKB rays limit to a pole of D_0; this will be explained in detail elsewhere.

REMARK 5.2. The same trick as above can help us tackle more general situations as follows. If $\chi = \chi(x)$ is any holomorphic function, then the change of the unknown variable $f \mapsto g$ given by $f = \chi g$ transforms the Riccati equation (36) into

$$\hbar \partial_x g = a' g^2 + b' g + c', \tag{86}$$

where

$$a' := \chi a \qquad b' := b - \hbar \partial_x \log \chi \qquad c' := \chi^{-1} c. \tag{87}$$

Note that since χ is independent of $\hbar$, the leading-order discriminant remains unchanged: $D_0' = (b_0')^2 - 4a_0' c_0' = b_0^2 - 4a_0 c_0 = D_0$. In summary, we have the following proposition, which in particular recovers Theorem 5.1 by taking $\chi = 1$ and Theorem 5.3 by taking $\chi = \sqrt{D_0}$.

PROPOSITION 5.4. *Assume all the hypotheses of Theorem 5.1, except hypothesis (2) is replaced with the following:*

(2′) There is a nonvanishing holomorphic function $\chi = \chi(x)$ on W such that a', b', c' defined by (87) admit Gevrey asymptotics as $\hbar \to 0$ along the closed arc $\bar{\mathsf{A}}_\theta = [\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}]$, with respect to the asymptotic scale $\sqrt{D_0}$, uniformly for all $x \in \mathsf{W}$:

$$a' \simeq \hat{a}', \quad b' \simeq \hat{b}', \quad c' \simeq \hat{c}' \qquad as\ \hbar \to 0\ along\ \bar{\mathsf{A}}_\theta,\ wrt\ \chi,\ unif.\ \forall x \in \mathsf{W}. \tag{88}$$

Then all the conclusions of Theorem 5.1, as well as Theorem 5.2, Lemma 5.2, and Proposition 5.2, hold verbatim. Furthermore, all the conclusions of Proposition 5.1 hold verbatim with the only exception that the asymptotic statement (43) must be replaced with the following asymptotic statement with respect to the asymptotic scale χ:
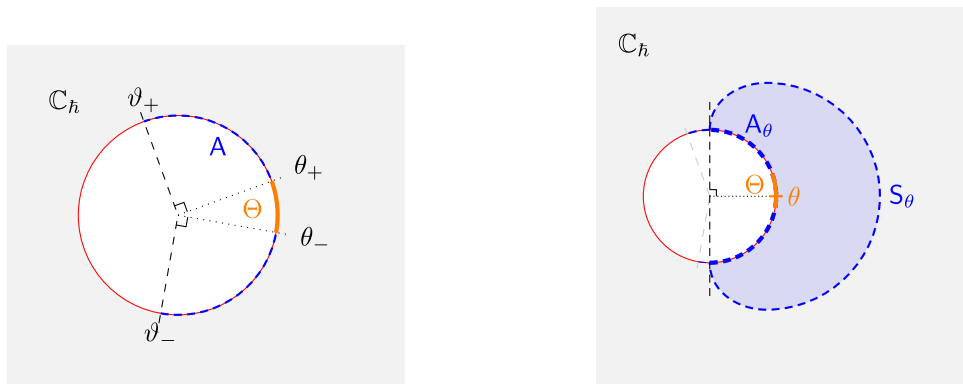
$$f_\alpha^\theta \simeq \hat{f}_\alpha \qquad as\ \hbar \to 0\ along\ \bar{\mathsf{A}}_\theta,\ wrt\ \chi,\ unif.\ \forall x \in \mathsf{W}_0. \tag{89}$$

5.5 Exact solutions in Wider sectors

The last general result we prove is about extending canonical exact solutions to sectors with wider openings. However, we not address here the question of extending canonical exact solutions to radially larger sectorial domains in $\mathbb{C}_\hbar$ or discussing the relationship between unequal canonical exact solutions for different values of θ. These questions will be examined in detail elsewhere.

Thus, suppose that $\pi \leq |\mathsf{A}| \leq 2\pi$. Let $\Theta := [\theta_-, \theta_+]$ be the closed arc such that $\mathsf{A} = (\theta_- - \frac{\pi}{2}, \theta_+ + \frac{\pi}{2})$; that is, $\theta_\pm := \vartheta_\pm \mp \frac{\pi}{2}$. See Figure 1a. This arc Θ is sometimes called the arc of *copolar directions* of A. For every $\theta \in \Theta$, let $\mathsf{A}_\theta := (\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}) \subset \mathsf{A}$ be the semicircular arc bisected by θ. See Figure 1b. Note that $\mathsf{A} = \mathbb{U}_{\theta \in \Theta} \mathsf{A}_\theta$.

PROPOSITION 5.5. *Consider the Riccati equation (36) whose coefficients a, b, c are holomorphic functions of $(x, \hbar) \in \mathsf{X} \times \mathsf{S}$ admitting locally uniform asymptotic expansions $\hat{a}, \hat{b}, \hat{c}$ as $\hbar \to 0$ along A. Assume that the leading-order discriminant $D_0 = b_0^2 - 4a_0 c_0$ is not identically zero. Fix a regular point $x_0 \in \mathsf{X}$, a square-root branch $\sqrt{D_0}$ near x_0, and a sign $\alpha \in \{+, -\}$. Fix a regular point $x_0 \in \mathsf{X}$, a square-root branch $\sqrt{D_0}$ near x_0, and a sign $\alpha \in \{+, -\}$. In addition, assume that hypotheses (1) and (2) in Theorem 5.1 are satisfied for every $\theta \in \Theta$.*

(a) The arc Θ of copolar directions of A.

(b) A Borel disc S_θ with opening A_θ.

Figure 1.

Then the Riccati equation (36) has a canonical local exact solution f_α^Θ near x_0 which is asymptotic to the formal solution $\hat{f}_\alpha$ as $\hbar \to 0$ in every direction $\theta \in \Theta$. Namely, there is a neighborhood $\mathsf{U}_0 \subset \mathsf{X}$ of x_0 and a sectorial domain $\mathsf{S}_0 \subset \mathsf{S}$ with the same opening A such that the Riccati equation (36) has a unique holomorphic solution f_α^Θ on $\mathsf{U}_0 \times \mathsf{S}_0$ which is Gevrey asymptotic to $\hat{f}_\alpha$ as $\hbar \to 0$ along the closed arc $\bar{\mathsf{A}}$ uniformly for all $x \in \mathsf{U}_0$:

$$f_\alpha^\Theta \simeq \hat{f}_\alpha \qquad \text{as } \hbar \to 0 \text{ along } \bar{\mathsf{A}}, \text{ unif. } \forall x \in \mathsf{U}_0. \tag{90}$$

Proof. By Theorem 5.2 (or more specifically by part (4) of Lemma 5.2), for every $\theta \in \Theta$, the canonical exact solution in the direction θ exists and can be written as

$$f_\alpha^\theta(x,\hbar) = f_0^\alpha(x) + \mathfrak{L}_\theta\big[\varphi_\alpha^\theta\big](x,\hbar) = f_0^\alpha(x) + \int_{e^{i\theta}\mathbb{R}_+} e^{-\xi/\hbar} \varphi_\alpha^\theta(x,\xi)\mathrm{d}\xi,$$

where φ_α^θ is the analytic Borel transform of f_α^θ in the direction θ. The explicit formula from Proposition 5.3 reveals that these analytic Borel transforms φ_α^θ for each $\theta \in \Theta$ together define a holomorphic function φ_α^Θ on an ϵ-neighborhood $\Xi_\Theta = \big\{\xi \mid \mathrm{dist}(\xi, \Sigma_\Theta) < \epsilon\big\}$ of the sector $\Sigma_\Theta := \big\{\xi \mid \arg(\xi) \in \Theta\big\}$ for a sufficiently small $\epsilon > 0$. Furthermore, φ_α^Θ has at most exponential growth at infinity in Ξ_Θ, which means that Cauchy–Goursat's theorem yields the identity $\mathfrak{L}_{\theta_+}\big[\varphi_\alpha^{\theta_+}\big] = \mathfrak{L}_{\theta_+}\big[\varphi_\alpha^\Theta\big] = \mathfrak{L}_{\theta_-}\big[\varphi_\alpha^\Theta\big] = \mathfrak{L}_{\theta_-}\big[\varphi_\alpha^{\theta_-}\big]$. □

§6. Examples and applications

The somewhat obscure technical hypotheses in Theorems 5.1 and 5.3 can be made more transparent in a number of special situations which we describe in this section. We also present the simplest explicit example in §6.2 where we construct a pair of canonical exact solutions by following all the steps in the proof of the main theorem. Finally, in §6.3, we give a very important application of our result in the context of the exact WKB analysis of Schrödinger equations.

6.1 Equations with mildly deformed coefficients

6.1.1. Undeformed coefficients

The simplest yet ubiquitous situation is when the coefficients of the Riccati equation (36) are independent of $\hbar$, in which case the asymptotic hypotheses in Theorem 5.1 dramatically simplify.

Thus, let us consider both the general Riccati equation (36) as well as a monic Riccati equation (80) on $\mathsf{X} \times \mathbb{C}_{\hbar}$ with $\hbar$-independent coefficients a, b, c, p, q. Their leading-order discriminants D_0 are simply $b^2 - 4ac$ and $p^2 - 4q$. The sectorial domain S can be taken to be any half-plane bisected by some direction in $\mathbb{C}_{\hbar}$.

Polynomial coefficients. The simplest case is when the coefficients are polynomials in x; that is, $a, b, c, p, q \in \mathbb{C}[x]$. Then $\mathsf{X} = \mathbb{C}_x$ and there are only finitely many turning points and singular WKB trajectories. All singular WKB trajectories either connect a turning point to infinity, or two turning points together. All WKB trajectories can be easily plotted using a computer, or even by hand in simple examples. See §6.2 where we examine the simplest example in detail.

The biggest advantage of this simple situation is that in order to check hypothesis (1) in either Theorem 5.1 or 5.3 that a given regular point x_0 is contained in a WKB half-strip, it is sufficient to only examine the WKB ray emanating from x_0 and check that it does not hit a turning point. If so, this WKB ray is necessarily either a closed WKB trajectory (i.e., a simple closed curve in the complement of the turning points) or it escapes to infinity. In either case, there is necessarily a WKB half-strip W containing x_0. For instance, we can take a small disk $\mathbb{D}$ centered at x_0 which is compactly contained in the complement of all singular WKB rays (for the same phase and sign), and let W be the union of all WKB rays emanating from $\mathbb{D}$. This disk $\mathbb{D}$ should be chosen small enough that W is compactly contained in the complement of the turning points; it ensures in particular that hypothesis (3) in Theorem 5.3 is satisfied.

If the WKB ray emanating from x_0 is a closed WKB trajectory, then hypothesis (2) in both theorems is automatic. On the other hand, if this WKB ray escapes to infinity, then hypothesis (2) in Theorem 5.1 is equivalent to saying that the polynomials a, b, c are all bounded at infinity by $\sqrt{D_0}$, and hypothesis (2) in Theorem 5.3 is equivalent to saying that p is bounded at infinity by $\sqrt{D_0}$ and q is bounded at infinity by D_0. As these are all polynomials in x, hypothesis (2) in both theorems therefore boils down to a condition on their degrees. In summary, we have the following.

PROPOSITION 6.1. *Consider either the general Riccati equation (36) or a monic Riccati equation (80) on $\mathbb{C}_x \times \mathbb{C}_{\hbar}$ with polynomial $\hbar$-independent coefficients a, b, c or $p, q \in \mathbb{C}[x]$. Fix a regular point $x_0 \in \mathbb{C}_x$ and a square-root branch $\sqrt{D_0}$ near x_0. Fix a sign $\alpha \in \{+, -\}$, a phase $\theta \in \mathbb{R}$, and let $\mathsf{S} := \{\hbar \mid \mathrm{Re}(e^{-i\theta}\hbar) > 0\}$. Assume that*

(1) *the WKB (θ, α)-ray $\Gamma_{\theta}^{\alpha} = \Gamma_{\theta}^{\alpha}(x_0)$ emanating from x_0 does not hit a turning point; assume in addition that a_0 is nonvanishing on Γ_{θ}^{α} if $\alpha = -$.*

If Γ_{θ}^{α} is a closed trajectory, then all the hypotheses of Theorem 5.1 or 5.3 are satisfied, and therefore their conclusions hold verbatim. If, on the other hand, Γ_{θ}^{α} escapes to infinity, then in addition we assume that either

(2) $\deg(a), \deg(b), \deg(c) \leq \frac{1}{2}\deg(D_0)$ $\quad$ *or*

$\deg(p) \leq \frac{1}{2}\deg(D_0)$ $\quad$ *and* $\quad$ $\deg(q) \leq \deg(D_0)$.

Then all the hypotheses of Theorem 5.1 or 5.3 are satisfied.

Rational coefficients. More generally, the coefficients a, b, c and p, q may be arbitrary rational functions of x. Then $\mathsf{X} = \mathbb{C}_x \setminus \{\text{poles}\}$ and again there are only finitely many turning points and singular WKB trajectories, all of which can be easily plotted using a computer. All singular WKB trajectories end either on a turning point or a simple pole of D_0.

As in the polynomial scenario, checking hypothesis (1) in both theorems boils down to examining a single WKB ray emanating from a regular point x_0. If this ray does not hit a turning point, it must be either a closed trajectory or limit to one of the poles x_∞ of D_0 including possibly the one at infinity. This WKB ray is infinite if and only if the pole order of D_0 at x_∞ is 2 or greater. Note that if $\mathrm{ord}(D_0) \geq 2$ at x_∞, then hypothesis (3) in Theorem 5.3 is automatic. Finally, similar to the polynomial scenario, the asymptotic hypothesis (2) in both theorems boils down to a boundedness condition near the pole x_∞ on the coefficients by an appropriate power of D_0. In summary, we have the following.

PROPOSITION 6.2. *Consider either the general Riccati equation (36) or a monic Riccati equation (80) on $\mathbb{C}_x \times \mathbb{C}_\hbar$ with rational $\hbar$-independent coefficients a, b, c or $p, q \in \mathbb{C}(x)$. Fix a sign $\alpha \in \{+, -\}$, a phase $\theta \in \mathbb{R}$, and let $\mathsf{S} := \{\hbar \mid \mathrm{Re}(e^{-i\theta}\hbar) > 0\}$. Fix a regular point $x_0 \in \mathsf{X}$ and a square-root branch $\sqrt{D_0}$ near x_0. Assume that*

(1) *the WKB (θ, α)-ray $\Gamma_\theta^\alpha = \Gamma_\theta^\alpha(x_0)$ emanating from x_0 does not hit a turning point; assume in addition that a_0 is nonvanishing on Γ_θ^α if $\alpha = -$.*

If Γ_θ^α is a closed trajectory, then all the hypothesis of Theorem 5.1 or 5.3 are satisfied, and therefore their conclusions hold verbatim. If, on the other hand, Γ_θ^α tends a pole $x_\infty \in \mathbb{C}_x \cup \{\infty\}$ of D_0 of order 2 or higher, then we also assume that either

(2) $\mathrm{ord}(a), \mathrm{ord}(b), \mathrm{ord}(c) \leq \frac{1}{2}\mathrm{ord}(D_0)$ *at x_∞* *or*
$\mathrm{ord}(p) \leq \frac{1}{2}\mathrm{ord}(D_0)$ *and* $\mathrm{ord}(q) \leq \mathrm{ord}(D_0)$ *at x_∞.*

Then all the hypotheses of Theorem 5.1 or 5.3 are satisfied.

General meromorphic coefficients. When a, b, c or p, q are more general not necessarily rational meromorphic functions, the WKB geometry is far more complicated to describe in general. However, if the WKB ray $\Gamma_\theta^\alpha(x_0)$ is closed or limits to a second- or higher-order pole of D_0, then a general but simplified version of both theorems can be stated as follows.

PROPOSITION 6.3. *Consider either the general Riccati equation (36) or a monic Riccati equation (80) on $\mathsf{X} \times \mathbb{C}_\hbar$ with $\hbar$-independent coefficients a, b, c or p, q. Fix a sign $\alpha \in \{+, -\}$, a phase $\theta \in \mathbb{R}$, and let $\mathsf{S} := \{\hbar \mid \mathrm{Re}(e^{-i\theta}\hbar) > 0\}$. Fix a regular point $x_0 \in \mathsf{X}$ and a square-root branch $\sqrt{D_0}$ near x_0. Then we make the following hypotheses:*

(1) *The WKB (θ, α)-ray $\Gamma_\theta^\alpha = \Gamma_\theta^\alpha(x_0)$ emanating from x_0 does not hit a turning point but instead limits to a pole $x_\infty \in \partial \mathsf{X}$ of D_0 of order 2 or greater. If $\alpha = -$, we also assume that a_0 is nonvanishing on Γ_θ^α and x_∞ is not an accumulation point of zeroes of a_0.*
(2) $\mathrm{ord}(a), \mathrm{ord}(b), \mathrm{ord}(c) \leq \frac{1}{2}\mathrm{ord}(D_0)$ *at x_∞* *or*
$\mathrm{ord}(p) \leq \frac{1}{2}\mathrm{ord}(D_0)$ *and* $\mathrm{ord}(q) \leq \mathrm{ord}(D_0)$ *at x_∞.*

Then all the hypotheses of Theorems 5.1 or 5.3 are satisfied.

6.1.2. Polynomially deformed coefficients

The next simplest situation is when the equations coefficients depend on $\hbar$ at most polynomially. Thus, let us consider both the general Riccati equation (36) and a monic Riccati equation (80) on $\mathsf{X} \times \mathbb{C}_\hbar$ where functions a, b, c in (36) or p, q in (80) are at most polynomials in $\hbar$ with holomorphic coefficients. The sectorial domain S can still be taken to be a half-plane in $\mathbb{C}_\hbar$.

The WKB geometry is fully determined by the leading-order part of the equation, so all the same considerations apply as explained in §6.1.1. The advantage of being given the coefficients a, b, c or p, q as polynomials in $\hbar$ rather than more general functions of $\hbar$ is that the assumptions on the $\hbar$-asymptotics reduce to simple bounds on the $\hbar$-polynomial coefficients of a, b, c or p, q of the kind we have already seen. In summary, we have the following.

PROPOSITION 6.4. *Consider either the general Riccati equation (36) or a monic Riccati equation (80) on $\mathsf{X} \times \mathbb{C}_\hbar$ with coefficients a, b, c or $p, q \in \mathcal{O}(\mathsf{X})[\hbar]$. Fix a sign $\alpha \in \{+, -\}$, a phase $\theta \in \mathbb{R}$, and let $\mathsf{S} := \{\hbar \mid \mathrm{Re}(e^{-i\theta}\hbar) > 0\}$. Fix a regular point $x_0 \in \mathsf{X}$ and a square-root branch $\sqrt{D_0}$ near x_0. Assume hypothesis (1) from Proposition 6.3, and instead of hypothesis (2), assume that*

(2) $\mathrm{ord}(a_k), \mathrm{ord}(b_k), \mathrm{ord}(c_k) \leq \frac{1}{2}\mathrm{ord}(D_0)$ *at x_∞ for every k*   *or*
  $\mathrm{ord}(p_k) \leq \frac{1}{2}\mathrm{ord}(D_0)$ *and* $\mathrm{ord}(q_k) \leq \mathrm{ord}(D_0)$ *at x_∞ for every k.*

Then all the hypotheses of Theorem 5.1 or 5.3 are satisfied.

6.2 The simplest explicit example: Deformed Airy

In this subsection, we illustrate the main constructions in this paper in the following explicit example. Consider the following Riccati equation on the domain $\mathbb{C}_x \times \mathbb{C}_\hbar$:

$$\hbar \partial_x f = f^2 - x. \tag{91}$$

Thus, in this example, $a = 1, b = 0, c = -x$, and $\mathsf{X} = \mathbb{C}_x$. Let us fix $\theta = 0$, so we will search for canonical exact solutions of (91) with prescribed asymptotics as $\hbar \to 0$ along the positive real axis $\mathbb{R}_+ \subset \mathbb{C}_\hbar$. Then the sectorial domain S can be taken as the complement of the negative real axis $\mathbb{R}_- \subset \mathbb{C}_\hbar$.

This Riccati equation arises in the WKB analysis of the Airy differential equation $\hbar^2 \partial_x^2 \psi(x, \hbar) = x\psi(x, \hbar)$ upon considering the WKB ansatz $\psi = \exp\left(-\int f/\hbar\right)$ (see [12] for more details). For this Riccati equation, it is known that exact solutions exist (see, e.g., [8, §2.2]). There, instead of solving the Riccati equation directly, the Borel–Laplace method is applied to the Airy differential equation. Here, we take a different approach by solving the Riccati equation directly. We have included this example because it is the simplest and most explicit standard example that nicely illustrates most constructions encountered in our paper.

6.2.1. Leading-order analysis.

Following §3.1, the leading-order equation (8) for the Riccati equation (91) is simply $f_0^2 - x = 0$. The leading-order discriminant given by formula (9) is $D_0(x) = 4x$. There is a single turning point at $x = 0$. Let $\sqrt{x}$ be the principal square-root branch (i.e., positive on the positive real axis) in the complement of a branch cut along, say, the negative real axis. Label the two leading-order solutions as

$$f_0^\pm(x) := \pm\sqrt{x} \qquad \text{so that} \qquad \sqrt{D_0} = 2\sqrt{x}. \tag{92}$$

The leading-order solutions $f_0^\pm$ are holomorphic on any simply connected domain $\mathsf{U} \subset \mathbb{C}_x^*$. However, $f_0^\pm$ are unbounded if U is unbounded: the coefficient $c = -x$ of (91) is not bounded by $\sqrt{D_0} = 2\sqrt{x}$, so not all the hypotheses of Lemma 3.1 are satisfied. This will be rectified later by regularizing the coefficients following §5.4.

6.2.2. Formal perturbation theory.

Now, we study the formal aspects of this Riccati equation following §3.2. By Theorem 3.1, the Riccati equation (91) has a pair of formal solutions $\hat{f}_\pm$ with leading-order terms $f_0^\pm$. Their coefficients $f_k^\pm$ for $k \geq 1$ are given by the recursive formula (12), which in this example reduces to

$$f_k^\pm = \pm \tfrac{1}{2\sqrt{x}} \partial_x f_{k-1}^\pm \mp \tfrac{1}{2\sqrt{x}} \sum_{\substack{i+j=k \\ }}^{i,j\neq k} f_i^\pm f_j^\pm. \tag{93}$$

The first few coefficients are

$$f_0^\pm = \pm\sqrt{x}, \qquad f_1^\pm = +\tfrac{1}{4}x^{-1}, \qquad f_2^\pm = \mp\tfrac{5}{32}x^{-5/2}, \qquad f_3^\pm = +\tfrac{15}{64}x^{-4}, \quad \dots. \tag{94}$$

In fact, if we set $d_0^\pm = \pm 1$, it is easy to show by induction that for all $k \geq 1$,

$$f_k^\pm(x) = d_k^\pm x^{-3k/2}\sqrt{x} \quad \text{where} \quad d_k^\pm := \frac{1}{2}\left((2-3k/2)d_{k-1}^\pm - \sum_{\substack{i+j=k}}^{i,j\neq k} d_i^\pm d_j^\pm \right). \tag{95}$$

Note that all $d_k^\pm$ are rational numbers.

6.2.3. The Liouville transformation and WKB trajectories.

Following §4, let us describe the geometry of WKB trajectories on $\mathbb{C}_x$ emerging from this Riccati equation. For any basepoint $x_0 \in \mathbb{C}_x$, the Liouville transformation is given, on the complement of the branch cut, by the simple formula

$$z = \Phi(x) = \int_{x_0}^x 2\sqrt{t}\,\mathrm{d}t = \tfrac{4}{3}(x^{3/2} - x_0^{3/2}). \tag{96}$$

It follows that, for example, the WKB $(0,+)$-ray emanating from any point x_0 with $\arg(x_0) \neq \pm 3\pi/2$ is complete. If, on the other hand, $\arg(x_0) = \pm 3\pi/2$, then the WKB $(0,+)$-ray emanating from x_0 hits the turning point in finite time. Likewise, the WKB $(0,-)$-ray of every point x_0 with $\arg(x_0) \neq 0$ is complete. See Figure. 2. We focus our attention now on the domain

$$\mathsf{U} := \left\{ x \mid 0 < \arg(x) < +3\pi/2 \right\}. \tag{97}$$

Its image under the Liouville transformation $\Phi_0(x) := \tfrac{4}{3}x^{3/2}$ is the upper half-plane $\mathsf{H} = \left\{ z \mid \mathrm{Im}(z) > 0 \right\}$. Clearly, the domain U is swept out by complete WKB trajectories and every point is contained in a WKB strip. Thus, for example, let us take $x_0 := \tfrac{3}{4}e^{i\pi/3}$, so $\Phi_0(x_0) = i$. Let U_0 be the preimage under Φ_0 of the horizontal strip $\mathsf{H}_0 := \left\{ z \mid \tfrac{1}{2} < \mathrm{Im}(z) < \tfrac{3}{2} \right\}$. The Liouville transformation based at x_0 is simply $\Phi = \Phi_0 - i$, and the image of U_0 is the WKB strip $\left\{ z \mid \mathrm{dist}(z, \mathbb{R}) < \tfrac{1}{2} \right\}$. However, in this example, it is more convenient to work with the Liouville transformation Φ_0.

6.2.4. Regularizing the coefficients.

The first observation is that the Riccati equation (91) does not satisfy hypothesis (2) of Theorem 5.1. This is because the rescaled coefficients $a/\sqrt{D_0}$, $b/\sqrt{D_0}$, and $c/\sqrt{D_0}$ are, respectively, $\tfrac{1}{2}x^{-1/2}, 0$, and $-\tfrac{1}{2}x^{+1/2}$, which are unbounded on U. This unboundedness is caused by two separate problems: one is that $x^{+1/2}$ is unbounded at infinity, and the other is
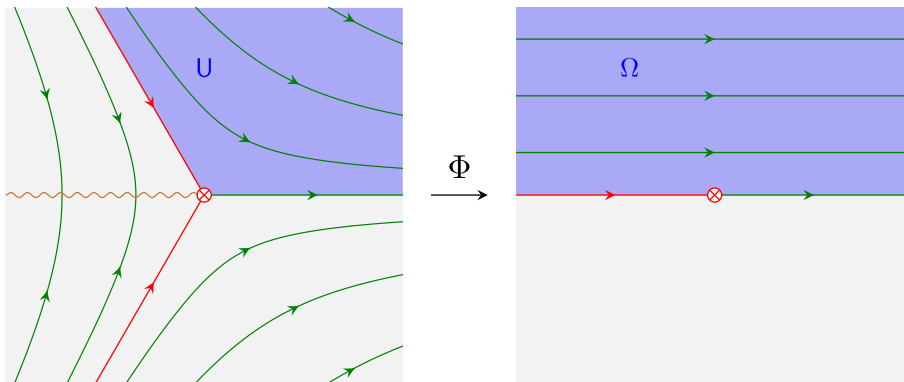
Figure 2.

Pictured are the complex planes $\mathbb{C}_x$ (left) and $\mathbb{C}_z$ (right) with Φ being the Liouville transformation with basepoint $x_0 = 0$. In $\mathbb{C}_x$, there is a turning point at the origin, indicated by a red circled cross. A few complete WKB trajectories on $\mathbb{C}_x$ are drawn in green, with arrows indicating the orientation with respect to the chosen square-root branch $\sqrt{D_0} = 2\sqrt{x}$, for which the branch cut is taken along the negative real axis. There are two special trajectories, indicated in red, which are not complete: they flow into the turning point in finite time. The domain U from (97) is shaded in blue.

that $x^{-1/2}$ is unbounded near the turning point at the origin. The latter problem is remedied by restricting to U_0. In order to remedy the first problem and proceed according to our method, it is necessary to regularize the coefficients of this Riccati equation as in §5.4.

If we make a change of the unknown variable $f \mapsto g = x^{-1/2}f$ over U, then the Riccati equation (91) gets transformed into

$$\hbar \partial_x g = \sqrt{x}\, g^2 - \tfrac{\hbar}{2x}g - \sqrt{x}. \tag{98}$$

This is equation (86) from §5.4 with

$$a' = \sqrt{x}, \qquad b' = \tfrac{1}{2}\hbar x^{-1}, \qquad c' = -\sqrt{x}, \qquad \chi = \sqrt{x} = \tfrac{1}{2}\sqrt{D}.$$

The Riccati equation (98) now satisfies hypothesis (3') from Proposition 5.4 with regularizing factor $\chi = \sqrt{x}$.

The coefficients of the formal solutions $\hat{g}_\pm$ of (98) are given by $g_k^\pm(x) = d_k^\pm x^{-3k/2}$. Explicitly, the first few coefficients are

$$g_0^\pm = \pm 1, \qquad g_1^\pm = +\tfrac{1}{4}x^{-3/2}, \qquad g_2^\pm = \mp\tfrac{5}{32}x^{-3}, \qquad g_3^\pm = +\tfrac{15}{64}x^{-9/2}, \quad \dots.$$

We now follow the step-by-step procedure in the proof of Theorem 5.1 in §5.1.

Step 0: Preliminary transformation.

We begin by performing the preliminary transformations (one for each of $\pm$) of the unknown variable $g \mapsto \tilde{g}$ given by

$$g = g_0^\pm + \hbar\big(g_1^\pm + \tilde{g}\big) = d_0^\pm + \hbar d_1^\pm x^{-3/2} + \hbar\tilde{g} = \pm 1 + \tfrac{1}{4}\hbar x^{-3/2} + \hbar\tilde{g}. \tag{99}$$

They transform the regularized Riccati equation (98) into a pair of Riccati equations

$$\tfrac{\hbar}{2\sqrt{x}}\partial_x \tilde{g} - \tilde{g} = \hbar\big(\tfrac{1}{2}\tilde{g}^2 - d_2^\pm x^{-3}\big) = \hbar\big(\tfrac{1}{2}\tilde{g}^2 \pm \tfrac{5}{32}x^{-3}\big). \tag{100}$$

This is equation (64) from §5.1 with $\tilde{a} = \frac{1}{2}$, $\tilde{b} = 0$, and $\tilde{c} = \pm\frac{5}{32}x^{-3}$. Applying the Liouville transformation Φ_0 to the Riccati equations (100), we get

$$\hbar\partial_z F - F = \hbar\left(\tfrac{1}{2}F^2 - d_2^{\pm}\left(3z/4\right)^{-2}\right) = \hbar\left(\tfrac{1}{2}F^2 \pm \tfrac{5}{18}z^{-2}\right). \tag{101}$$

The unknown variables $\tilde{g}$ and F are related by $\tilde{g}(x,\hbar) = F\left(\frac{4}{3}x^{4/3},\hbar\right)$. Equation (101) is equation (45) from §5.1 with $A_0 = \pm\frac{5}{18}z^{-2}$, $A_1 = 0$, and $A_2 = \frac{1}{2}$.

Step 1: The analytic Borel transform.

Since A_i are independent of $\hbar$, it follows that their Borel transforms α_i are zero, and so the Borel transform of (101) is the following PDE:

$$\partial_z\phi - \partial_\xi\phi = \tfrac{1}{2}\phi * \phi. \tag{102}$$

This is equation (49) from §5.1 with $\alpha_0 = \alpha_1 = \alpha_2 = 0$, $a_0 = \pm\frac{5}{18}z^{-2}$, $a_1 = 0$, and $a_2 = \frac{1}{2}$. In this case, the tubular neighborhood Ξ_+ can be taken arbitrarily large.

Step 2: The integral equation.

The PDE (102) is easy to transform into an integral equation:

$$\phi(z,\xi) = \phi_0^{\pm}(z) + \frac{1}{2}\int_0^\xi \phi * \phi(z+t,\xi-t)\mathrm{d}t, \tag{103}$$

where $\phi(x,0) = \phi_0^{\pm}(z) := a_0(z) = \pm\frac{5}{18}z^{-2}$. This is equation (50) from §5.1.

Step 3: Method of successive approximations.

The integral equation (103) is solved by the method of successive approximations. This method yields a sequence $\{\phi_n^{\pm}\}_{n=0}^\infty$ of holomorphic functions given by $\phi_0^{\pm} = a_0 = \pm\frac{5}{18}z^{-2}$, $\phi_1^{\pm} = 0$, and for $n \geq 2$,

$$\begin{aligned}
\phi_n^{\pm} &= \frac{1}{2}\sum_{i+j=n-2}\int_0^\xi \phi_i^{\pm} * \phi_j^{\pm}(z+t,\xi-t)\mathrm{d}t \\
&= \frac{1}{2}\sum_{i+j=n-2}\int_0^\xi\int_0^{\xi-t}\phi_i^{\pm}(z+t,\xi-t-y)\phi_j^{\pm}(z+t,y)\mathrm{d}y\mathrm{d}t.
\end{aligned}$$

This is equation (53) from §5.1. It is easy to see that $\phi_n = 0$ for all n odd because $\phi_1 = 0$. The first few even terms of this sequence are

$$\begin{aligned}
\phi_0^{\pm} &= \pm\tfrac{5}{18}z^{-2}; \\
\phi_2^{\pm} &= \tfrac{1}{12}\left(\pm\tfrac{5}{18}\right)^2\tfrac{3z+2\xi}{z^3(z+\xi)^2}\xi^2 \sim \tfrac{1}{6}\left(\pm\tfrac{5}{18}\right)^2 z^{-3}\xi \qquad \text{as } \xi\to+\infty; \\
\phi_4^{\pm} &= \tfrac{1}{48}\left(\pm\tfrac{5}{18}\right)^3\tfrac{\xi^4}{z^4(z+\xi)^2} \sim \tfrac{1}{48}\left(\pm\tfrac{5}{18}\right)^3 z^{-4}\xi^2 \qquad \text{as } \xi\to+\infty; \\
\phi_6^{\pm} &= \tfrac{1}{2}\left(\pm\tfrac{5}{18}\right)^4\left(\tfrac{\xi\left(16\xi^5+810z\xi^4+1{,}650\xi z^4+915\xi^2 z^3+70\xi^3 z^2-8\xi^4 z\right)}{4{,}320z^5(z+\xi)^2(2z+\xi)} + \log\left(\tfrac{z}{z+\xi}\right)\tfrac{7\xi^2+27z^2+28\xi z}{72z^2(2z+\xi)^2}\right) \\
&\sim \tfrac{1}{270}\left(\pm\tfrac{5}{18}\right)^4 z^{-5}\xi^3 \qquad \text{as } \xi\to+\infty; \\
\phi_8^{\pm} &\sim \tfrac{7}{35{,}640}\left(\pm\tfrac{5}{18}\right)^5 z^{-6}\xi^4 \qquad \text{as } \xi\to+\infty.
\end{aligned}$$

An exact expression for $\phi_8^{\pm}$ involves logarithms and dilogarithms; it is very long and not very useful, occupying almost half of this page. However, the pattern is clear:

$$\phi_{2n}^{\pm} \in O\left(\tfrac{L^n}{n!}z^{-2}\left(\xi/z^2\right)^n\right) \qquad \text{as } \xi\to+\infty,$$

for some constant $L > 0$ independent of n and z. It follows that the solution to the integral equation (103) satisfies

$$\phi_\pm(z,\xi) = \sum_{n=0}^\infty \phi_n(z,\xi) \quad \preccurlyeq \quad \sum_{n=0}^\infty \tfrac{1}{n!} z^{-2}\big(L\xi/z^2\big)^n = z^{-2} e^{L\xi/z^2} \qquad \text{as } \xi \to +\infty.$$

This asymptotic inequality yields the exponential estimate (55) from §5.1 with $A = 4$ and $K = 0$.

Step 4: Laplace transform.

Applying the Laplace transform to $\phi_\pm$, we obtain exact solutions $F_\pm$ of the two Riccati equations (101):

$$F_\pm(z,\xi) := \int_0^{+\infty} e^{-\xi/\hbar} \phi_\pm(z,\xi) \mathrm{d}\xi.$$

It is evident from the asymptotic behavior of $\phi_\pm$ as $\xi \to +\infty$ that this Laplace integral is uniformly convergent for all $z \in \mathsf{H}_0$ and all $\hbar \in \mathsf{S}_0 := \{\mathrm{Re}(1/\hbar) > L\}$. Note that it is not uniformly convergent for $z \in \mathsf{H}$, because the constant A in the estimate for $\phi_\pm$ grows like $|z|^{-2}$.

Finally, using the inverse Liouville transformation $\Phi_0^{-1} : z \mapsto \big(\tfrac{3}{4} z\big)^{2/3}$ to go back to the x-variable, we obtain two exact solutions of the Riccati equation (98):

$$g_\pm(x,\hbar) := \pm 1 + \tfrac{1}{4}\hbar x^{-3/2} + \hbar \int_0^{+\infty} e^{-\xi/\hbar} \phi_\pm\big(\tfrac{4}{3} x^{3/2},\xi\big) \mathrm{d}\xi.$$

Transforming back to the original Riccati equation (91) via the identities (99) and $f = x^{-1/2} g$, we obtain two exact solutions of the original Riccati equation (91):

$$f_\pm(x,\hbar) = \pm\sqrt{x} + \tfrac{1}{4x}\hbar + \hbar\sqrt{x} \int_0^{+\infty} e^{-\xi/\hbar} \phi_\pm\big(\tfrac{4}{3} x^{3/2},\xi\big) \mathrm{d}\xi.$$

These are the two canonical exact solutions on U_0.

6.3 Exact WKB solutions of Schrödinger equations

In this final subsection, we give an application of our existence and uniqueness result to deduce existence and uniqueness of the so-called *exact WKB solutions* of the complex one-dimensional stationary Schrödinger equation

$$\Big(\hbar^2 \partial_x^2 - q(x,\hbar)\Big)\psi(x,\hbar) = 0. \tag{104}$$

We keep the discussion here very brief; the details can be found in [12]. The *potential* function $q(x,\hbar)$ is defined on a domain in $\mathbb{C}_x \times \mathbb{C}_\hbar$ and usually assumed to have polynomial or even constant dependence on $\hbar$. In view of the work done in this article, we can assume a much more general $\hbar$-dependence, but for simplicity of presentation, let us suppose that q is a polynomial in $\hbar$:

$$q(x,\hbar) = q_0(x) + q_1(x)\hbar + \cdots + q_n(x)\hbar^n.$$

The WKB method begins by searching for a solution in the form of the *WKB ansatz*:

$$\psi(x,\hbar) = \exp\left(-\frac{1}{\hbar} \int_{x_*}^x f(t,\hbar)\mathrm{d}t\right), \tag{105}$$

where x_0 is a chosen basepoint, and $f = f(x, \hbar)$ is the unknown function to be solved for. Substituting this expression back into the Schrödinger equation, we find that the WKB ansatz (105) is a solution if the function f satisfies the singularly perturbed Riccati equation

$$\hbar \partial_x f = f^2 - q. \tag{106}$$

Locally in x, this Riccati equation has two formal solutions $\hat{f}_\pm$ with locally holomorphic leading-orders $f_0^\pm = \pm \sqrt{q_0}$. They give rise to a pair of *formal WKB solutions*, which by definition are the following formal expressions:

$$\hat{\psi}_\pm(x, \hbar) := \exp\left(-\frac{1}{\hbar} \int_{x_0}^x \hat{f}_\pm(t, \hbar) \mathrm{d}t \right). \tag{107}$$

An *exact WKB solution* is any analytic solution $\psi(x, \hbar)$ to the Schrödinger equation (104), which is asymptotic as $\hbar \to 0$ in the right half-plane to a formal WKB solution.

THEOREM 6.1 (Local existence of exact WKB solutions). *Consider a Schrödinger equation (104) with potential $q = q(x, \hbar)$ which is a polynomial in $\hbar$ whose coefficients are rational functions on $\mathbb{C}_x$. We make the following two assumptions:*

(1) *Suppose that the poles of q have order at least 2 and that they are completely specified in the leading-order term q_0. More precisely, if $\mathsf{D} \subset \mathbb{C}_x \cup \{\infty\}$ is the set of poles of q_0, we assume that every pole $x_\infty \in \mathsf{D}$ has order $\mathrm{ord}(q_0) \geq 2$; we assume furthermore that every q_k has no poles other than D and that for every $x_\infty \in \mathsf{D}$, we have $\mathrm{ord}(q_i) \leq \mathrm{ord}(q_0)$.*
(2) *Fix a basepoint point $x_0 \in \mathbb{C}_x$ which is neither a pole nor a zero of q_0, and assume that the real one-dimensional curve*

$$\Gamma(x_0) := \left\{ x \in \mathbb{C}_x \ \middle| \ \mathrm{Im}\left(\int_{x_0}^x \sqrt{q_0(t)} \mathrm{d}t \right) = 0 \right\} \tag{108}$$

limits at both ends into points of D (not necessarily distinct).

Then the Schrödinger equation (104) has a canonical local basis of exact WKB solutions $\psi_\pm$ normalized at x_0:

$$\psi_\pm(x_0, \hbar) = 1 \quad \text{and} \quad \psi_\pm(x, \hbar) \sim \hat{\psi}_\pm(x, \hbar) \quad \text{as } \hbar \to 0 \text{ in the right half-plane.} \tag{109}$$

Proof. We consider the corresponding Riccati equation (106). Its leading-order discriminant is simply $D_0 = 4q_0$. The assumptions on the pole orders of q and the fact that $\Gamma(x_0)$ flows into D at both ends imply that $\Gamma(x_0)$ is a generic WKB trajectory. Thus, all the hypotheses of Theorem 5.3 (or more specifically of Proposition 6.4) are satisfied, so (106) has a canonical pair of exact solutions $f_\pm$ defined for x near x_0 and asymptotic to $\hat{f}_\pm$ as $\hbar \to 0$ in the right half-plane. The exact WKB solutions $\psi_\pm$ are then defined as $\psi_\pm(x, \hbar) := \exp\left(-\hbar^{-1} \int_{x_0}^x f_\pm(t, \hbar) \mathrm{d}t \right)$. They form a basis of solutions near x_0 because the Wronskian of ψ_+ and ψ_- evaluated at $x = x_0$ is $f_+(x_0, \hbar) - f_-(x_0, \hbar) \neq 0$. $\qquad \square$

Appendix A. Some useful elementary estimates

Here, we collect some elementary estimates that are used in the proof of Lemma 5.1. Their proofs are straightforward.

LEMMA A.1. *For any $R \geq 0$, any $L \geq 0$, and any nonnegative integer n,*

$$\int_0^R \frac{r^n}{n!} e^{Lr} \mathrm{d}r \leq \frac{R^{n+1}}{(n+1)!} e^{LR}.$$

LEMMA A.2. *For any $R \geq 0$, and any integers $m, n \geq 0$,*

$$\int_0^R (R-r)^m r^n \mathrm{d}r = \frac{m!n!}{(m+n+1)!} R^{m+n+1}.$$

LEMMA A.3. *Let i, j be nonnegative integers, and let $f_i(\xi), f_j(\xi)$ be holomorphic functions on a tubular neighborhood $\Xi_+ := \{\xi \mid \mathrm{dist}(\xi, \mathbb{R}_+) < \epsilon\}$ of the positive real axis $\mathbb{R}_+ \subset \mathbb{C}_\xi$ for some $\epsilon > 0$. If there are constants $M_i, M_j, L \geq 0$ such that*

$$\left| f_i(\xi) \right| \leq M_i \frac{|\xi|^i}{i!} e^{L|\xi|} \qquad and \qquad \left| f_j(\xi) \right| \leq M_j \frac{|\xi|^j}{j!} e^{L|\xi|} \qquad \forall \xi \in \Xi_+,$$

then their convolution product satisfies the following bound:

$$\left| f_i * f_j(\xi) \right| \leq M_i M_j \frac{|\xi|^{i+j+1}}{(i+j+1)!} e^{L|\xi|} \qquad \forall \xi \in \Xi_+.$$

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