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Helices, Hasimoto Surfaces and Bäcklund Transformations

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Abstract. Travelling wave solutions to the vortex filament flow generated by elastica produce surfaces in $\mathbb{R}^3$ that carry mutually orthogonal foliations by geodesics and by helices. These surfaces are classified in the special cases where the helices are all congruent or are all generated by a single screw motion. The first case yields a new characterization for the Bäcklund transformation for constant torsion curves in $\mathbb{R}^3$, previously derived from the well-known transformation for pseudospherical surfaces. A similar investigation for surfaces in H^3 or S^3 leads to a new transformation for constant torsion curves in those spaces that is also derived from pseudospherical surfaces.

1 Introduction and Results for $\mathbb{R}^3$

A *Hasimoto surface* is the surface traced out by a curve γ in $\mathbb{R}^3$ as it evolves over time according to this evolution equation:

$$\frac{\partial\gamma}{\partial t} = \frac{\partial\gamma}{\partial s} \times \frac{\partial^2\gamma}{\partial s^2}. \tag{1}$$

Known as the *vortex filament flow* or *Localized Induction Equation* (LIE), this flow was formulated by L. Da Rios in 1906 as a model for how a one-dimensional "filament" of vortices moves in an incompressible fluid. It was studied in the 1970s by Hasimoto [7], who discovered that it is essentially equivalent to a well-known completely integrable PDE, the focussing cubic nonlinear Schrödinger equation. (See [10] for a detailed history of LIE, with copious references.)

We will be mainly interested in the *geometrical properties* of solutions to (1), focussing on the properties that Hasimoto surfaces exhibit when the filament evolves so as to retain its shape. Because of the semigroup property of the evolution equation, this means that the curve moves by a one-parameter subgroup of the group of rigid motions of $\mathbb{R}^3$. The planar curves that so evolve under (1) include circles (which sweep out right circular cylinders) and the figure-eight elastic curve, which rotates about its self-intersection (see Figure 1). Non-planar examples include helices, which sweep out cylinders, and non-planar elastica, which move by a combination of rotation and translation known as a screw motion (see Figure 1). In fact, it is known that *the filaments that move by rigid motions under LIE are precisely the elastica, i.e., the set of curves which are critical for some functional of the form*

$$\lambda_1 \int_\gamma ds + \lambda_2 \int_\gamma \kappa^2\, ds.$$

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(This connection was also first explored by Hasimoto [6].) More generally, elastic rod centerlines, which are critical for functionals that also involve the integral of torsion, move under LIE by a combination of rigid motion and sliding along the filament [9].

This paper deals with the answer to the following question, posed by Joel Langer.

Hasimoto surfaces that are generated by elastica have two interesting geometrical properties: first, they are foliated in the time direction by helices, each traced out by a point on the filament as the filament evolves. Next, because (1) can also be written as

$$\frac{\partial \gamma}{\partial t} = \kappa B,$$

the tangent plane of the surface is spanned by the unit tangent T and binormal B of the filament, and the filament is a geodesic in the surface. So, these Hasimoto surfaces possess two orthogonal foliations, by geodesics and by helices. How particular is this structure to elastica and the LIE? That is, if a surface carries foliations of this sort, must the geodesics be copies of some member of the finite-dimensional family of elastica, evolving under LIE? It turns out that the answer to this question is no: such surfaces form an infinite-dimensional family.

Because of the different extra assumptions one can make about the helices on the surface, there are two theorems one can state. (See Section 4 for what happens when the extra assumptions are weakened.)

Theorem 1 *Let S be a smooth surface in $\mathbb{R}^3$ foliated orthogonally by geodesics and helices. Assume in addition that all the helices are congruent. Then either*

1. *the helices degenerate to parallel lines, and the geodesics are copies of a single arbitrary planar curve;*
2. *S is a right circular cylinder, and the geodesics are helices; or*
3. *the helices degenerate to circles, and the geodesics are all Bäcklund transformations of a single arbitrary curve of nonzero constant torsion.*

In either case, the generating curve depends on one arbitrary function of one variable.

The Bäcklund transformation for curves of constant torsion, defined and studied in [3], is this: given a unit speed curve $\gamma(s)$ in $\mathbb{R}^3$, with curvature κ and nonzero constant torsion τ, obtain a solution β of the differential equation

$$d\beta/ds = \lambda \sin\beta - \kappa. \tag{2}$$

Using Frenet frame vectors T and N along γ, define

$$\tilde{\gamma} = \gamma + \frac{2\lambda}{\lambda^2 + \tau^2}(T\cos\beta + N\sin\beta). \tag{3}$$

Then $\tilde{\gamma}(s)$ is a new unit speed curve of constant torsion τ.

This transformation is obtained by restricting the classical Bäcklund transformation for pseudospherical surfaces to the asymptotic lines of those surfaces, which have constant torsion. (Discussions of Bäcklund's transformation appear in many of the old treatises on

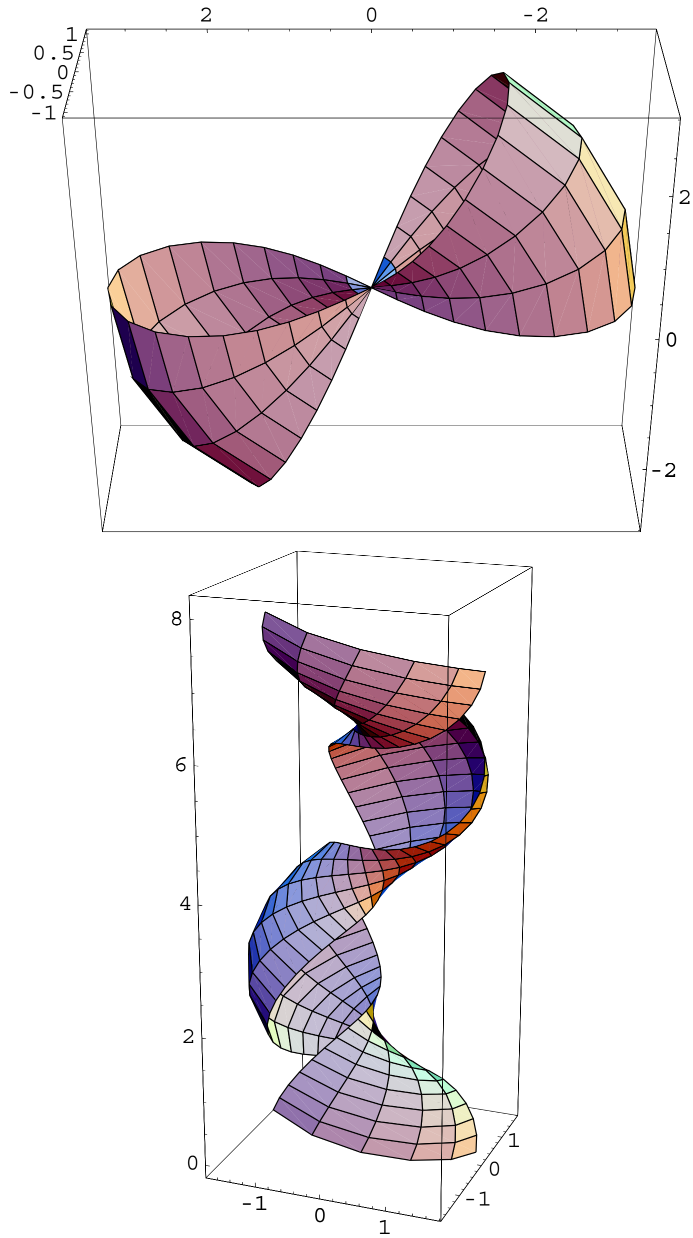

Figure 1: Hasimoto surfaces generated by planar and non-planar elastica.

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differential geometry, for example in [5]; a more recent account appears in the introduction to [4].) The form of the new curve depends in general on two parameters, λ and the initial condition for the ODE (2). Note that as the latter is varied, $\bar{\gamma}(s)$ swings around $\gamma(s)$ along the arc of a circle in the osculating plane of γ. It follows from the properties of the surface Bäcklund transformation that the the binormals of the old and new curves make a constant angle, which is equal to $2\arctan(\lambda/\tau)$, and the vector from $\gamma(s)$ to $\bar{\gamma}(s)$ is perpendicular to both binormals. In particular, when $\lambda = \pm\tau$, the binormal of $\bar{\gamma}$ points along the circles just mentioned, and thus the various curves $\bar{\gamma}$ sweep out a surface in which they are geodesics. Since the radius of the circles is maximized for this choice of λ, we will refer to this as a *maximal Bäcklund transformation*; of course, it is only this kind that appears in Theorem 1.

Figure 2 shows a helix and the "surface of circles" swept out by its Bäcklund transformations for $\lambda = \tau$.

We will pursue the ramifications of Theorem 1 later on. For now, we'll just note that the extra assumption on the helices was far too restrictive, causing them to degenerate. Instead, we will let the helices vary in radius but fix their axis and translational period.

Theorem 2 *Let S by a smooth surface in $\mathbb{R}^3$ foliated orthogonally by geodesics and helices. Assume in addition that the helices are generated by a common screw motion. Then the geodesics of S are all congruent under the screw motion, and depend on one arbitrary function of one variable.*

This theorem says that this class of surfaces is quite general. To construct one of them, suppose the screw motion moves j units along the z-direction for every counterclockwise rotation of 2π. (The torsion of the helices has the same sign as j.) If N and B are the normal and binormal of the helix at a point on the surface, let α be the angle such that $N\cos\alpha + B\sin\alpha$ is the tangent to the geodesic. Then α can be arbitrarily specified as a function of arclength s along the geodesic, and the cylindrical coordinates of the geodesic satisfy

$$\left.\begin{aligned} dr/ds &= -\cos\alpha \\ dz/ds &= r\sin\alpha/\sqrt{j^2+r^2} \\ d\theta/ds &= -(j/r^2)dz/ds \end{aligned}\right\} . \tag{4}$$

An example where the function α is chosen so that the geodesics are closed are shown in Figure 3.

The proofs of these theorems, which involve the use of moving frames and exterior differential systems, will be relegated to another section.

2 Results in Other Space Forms

The PDE underlying pseudospherical surfaces, the sine-Gordon equation, is connected to the hierarchy of flows built on the LIE (see [8] for how this connection works on the level of vector fields). Nevertheless, the appearance in Theorem 1 of the Bäcklund transformation for constant torsion curves in the answer to a seemingly unrelated question is still surprising. Constant torsion again appears in the generalization of Theorem 1 to surfaces in a three-dimensional space form (*i.e.*, either S^3 or H^3).

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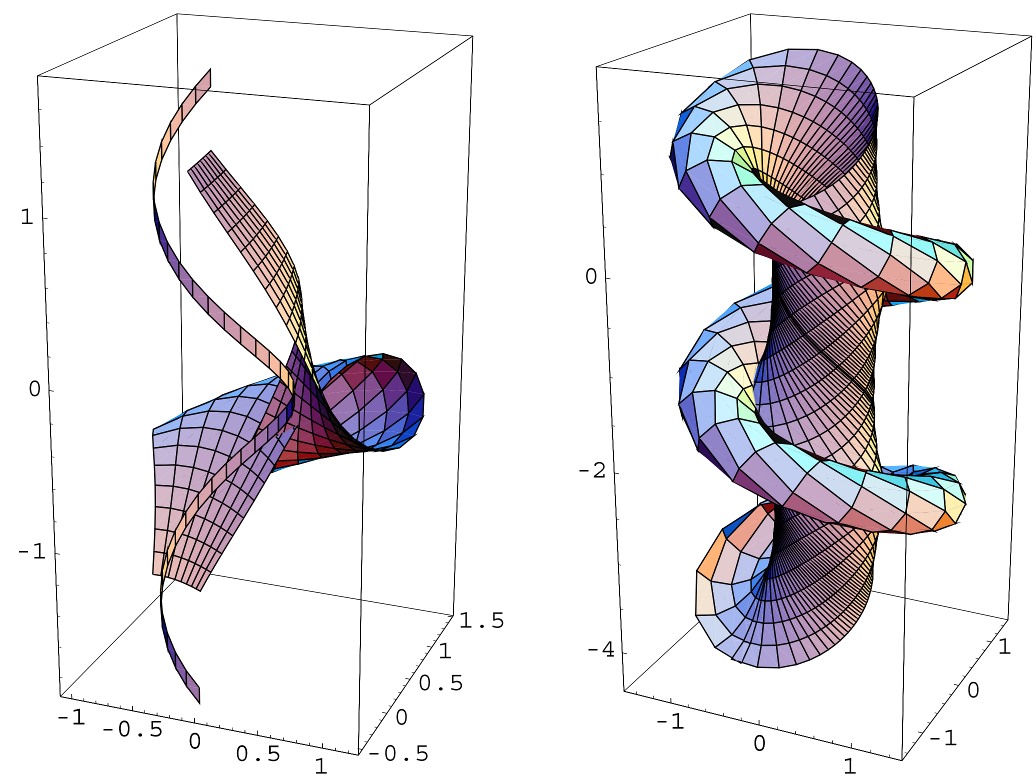

Figure 2: A helix and its surface of circles. In the left-hand picture, the helix is shown, thickened into a vertical strip, along with a small portion of its surface of circles. (The curves running up the surface represent individual Bäcklund transformations of the helix. Perpendicular to these are arcs of circles, which correspond to varying the initial angle for ODE (2).) This surface is extended in the right-hand picture.

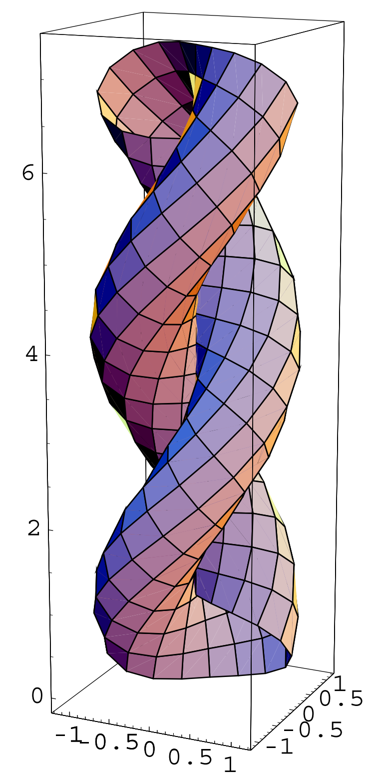

Figure 3: A "fake" Hasimoto surface of the type described in Theorem 2.

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Theorem 3 *Let $M^3(K)$ be the simply-connected space of constant sectional curvature $K \neq 0$, and suppose S is a surface in M carrying orthogonal foliations by helices and by geodesics. Assume that all the helices in S are congruent. Then either*

1. *$K > 0$ and the helices are great circles in S^3, part of a fixed Hopf fibration, and the geodesics have constant torsion $\tau = \pm\sqrt{K}$;*
2. *the geodesics are also helices, and S is a flat cylinder in M^3; or*
3. *the helices degenerate to circles, the geodesics have constant torsion satisfying $\tau^2 > K$, and are Bäcklund transformations of a single constant torsion curve in M.*

In fact, the proof of this theorem leads one to define a Bäcklund transformation for curves of constant torsion in M, albeit with the radii of the circles determined in terms of the torsion, as they are for maximal Bäcklund transformations in $\mathbb{R}^3$. However, it is easy to generalize this to a two-parameter family of Bäcklund transformations, defined as follows. Regard M as the hypersurface of points x satisfying $\langle x, x\rangle = 1/K$ in $\mathbb{R}^4$, using the appropriate signature for the inner product. Let γ be a curve in M of constant torsion τ, parametrized by arclength. Obtain a solution of $d\beta/ds = \lambda \sin\beta - k$ and let

$$\tilde{\gamma}(s) = \gamma(s)\cos\rho + \frac{\sin\rho}{\sqrt{K}}(T\cos\beta + N\sin\beta), \tag{5}$$

where ρ is real when $K > 0$ and pure imaginary when $K < 0$, and is related to λ by

$$\frac{\tan\rho}{\sqrt{K}} = \frac{2\lambda}{\lambda^2 + \tau^2 - K}.$$

Then $\tilde{\gamma}$ is another curve of constant torsion τ, along which s is still an arclength coordinate. This gives the transformations of Theorem 3 if and only if $\lambda^2 + K = \tau^2$. We will discuss (5) further in Section 5.

3 Proof of Theorems 1 and 3

As before, let M^3 be the simply-connected space form of constant curvature K, and let $\mathcal{F}$ be the bundle of oriented orthonormal frames (e_1, e_2, e_3) with basepoint e_0 on M. $\mathcal{F}$ is equipped with differential one-forms ω^i, ω^i_j, defined by

$$de_0 = e_i\omega^i$$
$$de_i = e_j\omega^j_i - Ke_0\omega^i.$$

Here, indices run from 1 to 3, we sum on repeated indices, and e_0 and e_i are vectors in $\mathbb{R}^3$ (or $\mathbb{R}^4$ when $K \neq 0$). These forms on $\mathcal{F}$ satisfy

$$d\omega^i = -\omega^i_j \wedge \omega^j, \quad d\omega^i_j = -\omega^i_k \wedge \omega^k_j.$$

Suppose we have a surface S in M foliated orthogonally by helices and by geodesics. Along the surface we can get a framing with e_1 tangent to the helices and e_2 tangent to the

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geodesics. This gives a lift of S into $\mathcal{F}$, to which ω^3 restricts to be zero, and ω^1 and ω^2 are linearly independent one-forms at each point. If $N = e_2 \cos\theta + e_3 \sin\theta$ is the Frenet normal to the helix, then the Frenet equations imply that, along the lift, $\omega_1^2 \equiv a\cos\theta\omega^1$ and $\omega_1^3 \equiv a\sin\theta\omega^1$ modulo ω^2, where a is the curvature of the helices. On the other hand, the geodesic condition is that $\omega_1^2 \equiv 0$ modulo ω^1. These considerations lead us to define the following exterior differential system on $\mathcal{F} \times S^1$:

$$\mathcal{I} = \begin{cases} \omega^3, \omega_1^2 - a\cos\theta\omega^1 \\ (\omega_1^3 - a\sin\theta\omega^1) \wedge \omega^2 \\ (d\theta + \omega_2^3 - b\omega^1) \wedge \omega^2 \\ \omega_1^3 \wedge \omega^1 + \omega_2^3 \wedge \omega^2 \\ \omega_2^3 \wedge \omega_1^3 - K\omega^1 \wedge \omega^2 + a\sin\theta d\theta \wedge \omega^1 - a^2\cos^2\theta\omega^1 \wedge \omega^2. \end{cases}$$

Here, a and b are constants giving the curvature and torsion of the helices. The last pair of two-forms are the exterior derivatives of the one-forms of $\mathcal{I}$ modulo the preceding forms.

If all the helices on S are congruent, then the aforementioned framing, together with the function θ, will give a lift into $\mathcal{F} \times S^1$ along which the forms of $\mathcal{I}$ restrict to be zero for some choice of a and b. This is known as an *integral surface* of $\mathcal{I}$; the fact that vectors e_1 and e_2 span the tangent space of S implies that $\omega^1 \wedge \omega^2$ restricts to be nonzero at every point of the integral surface.

An application of Cartan's Test [2] shows that system $\mathcal{I}$ is not involutive, so we must prolong. We adjoin new variables k, t, which will represent the curvature and torsion of the geodesics, we pull $\mathcal{I}$ up to $\mathcal{F} \times S^1 \times \mathbb{R}^2$, and adjoin new one-forms. The prolongation is generated by the following 1-forms (along with their exterior derivatives):

$$\hat{\mathcal{I}} = \begin{cases} \omega^3, \omega_1^2 - a\cos\theta\omega^1, \\ \omega_2^3 + t\omega^1 - k\omega^2, \omega_1^3 + t\omega^2 - a\sin\theta\omega^1, \\ d\theta + \omega_2^3 - b\omega^1 - (c + a\sin\theta)\omega^2 \end{cases} \tag{6}$$

where c must satisfy

$$K + a^2 + ac\sin\theta = t^2. \tag{7}$$

The exterior derivatives of forms in the last two rows of (6) give three new two-forms:

$$\begin{cases} dt \wedge \omega^1 - dk \wedge \omega^2 + 2at\cos\theta\omega^1 \wedge \omega^2 \\ dt \wedge \omega^2 + ac\cos\theta\omega^1 \wedge \omega^2 \\ dc \wedge \omega^2 + 2ab\cos\theta\omega^1 \wedge \omega^2. \end{cases} \tag{8}$$

Using the derivative of (7) modulo $\hat{\mathcal{I}}$ to relate dt and dc, we obtain the following linear combination of the last pair of two-forms:

$$a\cos\theta\big(c(b + 3t) - 2ab\sin\theta\big)\omega^1 \wedge \omega^2.$$

Since $\omega^1 \wedge \omega^2 \neq 0$ at each point of the surface, the coefficient here must vanish at each point.

If a vanishes identically, then the helices must be geodesics of M with $t^2 = K$. It is known that each such curve of constant torsion in S^3 is tangent to a contact structure whose planes are orthogonal to the leaves of a Hopf fibration. (Each of these fibrations is congruent to the standard one by the action of $U(2)$.) Furthermore, the curve osculates to the contact planes. This implies, in our case, that the helices are in fact the great circles of the Hopf fibration. In $\mathbb{R}^3$, the helices are parallel lines (since $de_1 = 0$) and the geodesics are copies of a single planar curve.

If $\cos\theta$ vanishes identically, then θ is constant, $t = -b$ is constant, $k = c + a\sin\theta$ is constant, and the surface has Gauss curvature zero. In fact, this surface is a right circular cylinder, generated by parallel geodesics of M perpendicular to congruent circles. (Of course, this will give a flat torus in S^3.)

Suppose then that $a\cos\theta \neq 0$ and

$$c(b+3t) - 2ab\sin\theta = 0 \tag{9}$$

identically. Differentiating this and using the derivative of (7) to eliminate dc gives

$$\big(2t(b+3t) + 3ac\sin\theta\big)dt - a\cos\theta\big(2ab\sin\theta + c(b+3t)\big)d\theta = 0.$$

Wedging with ω^2 gives

$$0 \equiv \big(2t(b+3t) + 3ac\sin\theta\big)(ac\cos\theta)\omega^1\wedge\omega^2 + a\cos\theta\big(2ab\sin\theta + c(b+3t)\big)(b+t)\omega^1\wedge\omega^2$$

modulo $\hat{\mathcal{J}}$. Dividing out by $a\cos\theta$ and using the identities (7) and (9) gives

$$0 = c\big(3(t^2 - K - a^2) + 2(b+t)(b+3t)\big).$$

Then either $c = 0$ or t is constant on integral surfaces. Since the latter would imply—from (8)—that $ac\cos\theta = 0$ anyway, we conclude that $c = 0$ and $b = 0$ (from (9)) identically. The geodesic foliation on the surface consists of curves of constant torsion t satisfying $t^2 = K + a^2$, from (7).

Since $b = 0$, the helices degenerate to circles in $M \subset \mathbb{R}^4$, and the centers of these circles can be normalized to give a curve in M:

$$\xi = \frac{a}{t}e_0 + \frac{1}{t}(e_2\cos\theta + e_3\sin\theta).$$

Now, by a calculation we will omit, one can show that ξ has constant torsion t, and that each constant torsion curve on S is a Bäcklund transformation of ξ, as defined in (5), when $\tau = t$, $\lambda = a$ and $\tan\rho = \sqrt{K}/a$.

4 Proof of Theorem 2

If S satisfies the hypotheses of Theorem 2, there is a framing (e_1, e_2, e_3) which is the Frenet frame for the helix at each point of S, and an angle θ such that

$$V = e_2\cos\theta + e_3\sin\theta$$

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is tangent to the geodesic foliation. Adjoining θ, the curvature a and the torsion b of the helices as auxiliary variables, we see that such a framing gives an integral surface of the following differential forms on $\mathcal{F} \times S^1 \times \mathbb{R}^2$:

$$\begin{cases} \omega^3 \cos\theta - \omega^2 \sin\theta, \\ \omega^3_1 \wedge \eta, (\omega^2_1 - a\omega^1) \wedge \eta, (\omega^3_2 - b\omega^2) \wedge \eta, da \wedge \eta, db \wedge \eta, \\ (\omega^1_2 \cos\theta + \omega^1_3 \sin\theta) \wedge \omega^1. \end{cases}$$

The first line says that e_1 and V are tangent to S, the second line encodes the Frenet equations for the helices, and says that a and b are constant along each helix, and the third line is the geodesic condition; for convenience, we have defined the one-form

$$\eta = \omega^2 \cos\theta + \omega^3 \sin\theta,$$

which is dual to V on the surface.

The point $e_0 + a/(a^2 + b^2)e_2$ will lie on the axis of the helix. The condition that the axes of the helices be fixed amounts to requiring that the derivative of this point be parallel to $be_1 + ae_3$, and this in turn means that the two one-forms

$$\omega^2 + d\big(a/(a^2 + b^2)\big), \quad a\omega^1 - b\omega^3 - \frac{a}{a^2 + b^2}(a\omega^2_1 + b\omega^3_2) \tag{10}$$

vanish on the integral surface. Furthermore, the translational period of a helix is given by $2\pi b/(a^2+b^2)$. If the helices are generated by a common screw motion, this will be constant, and we can set

$$a = C \sin 2\phi, \quad b = 2C\cos^2\phi \tag{11}$$

for some constant C, assumed nonzero, and some function ϕ on the surface.

Thus, if a surface S satisfies the hypotheses of Theorem 2, the specified framing, along with functions ϕ and θ, will give an integral surface of the following exterior differential system on $\mathcal{F} \times S^1 \times S^1$:

$$\begin{cases} \omega^3 \cos\theta - \omega^2 \sin\theta, \omega^2 + \frac{1}{2C}\sec^2\phi d\phi, \\ b\omega^3 + \dfrac{a}{a^2 + b^2}\big(a(\omega^2_1 - a\omega^1) + b(\omega^3_2 - b\omega^1)\big), \\ (\omega^1_2 \cos\theta + \omega^1_3 \sin\theta) \wedge \omega^1, \omega^3_1 \wedge \eta, \\ (\omega^2_1 - a\omega^1) \wedge \eta, (\omega^3_2 - b\omega^2) \wedge \eta, \\ (\omega^2_1 \sin\theta - \omega^3_1 \cos\theta) \wedge \omega^1 - (\omega^3_2 + d\theta) \wedge \eta, (\omega^2_1 + b\omega^3) \wedge \omega^1. \end{cases} \tag{12}$$

Here, the second and third one-forms come from (10) with (11) taken into account; the last pair of two-forms are the exterior derivatives of the first pair of one-forms, modulo the preceding forms.

Of course, it is necessary to adjoin the derivative of the third one-form in (12), but first we will derive some geometric consequences of the system as it stands. Let

$$\pi_1 = \omega^2_1 - a\omega^1 + b\omega^3;$$

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then two of the two-forms of the system are $\pi_1 \wedge \omega^1$ and $\pi_1 \wedge \eta$. Since the one-forms ω^1 and η must be linearly independent on any integral surface that comes from a surface in $\mathbb{R}^3$, we see that the form π_1 must vanish on any such integral surface. (Requiring that $\omega^1 \wedge \eta \neq 0$ at each point of the integral surface is called an *independence condition*.) Modulo π_1, the first pair of two-forms in (12) are congruent to

$$(\omega^1_3 \sin\theta + b\omega^3 \cos\theta) \wedge \omega^1, \quad \omega^1_3 \wedge \eta$$

so it follows that

$$\pi_2 = \omega^1_3 + b\omega^2$$

must be zero at every point of an integral surface satisfying the independence condition and $\sin\theta \neq 0$. (If $\sin\theta$ vanishes everywhere, then $V = \pm e_2$, the normal to the helices, and ω^1_2 and ω^3_2 are congruent to zero modulo ω^1. Thus, V is constant along the geodesics, the geodesics are straight lines through the axis of the screw motion, and the surface is a helicoid.) Finally, the third one-form in (12) can be expressed as

$$\pi_1 \sin^2\phi + \pi_3 \sin\phi\cos\phi$$

where

$$\pi_3 = \omega^3_2 - b\omega^1 + b^2/a\omega^3.$$

So, π_3 must vanish on surfaces that carry nondegenerate helices.

A geometric interpretation of the vanishing of π_1 and π_2 is this: the unit vector $e_1\cos\phi + e_3\sin\phi$ is parallel to the axis of the helices. Modulo the forms in (12), the derivative of this vector is

$$(e_1\sin\phi - e_3\cos\phi)\pi_2 + e_2(\pi_1\cos\phi - \pi_3\sin\phi).$$

So, it is not surprising that π_1, π_2, π_3 must vanish.

Notice that all of the two-forms in (12) are exterior multiples of the new one-forms π_1, π_2, π_3, except for the second to last two-form, which is congruent to $-d\theta \wedge \eta$ modulo the π's. With the new one-forms adjoined, the system is now generated by

$$\text{(13)} \qquad \begin{cases} \omega^3\cos\theta - \omega^2\sin\theta, \omega^2 + \frac{1}{2C}\sec^2\phi d\phi, \\ \pi_1, \pi_2, \pi_3 \\ d\theta \wedge \eta. \end{cases}$$

It is easily checked that this system is differentially closed and involutive, with Cartan character $s_1 = 1, s_2 = 0$. Thus, local solutions depend on one arbitrary function of one variable. To see, in a less abstract way, how one arbitrary function comes in to the picture, notice that the one-forms of (13) imply an ordinary differential equation for ϕ as a function of arclength s along the geodesics,

$$d\phi/ds = -2C\cos\theta\cos^2\phi,$$

but there is no corresponding ODE for θ. In fact, θ can be arbitrarily specified as a function of s along one geodesic. Since the curvature and torsion of that geodesic satisfy

$$\kappa = \frac{d\theta}{ds} - \frac{b^2}{a}\sin\theta, \quad \tau = -b,$$

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this roughly corresponds to specifying one of κ or τ. While it is difficult, in general, to explicitly solve the Frenet equations (obtaining a curve with a given κ and τ), in this instance we can solve simple ODEs for the curve in cylindrical coordinates. These are given in (4), where our function θ is re-labeled as α.

The involutivity of (13) implies that the space of "fake" Hasimoto surfaces is quite large, even with the extra assumptions made in Theorem 2. If one relaxes these assumptions, the space of solutions becomes larger. If the helices have a common translational period but are only required to have *parallel* axes, the surfaces depend on two functions of one variable. If only a common translational period is required, the surfaces depend on five functions of one variable.

5 Bäcklund Transformations for Surfaces in S^3 or H^3

The existence of a family of Bäcklund transformations is a hallmark of complete integrability. In fact, for AKNS systems (NLS, sine-Gordon, mKdV) the Bäcklund transformation is intimately related to the Lax pair [11]. Thus, our family (5) of transformations for constant torsion curves in $M = H^3$ or S^3 ought to be related to a completely integrable PDE.[1] It turns out that it is connected to the sine-Gordon equation in a way similar to our transformation (3) for curves in $\mathbb{R}^3$.

The generalization of Bäcklund's transformation to surfaces in S^3 and H^3 was discovered by Bianchi [1]. (Generalizations to higher dimensions were obtained by Tenenblat and Terng [12], [13].) To explain the connection with (5), we will briefly recapitulate this generalization.

Let M^3 be as before. A *linear Weingarten surface* in M is one whose Gauss curvature G and mean curvature H satisfy an equation of the form

$$A(G - K) + BH + C = 0, \tag{14}$$

where A, B, C are constants. The surface is hyperbolic if $B^2 - 4AC < 0$. In this case, one can show that every surface satisfying (14) gives rise to a solution of the sine-Gordon equation, in the form

$$\theta_{xt} = c_1 \sin\theta + c_2 \cos\theta,$$

where x and t are arclength coordinates along the asymptotic directions, θ is the angle between, and c_1, c_2 are constants depending on A, B, C, K.

We will call a hyperbolic linear Weingarten equation *pseudospherical* if $B = 0$; this just means that $G - K$ is a negative constant. The asymptotic lines on a pseudospherical surface have constant torsion τ satisfying $\tau^2 = K - G$. Moreover, such surfaces have a Bäcklund transformation that can be expressed geometrically using geodesic congruences in M^3—*i.e.*, two-parameter families of geodesics of M^3. By imitating [4], we can derive the Bäcklund transformation from geodesic congruences, as follows:

Suppose two surfaces $S, \bar{S}$ in M are related by a line congruence

$$\bar{x} = x\cos\phi + \frac{\sin\phi}{\sqrt{K}} V \tag{15}$$

[1] If one is unable to guess the PDE, it is still possible to recover it from the Bäcklund transformation. For example, one can expand (5) as a Laurent series in λ, yielding a hierarchy of commuting flows that preserve constant torsion. This technique will be discussed in a future paper.

such that ϕ is constant, V is a unit vector in $T_x S$, and the surface normals differ by a constant angle ψ. (The line from x to $\bar{x}$ is also assumed to be tangent to $\bar{S}$.) Then S and $\bar{S}$ have constant Gauss curvature $K(1 - \sin^2\psi/\sin^2\phi)$, less than K. (Again, ϕ is pure imaginary when $K < 0$.) The line congruence gives a Bäcklund transformation for such surfaces, and this transformation takes asymptotic lines to asymptotic lines. When restricted to a single asymptotic line, with constant torsion τ and Frenet frame T, N, B, this transformation takes the form (15) with $V = T\cos\beta + N\sin\beta$, where β satisfies $d\beta/ds = \lambda\sin\beta - \kappa$ along the curve, and

$$\frac{\tan\phi}{\sqrt{K}} = \frac{2\lambda}{\lambda^2 + \tau^2 - K}.$$

Thus we see that (5) is the restriction of the surface Bäcklund transformation to asymptotic lines.

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