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CONTRACTIBLE PERIODIC ORBITS OF LAGRANGIAN SYSTEMS

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Abstract

We consider a convex Lagrangian $L : TM \to \mathbb{R}$ quadratic at infinity with $L(x, 0) = 0$ for every $x \in M$ and such that the 1-form θ defined by $\theta_x(v) = L_v(x, 0)v$ is not closed. We show that for every number $a < 0$, there is a contractible (nonconstant) periodic orbit with action a. We also obtain estimates of the period and energy of such periodic orbits.

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1. Introduction

Let M be a closed connected smooth Riemannian manifold. Let

$$L : TM \to \mathbb{R}$$

be a smooth convex Lagrangian. This means that L restricted to each T_xM has positive-definite Hessian. The Lagrangian L is said to be quadratic at infinity if there exists $R > 0$ such that for each $x \in M$ and $|v|_x > R$, the function $L(x, v)$ has the form

$$L(x, v) = \tfrac{1}{2}|v|_x^2 + \theta_x(v) - V(x),$$

where θ is a smooth 1-form on M and $V : M \to \mathbb{R}$ is a smooth function. We shall also assume that $L(x, 0) = 0$ for every $x \in M$ and that the 1-form $\theta_x(v) = L_v(x, 0)v$ is not closed. Our assumptions are satisfied if $L(x, v)$ has the form $L(x, v) = \frac{1}{2}|v|_x^2 + \theta_x(v)$ for a nonclosed 1-form θ. The Euler–Lagrange flow of this Lagrangian is called exact magnetic flow and it models the movement of a particle under the effect of a magnetic field (see [4, 11, 13, 14, 19]).

Let Λ be the set of absolutely continuous contractible curves $x : [0, 1] \to M$, $x(0) = x(1)$, such that $\dot{x}$ has finite L^2-norm. It is well known [7, 15] that Λ has a Hilbert manifold structure compatible with the Riemannian metric on M.

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The free-time action $\mathcal{A}_L : \mathbb{R}^+ \times \Lambda \to \mathbb{R}$ (where $\mathbb{R}^+$ stands for the set of positive real numbers) is given by

$$\mathcal{A}_L(b, x) = \int_0^1 b\, L(x(t), \dot{x}(t)/b)\, dt.$$

If the Lagrangian is quadratic at infinity, arguments in [2, Proposition 3.1] show that $\mathcal{A}_L$ is a $C^{1,1}$ function (that is, a function with locally Lipschitz derivative).

Periodic orbits with energy larger than Mañé's critical value were obtained in [7]. In previous articles [16, 17], we showed the existence of periodic orbits, not necessarily contractible, with large enough abbreviated action and large enough action. The aim of this note is to obtain nontrivial (that is, nonconstant) *contractible* periodic orbits for every negative value of the action. We also obtain estimates of the period and energy of such periodic orbits. We prove the following theorem.

THEOREM 1.1. *Let L be a convex Lagrangian quadratic at infinity such that $L(x, 0) = 0$ for every $x \in M$ and such that the 1-form $\theta_x(v) = L_v(x, 0)v$ is not closed. Then there are $\gamma > 0$ and $a_1 < 0$ such that for every $a < 0$, the Lagrangian has a nontrivial contractible periodic orbit with action a so that its period T and energy e satisfy*

$$\begin{gathered} -\gamma^{-1} a \le T, \quad e \le \gamma \quad \text{for every } a < 0, \\ T \le -\gamma a, \quad -\gamma^{-1} a^{-1} \le e \quad \text{for every } a < a_1. \end{gathered}$$

We remark that nonconstant periodic orbits with action a are obtained for every $a < 0$. The first line of estimates holds for every $a < 0$ and the second line holds for every $a < a_1$. All estimates are satisfied if $a < a_1$.

In [5], it is shown that a particular class of Lagrangians has an infinite set of periodic orbits satisfying similar bounds. In [17], we obtained periodic orbits with large enough prescribed action but those orbits are not necessarily contractible and satisfy different estimates.

We outline here the proof of Theorem 1.1. In Lemma 4.5, we show that for every $a < 0$, the set X_a of those (b, x) such that $\mathcal{A}_L(b, x) = a$ is nonvoid. This is achieved by finding an upper bound of $\mathcal{A}_L(b, x)$ containing the integral of the 1-form θ. Since θ is not closed, we find a curve with negative action. By integrating many times along such a curve, we obtain arbitrarily large negative values of the action. On the other hand, under our assumptions, the value 0 is attained and hence the action takes every negative value (note that Λ is connected). In Lemma 4.3, we show that every $a < 0$ is a regular value of $\mathcal{A}_L$ and consequently X_a is a $C^{1,1}$ manifold.

Consider the function $\mathcal{T}_a : X_a \to \mathbb{R}$ given by

$$\mathcal{T}_a(b, x) = b.$$

In Proposition 2.3, we show that critical points of $\mathcal{T}_a$ correspond to periodic orbits of the Lagrangian. The existence of the desired periodic orbits follows by showing that for every $a < 0$, the map $\mathcal{T}_a$ has a critical point. The periodic orbits obtained are nonconstant because if $(b, x) \in X_a$, then b and the L^2-norm of $\dot{x}$ are bounded away from zero (see Lemma 4.4).

2. Contractible periodic orbits with prescribed action

The *energy* $E_L : TM \to \mathbb{R}$ is defined by

$$E_L(x, v) = \frac{\partial L}{\partial v}(x, v).v - L(x, v).$$

Since L is autonomous, E_L is a first integral of the Euler–Lagrange flow of L. Critical points of $\mathcal{A}_{L+k}$ correspond to periodic orbits with energy k.

Proposition 2.1. *If (b, x) is a critical point of $\mathcal{A}_{L+k}$, then $y : [0, b] \to M$ given by $y(s) = x(s/b)$ is a periodic solution of the Euler–Lagrange equation of L with energy k. (See* [2, 3, 8, 9]*.)*

Note that periodic orbits with energy different from k do not correspond to critical points of $\mathcal{A}_{L+k}$.

It is useful to define the *average energy function* $e : \mathbb{R}^+ \times \Lambda \to \mathbb{R}$ by

$$e(b, x) = \int_0^1 E_L(x(t), \dot{x}(t)/b) = \frac{1}{b}\int_0^b E_L(y(s), \dot{y}(s))\, ds.$$

Remark 2.2. It is easy to see that

$$\frac{\partial \mathcal{A}_L}{\partial b}(b, x) = -e(b, x).$$

Define $\mathcal{T}_a : X_a \to \mathbb{R}$ by

$$\mathcal{T}_a(b, x) = b$$

and note that $\mathcal{T}_a$ is the restriction to X_a of the canonical projection $\Pi : \mathbb{R}^+ \times \Lambda \to \mathbb{R}^+$. Then $\nabla\mathcal{T}_a(b, x)$ is the orthogonal projection of $\nabla\Pi = (1, 0)$ onto $T_{(b,x)}X_a$. We can find the projection by writing

$$\nabla\mathcal{T}_a = (v_b, v_x), \quad (1, 0) = \alpha\frac{\nabla\mathcal{A}_L}{\|\nabla\mathcal{A}_L\|} + (v_b, v_x).$$

It follows that

$$\alpha = \frac{1}{\|\nabla\mathcal{A}_L\|}\frac{\partial \mathcal{A}_L}{\partial b}, \quad v_b = 1 - \alpha^2, \quad v_x = -\frac{1}{\|\nabla\mathcal{A}_L\|^2}\frac{\partial \mathcal{A}_L}{\partial b}\frac{\partial \mathcal{A}_L}{\partial x}. \tag{2.1}$$

Note that $\nabla\mathcal{T}_a$ is locally Lipschitz.

Proposition 2.3. *If (b, x) is a critical point of $\mathcal{T}_a$, then (b, x) corresponds to a periodic orbit.*

Proof. Let (b, x) be such that $\nabla\mathcal{T}_a(b, x) = 0$. Then $v_b(b, x) = 0$ and $v_x(b, x) = 0$. Therefore, $\alpha \neq 0$ and hence

$$\frac{\partial \mathcal{A}_L}{\partial x}(b, x) = 0. \qquad \square$$

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3. Minimax principle

Definition 3.1. Let $f : X \to \mathbb{R}$ be a C^1 map, where X is an open set of a Hilbert manifold. We say that f satisfies the *Palais–Smale* condition at level c if every sequence $\{x_n\}$ with $f(x_n) \to c$ and $\|d_{x_n} f\| \to 0$ as $n \to \infty$ has a converging subsequence.

The following version of the minimax principle (Proposition 3.2 below) is a particular case of [7, Proposition 6.3] (which in turn is inspired by [12] (see also [18])). Let X be an open set in a Hilbert manifold and $f : X \to \mathbb{R}$ be a $C^{1,1}$ map. Observe that if X is not complete or the vector field $Y = -\nabla f$ is not globally Lipschitz, the gradient flow ψ_t of $-f$ is *a priori* only a local flow. Given $p \in X$ and $t > 0$, define

$$\alpha(p) := \sup\{a > 0 \mid s \mapsto \psi_s(p) \text{ is defined for } s \in [0, a]\}.$$

We say that a function $\tau : X \to [0, +\infty)$ is an *admissible time* if τ is differentiable and $0 \le \tau(x) < \alpha(x)$ for all $x \in X$. Given an admissible time τ and a subset $F \subset X$, define

$$F_\tau := \{\psi_{\tau(p)}(p) \mid p \in F\}.$$

Let $\mathcal{F}$ be a family of subsets $F \subset X$. We say that $\mathcal{F}$ is *forward invariant* if $F_\tau \in \mathcal{F}$ for all $F \in \mathcal{F}$ and any admissible time τ. Define

$$c(f, \mathcal{F}) = \inf_{F \in \mathcal{F}} \sup_{p \in F} f(p).$$

Proposition 3.2. *Let f be a $C^{1,1}$ function satisfying the Palais–Smale condition at level $c(f, \mathcal{F})$. Assume also that $\mathcal{F}$ is forward invariant under the gradient flow of $-f$. Suppose that $c = c(f, \mathcal{F})$ is finite and that there is an ε such that the gradient flow is relatively complete in the set $[c - \varepsilon \le f \le c + \varepsilon]$. Then $c(f, \mathcal{F})$ is a critical value of f.*

4. Proof of Theorem 1.1

Recall that we are assuming that our Lagrangian L is convex, quadratic at infinity and such that $L(x, 0) = 0$ for all x. Let θ_x be the 1-form $\theta_x(v) = L_v(x, 0)v$, where L_v is the derivative along the fibre. Recall that we are assuming that θ is not closed. Set $\Theta(x) = \int_x \theta$ and define

$$\sigma(L) = \sup_{\|\dot{x}\| \neq 0} \frac{|\Theta(x)|}{\|\dot{x}\|_{L^2}^2}.$$

Lemma 4.1. $0 < \sigma(L) < +\infty$.

Proof. Let $\ell(x)$ be the length of x and note that $\ell(x) \le \|\dot{x}\|$.

By [7, Lemma 5.1, page 369], there is a $\sigma_0 > 0$ such that $|\Theta(x)| \le \sigma_0 \ell(x)^2$ for every $x \in \Lambda$ and therefore $\sigma(L) < \infty$.

We claim that there is an $x \in \Lambda$ so that $\Theta(x) \neq 0$. Assume by contradiction that $\Theta(x) = 0$ for every $x \in \Lambda$. Consider $U \subset M$, a contractible open set. We have in particular that $\Theta(x) = \int_x \theta = 0$ for every x contained in U. Therefore, $\theta|_U$ is exact and hence $d\theta|_U = 0$. Since U is arbitrary, we conclude that θ is closed, contradicting the hypothesis of Theorem 1.1. This proves the claim, which implies that $\sigma > 0$. □

Lemma 4.2. *Set $\theta_x(v) = L_v(x,0)v$ and let, as before, $\Theta(x) = \int_x \theta$. If L is convex, quadratic at infinity and such that $L(x,0) = 0$ for every $x \in M$, there are positive constants A, A_1, B, B_1 such that*

$$\frac{A}{2b}\|\dot{x}\|^2_{L^2} + \Theta(x) \le \mathcal{A}_L(b,x) \le \frac{A_1}{2b}\|\dot{x}\|^2_{L^2} + \Theta(x), \tag{4.1}$$

$$\frac{A}{2b^2}\|\dot{x}\|^2_{L^2} \le e(b,x) \le \frac{A_1}{2b^2}\|\dot{x}\|^2_{L^2}, \tag{4.2}$$

$$A\frac{\|\dot{x}\|^2_{L^2}}{2b} - Bb \le \mathcal{A}_L(b,x) \le A_1\frac{\|\dot{x}\|^2_{L^2}}{2b} + B_1 b. \tag{4.3}$$

Proof. Let a, a_1 be positive numbers according to [7, Lemma 3.1] such that

$$\frac{a}{2}|v|^2_x + \theta_x(v) \le L(x,v) \le \frac{a_1}{2}|v|^2_x + \theta_x(v), \tag{4.4}$$

$$\frac{a}{2}|v|^2_x \le E_L(x,v) \le \frac{a_1}{2}|v|^2_x. \tag{4.5}$$

On the other hand, since L is quadratic at infinity, there are positive numbers $\bar{a}, B, \bar{a}_1, B_1$ such that

$$\frac{\bar{a}}{2}|v|^2 - B \le L(x,v) \le \frac{\bar{a}_1}{2}|v|^2 + B_1. \tag{4.6}$$

Take A and A_1 such that $0 < A < a$, $0 < A < \bar{a}$ and $0 < a_1 < A_1$, $0 < \bar{a}_1 < A_1$. The lemma follows from this choice on taking account of (4.4)–(4.6). $\square$

From (4.2) and (4.3),

$$\frac{A}{A_1}\left(\frac{\mathcal{A}_L(b,x)}{b} - B_1\right) \le e(b,x) \le \frac{A_1}{A}\left(\frac{\mathcal{A}_L(b,x)}{b} + B\right). \tag{4.7}$$

Let X_a be the set of those $(b,x) \in \mathbb{R}^+ \times \Lambda$ such that $\mathcal{A}_L(b,x) = a$.

Lemma 4.3. *Suppose that $a < 0$. Then X_a does not contain critical points of $\mathcal{A}_L$.*

Proof. Assume that the lemma is false. Then there is (b,x) such that

$$\frac{\partial \mathcal{A}_L}{\partial x}(b,x) = 0 \quad \text{and} \quad \frac{\partial \mathcal{A}_L}{\partial b}(b,x) = 0.$$

By Remark 2.2, $e(b,x) = 0$ and hence, by (4.2), $\|\dot{x}\|_{L^2} = 0$. By (4.1),

$$0 = \frac{A}{2b}\|\dot{x}\|^2_{L^2} + \Theta(x) \le \mathcal{A}_L(b,x) = a < 0,$$

which is a contradiction. $\square$

Note that Lemma 4.3 implies that X_a is a $C^{1,1}$ manifold for every $a < 0$ and hence we can apply the minimax principle to the function $\mathcal{T}_a : X_a \to \mathbb{R}$. If (b,x) is a critical point of $\mathcal{T}_a$, it satisfies $\partial\mathcal{A}_L/\partial x\,(b,x) = 0$ on account of Proposition 2.3 (and therefore it corresponds to a periodic orbit). However, such a point (b,x) cannot satisfy $\partial\mathcal{A}_L/\partial b\,(b,x) = 0$ at the same time.

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LEMMA 4.4. *Suppose that* $(b, x) \in X_a$ *and* $a < 0$. *Then:*

(1) $b \geq (A/2\sigma)$;
(2) $\|\dot{x}\|_{L^2}^2 \geq -a/\sigma$.

PROOF. By (4.1) and Lemma 4.1,

$$\frac{A}{2b}\|\dot{x}\|_{L^2}^2 - \sigma\|\dot{x}\|_{L^2}^2 \leq \frac{A}{2b}\|\dot{x}\|_{L^2}^2 + \Theta(x) \leq \mathcal{A}_L(b, x) = a < 0$$

and this gives the first item. To prove the second item, note that

$$-a \leq \left(\sigma - \frac{A}{2b}\right)\|\dot{x}\|_{L^2}^2 \leq \sigma\|\dot{x}\|_{L^2}^2. \qquad \square$$

Recall that $\mathcal{T}_a : X_a \to \mathbb{R}$ is defined by $\mathcal{T}_a(b, x) = b$. Let $\mathcal{F}_a$ be the family of sets $F = \{(b, x)\}$ such that $\mathcal{A}_L(b, x) = a$.

LEMMA 4.5. $\mathcal{F}_a$ *is nonvoid for every* $a < 0$ *and there are* $\gamma > 0$ *and* $a_1 < 0$ *such that*

$$-\gamma^{-1}a \leq c(\mathcal{T}_a, \mathcal{F}_a) \quad \text{for every } a < 0,$$
$$c(\mathcal{T}_a, \mathcal{F}_a) \leq -\gamma a \quad \text{for every } a < a_1.$$

PROOF. First we show that $\mathcal{A}_L$ takes arbitrarily large negative values. To see this, note that for nonconstant x,

$$\mathcal{A}_L(b, x) \leq \frac{A_1\|\dot{x}\|_{L^2}^2}{2b} + \Theta(x) = \|\dot{x}\|_{L^2}^2\left(\frac{A_1}{2b} + \frac{\Theta(x)}{\|\dot{x}\|_{L^2}^2}\right).$$

By Lemma 4.1, $\sigma > 0$ and hence there is an $x_1 \in \Lambda$ such that $\Theta(x_1) \neq 0$. (Recall that to show that $\sigma > 0$, we used the assumption that θ is not closed.) If $\Theta(x_1) < 0$, set $x_0 = x_1$ and, if $\Theta(x_1) > 0$, set $x_0(t) = x_1(1 - t)$ for $t \in [0, 1]$. This shows that there is an $x_0 \in \Lambda$ such that $\Theta(x_0) < 0$.

Take b_0 so that $\mathcal{A}_L(b_0, x_0) < 0$ and set

$$a_0 = \mathcal{A}_L(b_0, x_0).$$

For $n = 1, 2, \ldots,$ let $w^n : [0, n] \to M$ be the map defined by $w^n(t) = x_0(t - i)$ for $t \in [i, i + 1]$, where i is any integer such that $0 \leq i < n$. Let $x_0^n \in \Lambda$ be defined by $x_0^n(t) = w^n(tn)$. Then

$$\mathcal{A}_L(nb_0, x_0^n) = na_0, \tag{4.8}$$

which shows that $\mathcal{A}_L$ takes arbitrarily large negative values. On the other hand, since $\mathcal{A}_L$ takes the value 0, it takes every negative value since $\mathcal{A}_L$ is continuous and Λ is connected. This shows that X_a is nonvoid for every $a < 0$.

If $(b, x) \in X_a$, then, by (4.3),

$$-Bb \leq A\frac{\|\dot{x}\|_{L^2}^2}{2b} - Bb \leq \mathcal{A}_L(b, x) = a.$$

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Therefore,

$$-\frac{1}{B}a \le b$$

and hence

$$-\frac{1}{B}a \le c(\mathcal{T}_a, \mathcal{F}_a).$$

For the other inequality, let $a < a_0$ and keep it fixed for the rest of the proof. Take a positive integer n such that

$$(n+1)a_0 \le a < na_0.$$

Let $\Gamma : [0,1] \to \mathbb{R}^+ \times \Lambda$ be a continuous map given by

$$\Gamma(s) = (\beta_s, z_s)$$

such that $\beta_s = (n+s)b_0$, $z_0 = x_0^n$ and $z_1 = x_0^{n+1}$. By (4.8) and the continuity of $\mathcal{A}_L$, there is an $s_1 \in [0,1]$ such that

$$\mathcal{A}_L(\Gamma(s_1)) = a.$$

Then

$$\mathcal{T}_a(\Gamma(s_1)) = \beta_{s_1}$$

and consequently

$$c(\mathcal{T}_a, \mathcal{F}_a) \le \beta_{s_1}.$$

Therefore,

$$c(\mathcal{T}_a, \mathcal{F}_a) \le (n+1)b_0 = \frac{b_0}{-a_0}(-na_0 - a_0) < \frac{b_0}{-a_0}(-a - a_0).$$

We choose $a_1 < 0$ and $\gamma_1 > 0$ so that

$$\frac{b_0}{-a_0}(-a - a_0) < -\gamma_1 a \quad \text{for } a < a_1.$$

Finally, let $\gamma > B$ and $\gamma > \gamma_1$. □

Lemmas 4.6 and 4.7 below appear also in [17] and we include them here for the sake of completeness.

LEMMA 4.6. *The flow of* $-\nabla\mathcal{T}_a$ *is relatively complete in* $0 < c_1 \le \mathcal{T}_a \le c_2$.

PROOF. Arguing by contradiction, let $s \to \Gamma(s) = (b(s), x(s))$ be a flow semi-trajectory defined in the maximal interval $[0, \bar{s})$ and contained in $c_1 \le \mathcal{T}_a \le c_2$. Let $t_n \in [0, \bar{s})$ be a sequence converging to $\bar{s}$. By the same argument as in [7, Lemma 6.9], the sequence $\Gamma(t_n)$ is a Cauchy sequence, implying that $b(t_n)$ converges to $b_0 \in [0, \infty)$. Since $b(t_n) \ge c_1 > 0$, we know that $b_0 > 0$ and hence the sequence $\Gamma(t_n)$ converges in X_a, which allows the flow semi-trajectory to be extended. □

LEMMA 4.7. $\mathcal{T}_a$ *satisfies the Palais–Smale condition at level* $c(\mathcal{T}_a, \mathcal{F}_a)$.

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Proof. Let $\{(b_n, x_n)\}$ be a sequence in X_a such that $\mathcal{T}_a(b_n, x_n) \to c(\mathcal{T}_a, \mathcal{F}_a)$ and such that $\|\nabla\mathcal{T}_a(b_n, x_n)\| \to 0$. By Lemma 4.5, b_n is bounded and bounded away from zero. Then $\|\nabla A_L(b_n, x_n)\|$ is bounded away from zero since otherwise the arguments of [7, Proposition 3.12] (see also [6, 10]) show that $\{(b_n, x_n)\}$ has a subsequence converging to a critical point of $\mathcal{A}_L$, which is impossible because a is a regular value of $\mathcal{A}_L$. Since $\mathcal{A}_L(b_n, x_n) = a$ and b_n is bounded and bounded away from zero, we conclude by (4.3) that $\|\dot{x}_n\|_{L^2}$ is bounded and this implies, by (4.2), that $e(b_n, x_n) = -\partial\mathcal{A}_L/\partial b\,(b_n, x_n)$ is bounded.

Let v_b, v_x, α be as in (2.1). Then $v_b(b_n, x_n)$ converges to 0 and thus $\alpha(b_n, x_n)$ is bounded and bounded away from zero. This implies that $\|\nabla A_L(b_n, x_n)\|$ is bounded and $\partial\mathcal{A}_L/\partial b\,(b_n, x_n)$ is bounded away from zero.

On the other hand, since $\|v_x(b_n, x_n)\|$ also converges to 0,

$$\left\|\frac{\partial\mathcal{A}_L}{\partial x}(b_n, x_n)\right\| \to 0.$$

Hence, by the argument of [7, Proposition 3.12], the sequence $\{(b_n, x_n)\}$ has a convergent subsequence in X_a. In fact [7, Proposition 3.12] assumes that $\|d_{b_n,x_n}\mathcal{A}_L\|$ converges to zero but it is enough to have $\|\partial\mathcal{A}_L/\partial x\,(b_n, x_n)\| \to 0$ as is shown in [1, Lemma 5.3]. □

Proof of Theorem 1.1. Lemmas 4.3, 4.5–4.7 allow us to apply Proposition 3.2, completing the proof. The orbits obtained are nontrivial because of Lemma 4.4. The estimates for the period follow from Lemma 4.5. The upper bound of the energy is obtained from the estimates for the period and (4.7) (possibly after taking a bigger γ). The lower bound of the energy is obtained from the estimates for the period and Lemma 4.4. □

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