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A Berry-Esseen Type Theorem on Nilpotent Covering Graphs

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Abstract. We prove an estimate for the speed of convergence of the transition probability for a symmetric random walk on a nilpotent covering graph. To obtain this estimate, we give a complete proof of the Gaussian bound for the gradient of the Markov kernel.

1 Introduction

Let $X = (V, E)$ be a locally finite connected graph, V being the set of vertices and E being the set of oriented edges. For $e \in E$, the origin and the end of e are denoted by $o(e)$ and $t(e)$, respectively, and the inverse edge is denoted by $\bar{e}$. We suppose that X is a *nilpotent covering graph*, namely a covering of a finite graph X_0 whose covering transformation group Γ is a finitely generated nilpotent group. Furthermore, we assume that Γ is torsion free.

A *symmetric random walk* on X with a weight $m\colon V \longrightarrow \mathbb{R}_{>0}$ is given by a positive valued function p on E satisfying $\sum_{e \in E_x} p(e) = 1$ and $p(e)m(o(e)) = p(\bar{e})m(t(e))$, where $E_x = \{e \in E \mid o(e) = x\}$. We assume that m and p are Γ-invariant. We consider $p(e)$ the probability that a particle placed at $o(e)$ moves to the terminus $t(e)$ along the edge e in one unit time. The transition probability that a particle starting at x reaches y at time n is given by

$$p_n(x, y) = \sum_{c=(e_1,e_2,\dots,e_n)} p(e_1)p(e_2)\cdots p(e_n),$$

where the sum is taken over all path $c = (e_1, e_2, \dots, e_n)$ of length n whose origin $o(c) = x$ and terminus $t(c) = y$. The transition operator L associated with the random walk is the operator acting on functions on V defined by

$$Lf(x) = \sum_{e \in E_x} f\big(t(e)\big)\, p(e).$$

It is easy to check that the function $k_n(x, y) = p_n(x, y)m(y)^{-1}$ is the kernel function of L^n, namely $L^n f(x) = \sum_{y \in V} k_n(x, y) f(y)m(y)$. The hypothesis of m and p implies $k_n(x, y) = k_n(y, x)$.

By a theorem of A. I. Mal'cev [11], there exists a connected and simply connected nilpotent Lie group G_Γ such that Γ is a cocompact lattice in G_Γ (see also M. S. Raghunathan [13]). The purpose of this article is to prove a Berry-Esseen type theorem, an

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estimate for the speed of convergence of the transition probability to the heat kernel corresponding to a sub-Laplacian on G_Γ as n goes to infinity. We remark that G. Alexopoulos proved a Berry-Esseen type theorem for convolution powers on a discrete group of polynomial growth Γ ([1]). To explain, let μ be a symmetric probability measure on Γ such that its support is finite and generates Γ with $\mu(e) > 0$. Then the transition probability p_n is defined by $p_n(x, y) = \mu^{*n}(y^{-1}x)$ $(x, y \in \Gamma)$. Let h_t be the heat kernel of the limit operator associated to μ on the nilpotent Lie group G_Γ (see [1]). Then,

Theorem ([1, Theorem 10]) *Let Γ have polynomial volume growth of order D. Then, there exists a constant $C > 0$ such that*

$$\sup_{x,y\in\Gamma} \left| p_n(x, y) - |G_\Gamma/\Gamma| h_n(x, y)\right| \leq Cn^{-\frac{D+1}{2}}.$$

On the other hand, when X is a *crystal lattice*, that is, a covering graph whose covering transformation group Γ is abelian, a local central limit theorem is proved by M. Kotani and T. Sunada [10]. In that case, the notion of *harmonic realization* from X to the abelian group $\Gamma \otimes \mathbb{R}$ is closely related to the asymptotics (see [10, 9]). We also remark that, as a convergence of a transition operator, an operator-theoretic central limit theorem on a nilpotent covering graph is obtained in [6]. Furthermore, a central limit theorem for magnetic schrödinger operator on a crystal lattice is proved by M. Kotani [7].

Our strategy for the proof of a Berry-Esseen type theorem on a nilpotent covering graph is much inspired by G. Alexopoulos [1]. Before describing our results, we will introduce some notations. Let $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$ be subspaces of the Lie algebra of G_Γ (see Section 2). We assume that $\Phi\colon X \to G_\Gamma$ is a Γ-equivariant map satisfying

$$\sum_{e\in E_x} p(e) \exp^{-1} \Phi\big(o(e)\big)^{-1}\Phi\big(t(e)\big)\,|_{\mathfrak{g}^{(1)}} = 0 \quad (x \in V).$$

This condition on Φ is equivalent to $\exp^{-1}\Phi|_{\mathfrak{g}^{(1)}} : X \to \mathfrak{g}^{(1)}$ is a harmonic realization (see [6]). Let p_n be the transition probability on X and h_t the heat kernel of the sub-Laplacian Ω for the Albanese metric (see [6, 9]) which is defined by

$$\Omega = -\frac{1}{2m(X_0)} \sum_{e\in E_0} m(e) X_e^2,$$

where $m(e) = p(e)m(o(e))$ and X_e is a left invariant vector field identified with $\exp^{-1}\Phi(o(e))\Phi(t(e))|_{\mathfrak{g}^{(1)}}$. Then we have

Theorem 1 (Berry-Esseen type theorem) *Let X be a nilpotent covering graph whose covering transformation group is Γ. The order of polynomial growth of Γ is denoted by D. Then, for any $0 < \epsilon < 1/2$, there exists a constant $C_\epsilon > 0$ such that*

1. *if X is a non-bipartite graph, then*

$$\sup_{x,y\in V}\left| p_n(x, y)m(y)^{-1} - \frac{|G_\Gamma/\Gamma|}{m(X_0)} h_n\big(\Phi(x), \Phi(y)\big)\right| \leq C_\epsilon n^{-\frac{D+1/2-\epsilon}{2}}.$$

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2. *If X is a bipartite graph with a bipartition $V = A \coprod B$, and*
   (a) *if $x, y \in A$ or $x, y \in B$, then $p_n(x, y) = 0$ for odd n and*

$$\sup_{x,y}\Big| p_n(x, y)m(y)^{-1} - 2\frac{|G_\Gamma/\Gamma|}{m(X_0)}h_n\big(\Phi(x), \Phi(y)\big)\Big| \le C_\epsilon n^{-\frac{D+1/2-\epsilon}{2}}$$

   for even n;
   (b) *if $x \in A$, $y \in B$ or $x \in B$, $y \in A$, then $p_n(x, y) = 0$ for even n and*

$$\sup_{x,y}\Big| p_n(x, y)m(y)^{-1} - 2\frac{|G_\Gamma/\Gamma|}{m(X_0)}h_n\big(\Phi(x), \Phi(y)\big)\Big| \le C_\epsilon n^{-\frac{D+1/2-\epsilon}{2}}$$

   for odd n.

In our approach, we have not been able to improve the speed of this convergence more than $C_\epsilon n^{-\frac{D+1/2-\epsilon}{2}}$, in general. However, if

(1) $\quad \sum_{e\in E_x} p(e) \exp^{-1} \Phi\big(o(e)\big)^{-1}\Phi\big(t(e)\big)\Big|_{\mathfrak{g}^{(2)}} = 0 \quad (x \in V)$

and

(2) the second order differential operator on G_Γ $\sum_{e\in E_x} p(e)X_e^2$ is independent of the choice of $x \in V$,

then the speed of convergence is estimated by $Cn^{-\frac{D+1}{2}}$ for each case. Indeed, a simple random walk on a Cayley graph of Γ satisfies (1) and (2). Triangular lattice and hexagonal lattice (see [10]) also satisfy these conditions. However, there exist graphs which do not satisfy them. For example, Kagome lattice (see [10]) does not satisfy (2).

In the proof of Theorem 1, we use Gaussian upper bounds for the kernel function k_n of L^n and its gradient on a nilpotent covering graph. The definition of a gradient of k_n is given as follows:

1. if X is a non-bipartite graph,

$$\nabla^y k_n(x, y) = \sup_{d_X(y,z)=1} |k_n(x, z) - k_n(x, y)|.$$

2. If X is a bipartite graph,

$$\nabla^y k_n(x, y) = \sup_{d_X(y,z)=2} |k_n(x, z) - k_n(x, y)|,$$

where $d_X(x, y)$ is the length of the shortest path from x to y. We note that W. Hebisch and L. Saloff-Coste gave Gaussian bounds for k_n and ∇k_n on a Cayley graph of Γ in [5]. Furthermore, if the growth rate of a graph is $V(n) \sim n^D$, then L. Saloff-Coste showed $k_{2n}(x, x) < Cn^{-D/2}$ in [14]. After that, C. Pittet and L. Saloff-Coste proved that the long run behavior of the probability of return to the beginning after $2n$-steps is left invariant by *quasi-isometry* in [12]. Since a nilpotent covering graph

X has polynomial growth and X is quasi-isometric to its transformation group Γ, the Gaussian upper bound for k_n is deduced:

Theorem ([14, 12], cf. [5]) *Let X be a non-bipartite graph. Then there exist two constants C and $C' > 0$ such that*

$$k_n(x, y) \leq Cn^{-\frac{D}{2}} e^{-d_X(x,y)^2/C'n} \tag{3}$$

for all $x, y \in V$, and all $n = 1, 2, \ldots$.

In this paper, for the sake of completeness, we give a proof of Gaussian bound for ∇k_n on X by following the argument by W. Hebisch and L. Saloff-Coste [5] in which the symmetry $\mu^{*n}(x) = \mu^{*n}(x^{-1})$ for a probability measure μ on Γ plays a crucial role. In our case, instead of this symmetry, we use an invariance for the action of Γ and a symmetry of k_n, namely $k_n(\gamma x, \gamma y) = k_n(x, y)$ and $k_n(x, y) = k_n(y, x)$, respectively. Then we have

Theorem 2 (Cf. [5]) *There exist two constants C and $C' > 0$ such that*

1. *if X is a non-bipartite graph,*

$$\nabla^y k_n(x, y) \leq Cn^{-\frac{D+1}{2}} e^{-d_X(x,y)^2/C'n} \tag{4}$$

   for all $x, y \in V$, and all $n = 1, 2, \ldots$.
2. *If X is a bipartite graph with a bipartition $V = A \coprod B$, and*
   (a) *if $x, y \in A$ or $x, y \in B$, then $k_n(x, y) = 0$ for odd n and*

$$\nabla^y k_n(x, y) \leq Cn^{-\frac{D+1}{2}} e^{-d_X(x,y)^2/C'n}$$

   for even n,

   (b) *if $x \in A$, $y \in B$ or $x \in B$, $y \in A$, then $k_n(x, y) = 0$ for even n and*

$$\nabla^y k_n(x, y) \leq Cn^{-\frac{D+1}{2}} e^{-d_X(x,y)^2/C'n}$$

   for odd n.

We note that various applications of these estimates have been discussed (for instance, see [2, 3, 4, 16, 18]).

Throughout this article, different constants may be denoted by the same letter C. When their dependence or independence is significant, it will be clearly stated.

2 Berry-Esseen Type Theorem

As we already mentioned in the introduction, G. Alexopoulos proved a Berry-Esseen type theorem for convolution powers on a discrete group of polynomial growth [1]. In that proof, the following three results play a crucial role:

R1 An estimate established in [1, Corollary 7].

R2 Gaussian bounds for the heat kernel on a nilpotent Lie group (N. Th. Varopoulos [17, Theorem IV.4.2]) .

R3 Gaussian bounds for the convolution powers on a discrete group of polynomial growth (W. Hebisch, L. Saloff-Coste [5, Theorem 5.1]).

Hence we will consider an analogue of these results on a nilpotent covering graph.

Let $\mathfrak{g}$ be the Lie algebra of G_Γ and $\exp\colon \mathfrak{g} \to G_\Gamma$ the exponential map. We set $n_1 = \mathfrak{g}$ and $n_{i+1} = [\mathfrak{g}, n_i]$ for $i \geq 1$. Since $\mathfrak{g}$ is nilpotent, we have the filtration:

$$\mathfrak{g} = n_1 \supset n_2 \supset \cdots \supset n_r \neq \{0\} \supset n_{r+1} = \{0\}.$$

We consider subspaces $\mathfrak{g}^{(1)}, \dots, \mathfrak{g}^{(r)} \subset \mathfrak{g}$ such that

$$n_k = \mathfrak{g}^{(k)} \oplus n_{k+1}. \tag{5}$$

Let $\{X_1^{(k)}, X_2^{(k)}, \dots, X_{d_k}^{(k)}\}$ be a basis of $\mathfrak{g}^{(k)}$. Then we have an identification of G_Γ with $\mathbb{R}^n$ as differential manifold given by

$$(x_{d_r}^{(r)}, x_{d_r-1}^{(r)}, \dots, x_1^{(1)}) \mapsto \exp x_{d_r}^{(r)} X_{d_r}^{(r)} \cdot \exp x_{d_r-1}^{(r)} X_{d_r-1}^{(r)} \cdots \exp x_1^{(1)} X_1^{(1)},$$

which is called the *canonical coordinates of the second kind* (see [1, 13]). For $x \in G_\Gamma$, we denote $P_i^{(k)}(x) = x_i^{(k)}$. We define $(i_1, k_1) > (i_2, k_2)$ if $k_1 > k_2$ or $k_1 = k_2$, $i_1 > i_2$. By the Campbell-Hausdorff formula, we remark that

$$\begin{aligned} P_i^{(1)}(xy) &= P_i^{(1)}(x) + P_i^{(1)}(y), \\ P_i^{(2)}(xy) &= P_i^{(2)}(x) + P_i^{(2)}(y) + \sum_{i_1 < i_2} [X_{i_1}^{(1)}, X_{i_2}^{(1)}]|_{X_i^{(2)}} P_{i_1}^{(1)}(x) P_{i_1}^{(1)}(y) \end{aligned}$$

and for $k \geq 3$,

$$P_i^{(k)}(xy) = P_i^{(k)}(x) + P_i^{(k)}(y) + \sum_{|K_1|+|K_2| \leq k} C_{K_1 K_2} [X^{K_1}, X^{K_2}]|_{X_i^{(k)}} P^{K_1}(x) P^{K_2}(y),$$

where K_1 and K_2 are multi-indices (see [6]).

Let h_t be the heat kernel of a sub-Laplacian on a nilpotent Lie group G_Γ. Then we can use the following same result as **R2**:

Theorem ([17, Theorem IV.4.2]) *Let $|K| = k_1 + k_2 + \cdots + k_\ell$. Then*

$$\left| \partial_t^s X_{i_1}^{(k_1)} X_{i_2}^{(k_2)} \cdots X_{i_\ell}^{(k_\ell)} h_t(g_1, g_2) \right| \leq C t^{\frac{D+2s+|K|}{2}} \exp(-d(g_1, g_2)^2 / c' t), \tag{6}$$

where $d(g_1, g_2)$ is a Carnot-Carathèodory distance on G_Γ (see [17]).

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We will show **R3** on a nilpotent covering graph in the next section. Now we try to create **R1** in our case.

For $u \in C^\infty(\mathbb{R}_{\geq 0} \times G_\Gamma)$, let $\partial_N u(t, \Phi(x)) = u(t+N, \Phi(x)) - u(t, \Phi(x))$ and $\Phi^* u(t,x) = u(t, \Phi(x))$. We denote

$$C_{x,n} = \{(e_1, e_2, \ldots, e_n) \mid e_i \in E, o(e_1) = x, t(e_i) = o(e_{i+1})\}$$

and $t(c) = t(e_n)$ for $c = (e_1, e_2, \ldots, e_n) \in C_{x,n}$. As an analogue of **R1**, we have

Lemma 2.1 (Cf. [1, Corollary 7], [6, Lemma 2.2], [7, Theorem 3]) *For any $J \geq 4$, there exists a constant $C_J > 0$ such that*

$$\begin{aligned}(7)\quad &\Big|\big(\partial_N + (I - L^N)\big)\, \Phi^* u(t,x) - N\,(\partial_t + \Omega)\, u\big(t, \Phi(x)\big)\Big| \\ &\leq C_J \sup_{\theta \in [0,1],\, g \in U_N} \Big(N^2 \Big|\frac{\partial^2}{\partial t^2} u\big(t + \theta N, \Phi(x)\big)\Big| + X^2 u\big(t, \Phi(x)\big) \\ &\qquad + \sum_{j=3}^{J-1} N^{j-1} X^j u\big(t, \Phi(x)\big) + \sum_{k=J}^{Jr} N^k X^k u(t, \Phi(x) g)\Big),\end{aligned}$$

where

$$X^k u\big(t, \Phi(x)\big) = \sum_{\ell=1}^{k} \sum_{k_1 + k_2 + \cdots + k_\ell = k} \Big| X_{i_1}^{(k_1)} X_{i_2}^{(k_2)} \cdots X_{i_\ell}^{(k_\ell)} u\big(t, \Phi(x)\big)\Big|$$

and U_N is a set of all $g \in G_\Gamma$ satisfying that there exists $c \in C_{x,N}$ such that

$$\big|P_i^{(k)}(g)\big| \leq \big|P_i^{(k)}\big(\Phi(x)^{-1}\Phi(t(c))\big)\big| \qquad \textit{for all } (i,k).$$

Proof Let $u'(t,g) = u(t, \Phi(x)g)$. By Taylor's formula with respect to the canonical coordinates of the second kind, there exist $\theta \in [0,1]$ and $g_c \in U_N$ such that

$$\begin{aligned}\big(\partial_N + (I - L^N)\big)\, \Phi^* u(t,x) &= N \frac{\partial u}{\partial t}\big(t, \Phi(x)\big) + \frac{N^2}{2} \frac{\partial^2 u}{\partial t^2}\big(t + \theta N, \Phi(x)\big) \\ &+ \sum_{c \in C_{x,N}} p(c) \Big\{ -\frac{\partial u'}{\partial x_i^{(k)}}(t,e) P_i^{(k)}\big(\Phi(x)^{-1}\Phi\big(t(c)\big)\big) \\ &\quad - \frac{1}{2} \frac{\partial^2 u'}{\partial x_{i_1}^{(k_1)} \partial x_{i_2}^{(k_2)}}(t,e) P_{i_1}^{(k_1)}\big(\Phi(x)^{-1}\Phi\big(t(c)\big)\big) P_{i_2}^{(k_2)}\big(\Phi(x)^{-1}\Phi\big(t(c)\big)\big) \\ &\quad - \sum_{j=3}^{J-1} \frac{1}{j!} \frac{\partial^j u'}{\partial x_{i_1}^{(k_1)} \partial x_{i_2}^{(k_2)} \cdots \partial x_{i_j}^{(k_j)}}(t,e) P_{i_1}^{(k_1)}\big(\Phi(x)^{-1}\Phi\big(t(c)\big)\big) \\ &\qquad \times P_{i_2}^{(k_2)}\big(\Phi(x)^{-1}\Phi\big(t(c)\big)\big) \cdots P_{i_j}^{(k_j)}\big(\Phi(x)^{-1}\Phi\big(t(c)\big)\big)\end{aligned}$$

$$- \frac{1}{J!} \frac{\partial^J u'}{\partial x_{i_1}^{(k_1)} \partial x_{i_2}^{(k_2)} \cdots \partial x_{i_J}^{(k_J)}} (t, g_c) P_{i_1}^{(k_1)} \big(\Phi(x)^{-1} \Phi\big(t(c)\big) \big)$$
$$\times P_{i_2}^{(k_2)} \big(\Phi(x)^{-1} \Phi\big(t(c)\big) \big) \cdots P_{i_J}^{(k_J)} \big(\Phi(x)^{-1} \Phi\big(t(c)\big) \big) \Big\} .$$

We observe now that

$$\frac{\partial u'}{\partial x_i^{(k)}}(t, e) = X_i^{(k)} u\big(t, \Phi(x)\big),$$
$$\frac{\partial^2 u'}{\partial x_{i_1}^{(k_1)} \partial x_{i_2}^{(k_2)}}(t, e) = X_{i_1}^{(k_1)} X_{i_2}^{(k_2)} u\big(t, \Phi(x)\big) \quad (i_1, k_1) \geq (i_2, k_2).$$

Hence we have

$$\big(\partial_N + (I - L^N) \big) \Phi^* u(t, x) = N \frac{\partial u}{\partial t}\big(t, \Phi(x)\big) + \frac{N^2}{2} \frac{\partial^2 u}{\partial t^2}\big(t + \theta N, \Phi(x)\big)$$
$$- \sum_{(i,k)} X_i^{(k)} u\big(t, \Phi(x)\big) \sum_{c \in C_{x,N}} p(c) P_i^{(k)} \big(\Phi(x)^{-1} \Phi\big(t(c)\big) \big)$$
$$- \frac{1}{2} \Big(\sum_{(i_1,k_1) \geq (i_2,k_2)} X_{i_1}^{(k_1)} X_{i_2}^{(k_2)} + \sum_{(i_2,k_2) > (i_1,k_1)} X_{i_2}^{(k_2)} X_{i_1}^{(k_1)} \Big) u\big(t, \Phi(x)\big)$$
$$\times \sum_{c \in C_{x,N}} p(c) P_{i_1}^{(k_1)} \big(\Phi(x)^{-1} \Phi\big(t(c)\big) \big) P_{i_2}^{(k_2)} \big(\Phi(x)^{-1} \Phi\big(t(c)\big) \big)$$
$$- \sum_{j=3}^{J-1} \frac{1}{j!} \frac{\partial^j u'}{\partial x_{i_1}^{(k_1)} \partial x_{i_2}^{(k_2)} \cdots \partial x_{i_j}^{(k_j)}} (t, e) \sum_{c \in C_{x,N}} p(c) P_{i_1}^{(k_1)} \big(\Phi(x)^{-1} \Phi\big(t(c)\big) \big)$$
$$\times P_{i_2}^{(k_2)} \big(\Phi(x)^{-1} \Phi\big(t(c)\big) big \cdots P_{i_j}^{(k_j)} \big(\Phi(x)^{-1} \Phi\big(t(c)\big) \big)$$
$$- \frac{1}{J!} \sum_{c \in C_{x,N}} p(c) \frac{\partial^J u'}{\partial x_{i_1}^{(k_1)} \partial x_{i_2}^{(k_2)} \cdots \partial x_{i_J}^{(k_J)}} (t, g_c) P_{i_1}^{(k_1)} \big(\Phi(x)^{-1} \Phi\big(t(c)\big) \big)$$
$$\times P_{i_2}^{(k_2)} \big(\Phi(x)^{-1} \Phi\big(t(c)\big) \big) \cdots P_{i_J}^{(k_J)} \big(\Phi(x)^{-1} \Phi\big(t(c)\big) \big) .$$

From the harmonicity of Φ,

$$\sum_{c \in C_{x,N}} p(c) P_i^{(1)} \big(\Phi(x)^{-1} \Phi\big(t(c)\big) \big) = 0.$$

By using the ergodicity (see [6, 7]) and the harmonicity of Φ, there exists $C > 0$ independent of N such that

$$(8) \quad \Big| X_i^{(2)} u(t, \Phi(x)) \sum_{k=0}^{N-1} \sum_{c \in C_{x,k}} p(c) \sum_{e \in E_{t(c)}} p(e) \exp^{-1} \Phi(o(e))^{-1} \Phi(t(e)) \big|_{X_i^{(2)}} \Big|$$
$$\leq C X^2 u(t, \Phi(x))$$

and

$$(9)\quad \Big| -\frac{1}{2}\sum_{i_1,i_2\le d_1}\Big\{X^{(1)}_{i_1}X^{(1)}_{i_2}u(t,\Phi(x))\sum_{k=0}^{N-1}\sum_{c\in C_{x,k}}p(c)\sum_{e\in E_{t(c)}}p(e)$$

$$\times \exp^{-1}\Phi(o(e))^{-1}\Phi(t(e))\big|_{X^{(1)}_{i_1}}\exp^{-1}\Phi(o(e))^{-1}\Phi(t(e))\big|_{X^{(1)}_{i_2}}\Big\} - N\Omega f(\Phi(x))\Big|$$

$$\le CX^2u(t,\Phi(x)).$$

By the harmonicity of Φ and the definition of $P_i^{(k)}$ (see also [6]), we have

$$\sum_{c\in C_{x,N}} p(c)P^{(k_1)}_{i_1}\big(\Phi(x)^{-1}\Phi(t(c))\big)\cdots P^{(k_j)}_{i_j}\big(\Phi(x)^{-1}\Phi(t(c))\big) \le CN^{|K|-1},$$

where $|K| = k_1 + k_2 + \cdots + k_j$. Since $g_c \in U_N$, there exists a constant $C'_J > 0$ such that

$$\Big|\frac{\partial^J u'}{\partial x^{(k_1)}_{i_1}\partial x^{(k_2)}_{i_2}\cdots\partial x^{(k_J)}_{i_J}}(t,g_c)\Big| \le C'_J\sum_{k\ge k_1+k_2+\cdots+k_J}^{Jr} N^{k-k_1-k_2\cdots-k_J}X^k u(t,\Phi(x)g_c).$$

Hence the lemma follows. ∎

Remark 2.2 If (1) and (2) are satisfied, then (8) and (9) are zero, so that $X^2u(t,\Phi(x))$ vanishes in (7).

For the proof of Theorem 1, we introduce some notations. We define

$$S_t(x,y) = \frac{|G_\Gamma/\Gamma|}{m(X_0)}h_t\big(\Phi(x),\Phi(y)\big) \quad (x,y\in V),$$

$$S'_t(x,y) = \frac{1}{m(X_0)}\int_F h_t\big(\Phi(x)\eta,\Phi(y)\big)\,d\eta \quad (x,y\in V),$$

where F is a fundamental domain in G_Γ for the action of Γ. We shall denote

$$k\cdot S(x,y) = \sum_{z\in V}k(x,z)S(z,y)m(z).$$

Let us also denote, for $T\ge 0$,

$$\delta(n) = \sup_{x,y\in V}|k_n(x,y) - S_n(x,y)|,$$

$$\delta_T(n) = \sup_{x,y\in V}|(k_n - S_n)\cdot S'_T(x,y)|.$$

By using Gaussian bounds for k_n, ∇k_n (Theorem 2) and h_t ([17]), we have

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Lemma 2.3 (*Cf.* [1, Lemma 11], [15, Lemma 1]) *Assume that X is a non-bipartite graph. Then, there are constants $\alpha, \beta \geq 0$ independent of n and T such that*

$$\delta(n) \leq \alpha\delta_T(n) + \beta\sqrt{T}n^{-\frac{D+1}{2}}.$$

As an analogue of [1, Proposition 12], we have

Lemma 2.4 *Assume that X is a non-bipartite graph. Let $q > 0$ and $J \geq 4$. If there exists a constant $A > 0$ such that*

$$\delta(i) \leq Ai^{-\frac{D+q}{2}} \tag{10}$$

for all $i = 1, 2, \ldots, n-1$, then there exists a constant $C_J > 0$ such that

$$\begin{aligned}\delta(n) \leq C_J\Big(& n^{-\frac{D+1}{2}} + N^{-1}n^{-\frac{D}{2}} + \sum_{j=3}^{J-1} N^{j-2}n^{-\frac{D+j-2}{2}} + \sum_{k=J}^{Jr} N^{k-1}n^{-\frac{D+k-2}{2}} \\ & + \sum_{j=3}^{J-1} N^{j-1}n^{-\frac{D+j}{2}} + \sum_{k=J}^{Jr} N^{k}n^{-\frac{D+k}{2}} + T^{\frac{1}{2}}n^{-\frac{D+1}{2}} \\ & + An^{-\frac{D+q}{2}}\Big[N^{-1}\log(n+T) + \sum_{j=3}^{J-1} N^{j-2}T^{-\frac{j-2}{2}} + \sum_{k=J}^{Jr} N^{k-1}T^{-\frac{k-2}{2}}\exp\Big(\frac{N^2}{c'T}\Big) \\ & + \sum_{j=3}^{J-1} N^{j-1}T^{-\frac{j}{2}} + \sum_{k=J}^{Jr} N^{k}T^{-\frac{k}{2}}\exp\Big(\frac{N^2}{c'T}\Big)\Big]\Big)\end{aligned}$$

for sufficiently smaller $N \in \mathbb{N}$ than n and $T \in \mathbb{N}$.

Proof By the previous lemma, we will consider $\delta_T(n)$. First we prove

$$\|S_{n+T} - S_n \cdot S'_T\|_\infty \leq Cn^{-\frac{D+1}{2}}. \tag{11}$$

Let R be a fundamental domain in X for the action of Γ such that $\Phi(R) \subset F$. Since Φ is Γ-equivariant, we get

$$\begin{aligned} & S_{n+T}(x,y) - S_n \cdot S'_T(x,y) \\ & = \frac{|G_\Gamma/\Gamma|}{m(X_0)} \sum_{\gamma\in\Gamma, z_0\in R} \Big[\frac{1}{m(X_0)}\int_F \Big(h_n\big(\Phi(x), \gamma\Phi(z_0)\eta\big)\, h_T\big(\gamma\Phi(z_0)\eta, \Phi(y)\big) \\ & \quad - h_n(\Phi(x), \gamma\Phi(z_0))h_T\big(\gamma\Phi(z_0)\eta, \Phi(y)\big)\Big)\, d\eta\Big]\, m(z_0) \\ & \leq \frac{|G_\Gamma/\Gamma|}{m(X_0)^2} \sum_{\gamma\in\Gamma, z_0\in R} \Big[\sup_{\eta\in F}\big|h_n\big(\Phi(x), \gamma\Phi(z_0)\eta\big) - h_n\big(\Phi(x), \gamma\Phi(z_0)\big)\big| \\ & \quad \times \int_F h_T\big(\gamma\Phi(z_0)\eta, \Phi(y)\big)\, d\eta\Big]\, m(z_0) \\ & \leq Cn^{-\frac{D+1}{2}}.\end{aligned}$$

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Hence it is enough to estimate $\|S_{n+T} - k_n S'\|_\infty$. Let $I \in \mathbb{N}$ be a quotient of n by N. Then we have

$$\begin{aligned}
&S_{n+T}(x,y) - k_n S'_T(x,y)\\
&= \sum_{0\le i\le I-2} \left\{ k_{iN} S_{n-iN+T} - k_{(i+1)N} S_{n-(i+1)N+T} \right\}(x,y)\\
&\quad + k_{(I-1)N} S_{n-(I-1)N+T}(x,y) - k_n \cdot S'_T(x,y)\\
&= \sum_{0\le i\le \frac{I-2}{2}} k_{iN}\left(S_{n-iN+T} - k_N S_{n-(i+1)N+T}\right)(x,y)\\
&\quad + \sum_{\frac{I-2}{2}<i\le I-2} \left(k_{iN} - S_{iN}\right)\left(S_{n-iN+T} - k_N S_{n-(i+1)N+T}\right)(x,y)\\
&\quad + \sum_{\frac{I-2}{2}<i\le I-2} S_{iN}\left(S_{n-iN+T} - k_N S_{n-(i+1)N+T}\right)(x,y)\\
&\quad + \left(k_{(I-1)N} - S_{(I-1)N}\right)\left(S_{n-(I-1)N+T} - k_{n-(I-1)N} S'_T\right)(x,y)\\
&\quad + S_{(I-1)N}\left(S_{n-(I-1)N+T} - k_{n-(I-1)N} S'_T\right)(x,y)\\
&= E_1 + E_2 + E_3 + E_4 + E_5.
\end{aligned}$$

Using Hölder's inequality,

$$E_1 \le \sum_{0\le i\le \frac{I-2}{2}} \|k_{iN}(x,\,\cdot\,)\|_{L^1} \left\|\left(S_{n-iN+T} - k_N S_{n-(i+1)N+T}\right)(\,\cdot\,,y)\right\|_\infty .$$

By using (6) and (7), we have

$$\begin{aligned}
E_1 \le \sum_{0\le i\le \frac{I-2}{2}} C &\Big\{ N^2(n-(i+1)N+T)^{-\frac{D+4}{2}} + (n-(i+1)N+T)^{-\frac{D+2}{2}}\\
&+ \sum_{j=3}^{J-1} N^{j-1}(n-(i+1)N+T)^{-\frac{D+j}{2}} + \sum_{k=J}^{Jr} N^k(n-(i+1)N+T)^{-\frac{D+k}{2}} \Big\}.
\end{aligned}$$

Since $(\frac{I-2}{2}+1)N = \frac{IN}{2} < \frac{n}{2}$, we get

$$E_1 \le C'_J\Big(Nn^{-\frac{D+2}{2}} + N^{-1}n^{-\frac{D}{2}} + \sum_{j=3}^{J-1} N^{j-2} n^{-\frac{D+j-2}{2}} + \sum_{k=J}^{Jr} N^{k-1} n^{-\frac{D+k-2}{2}}\Big).$$

To estimate E_2, using Hölder's inequality and (10),

$$\begin{aligned}
E_2 &\le \sum_{\frac{I-2}{2}<i\le I-2} \|(k_{iN} - S_{iN})(x,\,\cdot\,)\|_\infty \left\|\left(S_{n-iN+T} - k_N S_{N-(i+1)N+T}\right)(\,\cdot\,,y)\right\|_{L^1}\\
&\le \sum_{\frac{I-2}{2}<i\le I-2} A(iN)^{-\frac{D+q}{2}} \left\|\{\partial_N + (I-L^N)\} S_{n-(i+1)N+T}(\,\cdot\,,y)\right\|_{L^1}.
\end{aligned}$$

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By using (6) and (7), we have

$$
\begin{aligned}
&\|\{\partial_N + (I - L^N)\}S_{n-(i+1)N+T}(\,\cdot\,, y)\|_{L^1} \\
&\le C'_J\Big(\sup_{\theta\in[0,1]} N^2 \Big| \frac{\partial^2}{\partial t^2} h_{n-(i+1)N+T+\theta N}\big(\Phi(z), \Phi(y)\big)\Big| \\
&\quad + X^2 h_{n-(i+1)N+T}(\Phi(z), \Phi(y)) + \sum_{j=3}^{J-1} N^{j-1} X^j h_{n-(i+1)N+T}\big(\Phi(z), \Phi(y)\big) \\
&\quad + \sup_{g\in U_N} \sum_{k=J}^{Jr} N^k X^k h_{n-(i+1)N+T}\big(\Phi(z)g, \Phi(y)\big)\Big) m(z) \\
&\le C'_J \sum_{z\in V}\Big[N^2 (n-(i+1)N+T)^{-\frac{D+4}{2}} \exp\Big(- \frac{d(\Phi(z), \Phi(y))^2}{c'(n-(i+1)N+T)}\Big) \\
&\quad + (n-(i+1)N+T)^{-\frac{D+2}{2}} \exp\Big(- \frac{d(\Phi(z), \Phi(y))^2}{c'(n-(i+1)N+T)}\Big) \\
&\quad + \sum_{j=3}^{J-1} N^{j-1} (n-(i+1)N+T)^{-\frac{D+j}{2}} \exp\Big(- \frac{d(\Phi(z), \Phi(y))^2}{c'(n-(i+1)N+T)}\Big) \\
&\quad + \sup_{g\in U_N} \sum_{k=J}^{Jr} N^k (n-(i+1)N+T)^{-\frac{D+k}{2}} \exp\Big(- \frac{d(\Phi(z)g, \Phi(y))^2}{c'(n-(i+1)N+T)}\Big)\Big] m(z).
\end{aligned}
$$

Since the order of polynomial growth of X is D, there exists a constant $C > 0$ independent of n, i, N, T and $\Phi(y)$ such that

$$
\begin{aligned}
\big(n-(i+1)N+T\big)^{-\frac{D}{2}} \sum_{z\in V} \exp\Big(- \frac{d(\Phi(z), \Phi(y))^2}{c'(n-(i+1)N+T)}\Big) &\le C, \\
\sup_{g\in U_N} \big(n-(i+1)N+T\big)^{-\frac{D}{2}} \sum_{z\in V} \exp\Big(- \frac{d(\Phi(z)g, \Phi(y))^2}{c'(n-(i+1)N+T)}\Big) &\le C \exp\Big(\frac{N^2}{c'T}\Big).
\end{aligned}
$$

These imply

$$
\begin{aligned}
\Big\| \{\partial_N + (I - L^N)\}S_{n-(i+1)N+T}(\cdot, y)\Big\|_{L^1} &\le C'_J\Big(N^2 (n-(i+1)N+T)^{-\frac{4}{2}} \\
&\quad + (n-(i+1)N+T)^{-\frac{2}{2}} + \sum_{j=3}^{J-1} N^{j-1} (n-(i+1)N+T)^{-\frac{j}{2}} \\
&\quad + \sum_{k=J}^{Jr} N^k (n-(i+1)N+T)^{-\frac{k}{2}} \exp\Big(\frac{N^2}{c'T}\Big)\Big).
\end{aligned}
$$

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Hence we conclude

$$\begin{aligned}E_2 &\le C_J' A(n-2N)^{-\frac{D+q}{2}} \int_{\frac{I}{2}-1}^{I-1} \Big\{ N^2(n-(x+1)N+T)^{-2} \\ &\quad + (n-(x+1)N+T)^{-1} + \sum_{j=3}^{J-1} N^{j-1}(n-(x+1)N+T)^{-j/2} \\ &\quad + \sum_{k=J}^{Jr} N^k (n-(x+1)N+T)^{-\frac{k}{2}} \exp\Big(\frac{N^2}{c'T}\Big)\Big\}\,dx \\ &\le C_J' A(n-2N)^{-\frac{D+q}{2}} \Big(NT^{-1} + N^{-1}\log(n+T) \\ &\quad + \sum_{j=3}^{J-1} N^{j-2} T^{-\frac{j-2}{2}} + \sum_{k=J}^{Jr} N^{k-1} T^{-\frac{k-2}{2}} \exp\Big(\frac{N^2}{c'T}\Big)\Big).\end{aligned}$$

E_4 is estimated by

$$\begin{aligned}E_4 &\le \|(k_{(I-1)N} - S_{(I-1)N})(x,\,\cdot\,)\|_\infty \left\|\left(S_{n-(I-1)N+T} - k_{n-(I-1)N}\cdot S_T'\right)(\,\cdot\,,y)\right\|_{L^1} \\ &\le A((I-1)N)^{-\frac{D+q}{2}} \left\|\left(S_{n-(I-1)N+T} - k_{n-(I-1)N}\cdot S_T'\right)(\,\cdot\,,y)\right\|_{L^1}.\end{aligned}$$

By using Gaussian bounds for h_t [17, Theorem IV.4.2], we have

$$\begin{aligned}&\left\|\left(S_{n-(I-1)N+T} - k_{n-(I-1)N}\cdot S_T'\right)(\,\cdot\,,y)\right\|_{L^1} \\ &= \sum_{x\in V} \frac{1}{m(X_0)} \int_F \Big(h_{n-(I-1)N+T}\big(\Phi(x),\Phi(y)\big) - h_{n-(I-1)N+T}\big(\Phi(x)\eta,\Phi(y)\big) \\ &\quad + \{\partial_{n-(I-1)N} + (I - L^{n-(I-1)N})\} h_T\big(\Phi(\,\cdot\,)\eta,\Phi(y)\big)\,|_x\Big)\,d\eta \\ &\le C_J' \sup_{\substack{\eta\in F'\\ g\in U_N}} \sum_{\gamma\in\Gamma, x_0\in R} \Big[(n-(I-1)N+T)^{-\frac{D+1}{2}} \exp\Big(-\frac{d(\gamma\Phi(x_0)\eta,\Phi(y))^2}{c'(n-(I-1)N+T)}\Big) \\ &\quad + (n-(I-1)N)^2 T^{-\frac{D+4}{2}} \exp\Big(-\frac{d(\gamma\Phi(x_0)\eta,\Phi(y))^2}{c'T}\Big) \\ &\quad + T^{-\frac{D+2}{2}} \exp\Big(-\frac{d(\gamma\Phi(x_0)\eta,\Phi(y))^2}{c'T}\Big) \\ &\quad + \sum_{j=3}^{J-1} (n-(I-1)N)^{j-1} T^{-\frac{D+j}{2}} \exp\Big(-\frac{d(\gamma\Phi(x_0)\eta,\Phi(y))^2}{c'T}\Big) \\ &\quad + \sum_{k=J}^{Jr} (n-(I-1)N)^k T^{-\frac{D+k}{2}} \exp\Big(-\frac{d(\gamma\Phi(x_0)g\eta,\Phi(y))^2}{c'T}\Big)\Big]\end{aligned}$$

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$$\leq C'_J\Big(T^{-\frac{1}{2}} + N^2T^{-2} + T^{-1} + \sum_{j=3}^{J-1} N^{j-1}T^{-\frac{j}{2}} + \sum_{k=J}^{Jr} N^k T^{-\frac{k}{2}} \exp\big(\frac{N^2}{c'T}\big)\Big),$$

where F' is a compact subset in G_Γ.

Next, we consider $E_3 + E_5$. Let $[a]$ be the greatest integer not greater than a. Then,

$$\begin{aligned}
E_3 + E_5 &= \big(S_{[\frac{I}{2}]N} \cdot S_{n-[\frac{I}{2}]N+T} - S_{(I-1)N} \cdot k_{n-(I-1)N} \cdot S'_T\big)(x,y) \\
&\quad + \sum_{\frac{I-2}{2}<i\leq I-2} (S_{(i+1)N} - S_{iN} \cdot k_N) \cdot S_{n-(i+1)N+T}(x,y) \\
&= E'_3 + E'_5.
\end{aligned}$$

By using Hölder's inequality,

$$\begin{aligned}
E'_5 &\leq \sum_{\frac{I-2}{2}<i\leq I-2} \|(S_{(i+1)N} - S_{iN} \cdot k_N)(x,\cdot)\|_\infty \|S_{n-(i+1)N+T}(\cdot,y)\|_{L^1} \\
&\leq C'_J \sum_{\frac{I-2}{2}<i\leq I-2} \Big(N^2(iN)^{-\frac{D+4}{2}} + (iN)^{-\frac{D+2}{2}} + \sum_{j=3}^{J-1} N^{j-1}(iN)^{-\frac{D+j}{2}} \\
&\quad + \sum_{k=J}^{Jr} N^k(iN)^{-\frac{D+k}{2}}\Big) \\
&\leq C'_J n\Big(N(n-2N)^{-\frac{D+4}{2}} + N^{-1}(n-2N)^{-\frac{D+2}{2}} + \sum_{j=3}^{J-1} N^{j-2}(n-2N)^{-\frac{D+j}{2}} \\
&\quad + \sum_{k=J}^{Jr} N^{k-1}(n-2N)^{-\frac{D+k}{2}}\Big).
\end{aligned}$$

E'_3 is estimated by

$$\begin{aligned}
E'_3 \leq \|S_{[\frac{I}{2}]N} S_{n-[\frac{I}{2}]N+T} - S_{n+T}\|_\infty + \|S_{n+T} - S_n \cdot S'_T\|_\infty& \\
+ \|(S_n - S_{(I-1)N} \cdot k_{n-(I-1)N}) \cdot S'_T\|_\infty&.
\end{aligned}$$

Then we have

$$\begin{aligned}
&(S_{[\frac{I}{2}]N} S_{n-[\frac{I}{2}]N+T} - S_{n+T})(x,y) \\
&= \frac{|G_\Gamma/\Gamma|}{m(X_0)^2} \sum_{\gamma\in\Gamma, z_0\in R} \int_F \Big[h_{[\frac{I}{2}]N}\big(\Phi(x), \gamma\Phi(z_0)\big)\, h_{n-[\frac{I}{2}]N+T}\big(\gamma\Phi(z_0), \Phi(y)\big) \\
&\qquad - h_{[\frac{I}{2}]N}(\Phi(x), \gamma\eta) h_{n-[\frac{I}{2}]N+T}\big(\gamma\eta, \Phi(y)\big)\Big]\, d\eta\, m(z_0)
\end{aligned}$$

$$\leq \frac{|G_\Gamma/\Gamma|}{m(X_0)^2} \sum_{\gamma\in\Gamma, z_0\in R} \Big[\sup_{\eta\in F} \big|h_{n-[\frac{I}{2}]N+T}(\gamma\Phi(z_0), \Phi(y)) - h_{n-[\frac{I}{2}]N+T}(\gamma\eta, \Phi(y))\big|$$

$$\times \int_F h_{[\frac{I}{2}]N}(\Phi(x), \gamma\Phi(z_0))\, d\eta + \sup_{\eta\in F} \big| h_{[\frac{I}{2}]N}\big(\Phi(x), \gamma\Phi(z_0)\big) - h_{[\frac{I}{2}]N}(\Phi(x), \gamma\eta)\big|$$

$$\times \int_F h_{n-[\frac{I}{2}]N+T}\big(\gamma\eta, \Phi(y)\big)\, d\eta\Big]\, m(z_0)$$

$$\leq C_J'\Big(\big(\frac{n}{2}\big)^{-\frac{D+1}{2}} + \big(\frac{n}{2} - \frac{3}{2}N\big)^{-\frac{D+1}{2}}\Big).$$

By (11), $\|S_{n+T} - S_n S_T'\|_\infty \leq C n^{-\frac{D+1}{2}}$. So $\|(S_n - S_{(I-1)N} k_{n-(I-1)N}) S_T'\|_\infty$ is estimated by

$$\big(S_n - S_{(I-1)N} k_{n-(I-1)N}\big)\, S_T'(x, y)$$

$$\leq \|\big(S_n - S_{(I-1)N} \cdot k_{n-(I-1)N}\big)\,(x, \cdot)\|_\infty \|S_T'(\,\cdot\,, y)\|_{L^1}$$

$$\leq C_J'\Big[N^2(n-2N)^{-\frac{D+4}{2}} + (n-2N)^{-\frac{D+2}{2}} + \sum_{j=3}^{J-1} N^{j-1}(n-2N)^{-\frac{D+j}{2}}$$

$$+ \sum_{k=J}^{Jr} N^k (n-2N)^{-\frac{D+k}{2}}\Big].$$

By the hypothesis of N, the lemma follows. ■

Proof of Theorem 1

First, we will consider the case that X is a non-bipartite graph. We note that if (1) and (2) are satisfied, then the terms $N^{-1}n^{-\frac{D}{2}}$ and $N^{-1}\log(n+T)$ in Lemma 2.4 vanish. Hence we can use the same arguments as Alexopoulos [1] by putting $N = 1$ and $q = 1$. However, if (1) and (2) are not satisfied, then we put $N = [n^{(J-2)/(4J-6)}]$, $T = T_0[n^{(J-1)/(2J-3)}]$ ($T_0 \in \mathbb{N}$) and $q = (J-2)/(2J-3)$. In this case, if $\delta(i) \leq Ai^{-\frac{D+(J-2)/(2J-3)}{2}}$ for $i = 1, 2, \dots n-1$, then there exists a constant $\alpha_J > 1$ and a sequence $\{\beta_{T_0}(n)\}_{n\in\mathbb{N}}$ which converges to zero as $n \uparrow \infty$ such that

$$\delta(n) \leq \alpha_J \Big(1 + T_0^{1/2} + A\big(\beta_{T_0}(n) + T_0^{-(J-2)/2}\exp(1/c'T_0)\big)\Big)\, n^{-\frac{D+(J-2)/(2J-3)}{2}}.$$

Hence we will use the induction for n. Fix $s_J \in \mathbb{R}$ such that $1 - 1/\alpha_J < s_J < 1$. Let K_J and T_J be positive integers such that

$$\big(\beta_{T_J}(n) + T_J^{-(J-2)/2}\exp(1/c'T_J)\big) < 1 - s_J$$

for all $n \geq K_J$. Since $\delta(n)$ is uniformly bounded, there exists a constant $A_J > 0$ such that

$$\delta(n) \leq A_J n^{-\frac{D+(J-2)/(2J-3)}{2}}$$

for all $n < K_J$. By the previous lemma and the assumption of K_J, we have

$$\delta(K_J) \le \alpha_J\big(1 + T_J^{1/2} + A_J(1 - s_J)\big)\, K_J^{-\frac{D+(J-2)/(2J-3)}{2}}.$$

Put $C_J = \max\{A_J, (1 + T_J^{1/2})(1/\alpha_J + s_J - 1)^{-1}\}$. Then clearly we have

$$\delta(n) \le C_J n^{-\frac{D+(J-2)/(2J-3)}{2}}$$

for all $n \le K_J$.

When $n > K_s$, we assume that

$$\delta(i) \le C_J i^{-\frac{D+(J-2)/(2J-3)}{2}}$$

for $i = 1, 2, \dots n - 1$. By the previous lemma and the definition of C_J, we conclude

$$\delta(n) \le \alpha_J\big(1 + T_J^{1/2} + C_J(1 - s_J)\big)\, n^{-\frac{D+(J-2)/(2J-3)}{2}} \le C_J n^{-\frac{D+(J-2)/(2J-3)}{2}}.$$

Next, we will consider the case that X is a bipartite graph. Suppose that m and p are a weight and a transition probability on X which gives a symmetric random walk. The bipartition of V is denoted by $V = A \coprod B$. Let $X_A = (A, E_A)$ be an oriented graph, where $E_A = \{(e_1, e_2) \in C_{x,2} \mid x \in A\}$. For $e = (e_1, e_2) \in E_A$, let $o(e) = o(e_1)$, $t(e) = t(e_2)$ and $\overline{e} = (\overline{e_2}, \overline{e_1})$. Then a weight m_A and a transition probability p^A is denoted by

$$m_A(x) = m(x) \quad x \in A,$$
$$p^A(e) = p(e_1)p(e_2) \quad e = (e_1, e_2) \in E_A,$$

respectively. It is easy to show that m_A and p^A give a symmetric random walk on X_A. The transition probability starting at x reaches y at time n on X_A is denoted by $p_n^A(x, y)$. Then the kernel function k_n^A of the transition operator on X_A is written by $k_n^A(x, y) = p_n^A(x, y)m_A(y)^{-1}$. By using the argument of [8], X_A is also a nilpotent covering graph of a finite graph X_{A1} whose covering transformation group Γ_1 is Γ or a subgroup of Γ of index two. We note that X_A have a loop for each vertex. Hence we conclude

$$\sup_{x,y\in A} \Big| p_n^A(x, y)m(y)^{-1} - \frac{|G_\Gamma/\Gamma_1|}{m(X_{A1})} h_n^A\big(\Phi(x), \Phi(y)\big) \Big| \le C_\epsilon n^{-\frac{D+1/2-\epsilon}{2}},$$

where h_n^A is the heat kernel with respect to m_A and p^A. Since $p_n^A = p_{2n}$, $h_n^A = h_{2n}$, and $\frac{|G_\Gamma/\Gamma_1|}{m(X_{A1})} = 2\frac{|G_\Gamma/\Gamma|}{m(X_0)}$, the theorem is proved when $x, y \in A$ for even n. If $x \in A$, $y \in B$

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or $x \in B$, $y \in A$, then we have

$$\begin{aligned}
&p_{2n+1}(x,y)m(y)^{-1} - 2\frac{|G_\Gamma/\Gamma|}{m(X_0)}h_{2n+1}\big(\Phi(x),\Phi(y)\big)\\
&\quad= \sum_{z\in A} k_{2n}(x,z)k(z,y)m(z) - 2\frac{|G_\Gamma/\Gamma|}{m(X_0)}h_{2n+1}\big(\Phi(x),\Phi(y)\big)\\
&\quad= \sum_{z\in A}\Big(k_{2n}(x,z) - 2\frac{|G_\Gamma/\Gamma|}{m(X_0)}h_{2n}(\Phi(x),\Phi(z))\Big)k(z,y)m(z)\\
&\qquad+ \sum_{z\in A} 2\frac{|G_\Gamma/\Gamma|}{m(X_0)}h_{2n}(\Phi(x),\Phi(y))k(z,y)m(z) - 2\frac{|G_\Gamma/\Gamma|}{m(X_0)}h_{2n+1}(\Phi(x),\Phi(y))\\
&\quad\le C_\epsilon n^{-\frac{D+1/2-\epsilon}{2}} + |(\partial_1 + (I-L_y))S_{2n}(x,y)|\\
&\quad\le C_\epsilon n^{-\frac{D+1/2-\epsilon}{2}} + Cn^{-\frac{D+2}{2}} \le C_\epsilon n^{-\frac{D+1/2-\epsilon}{2}}.
\end{aligned}$$

Hence we complete the proof of Theorem 1.

3 Gaussian Upper Bound for ∇k_n

First, we assume that X is a non-bipartite graph. For our proof of the Gaussian upper bound for ∇k_n, we introduce next two lemmas.

Lemma 3.1 (Cf. [5, Lemma 3.2]) *Let $\ell, n \in \mathbb{N}$ and $f \in L^2(X)$. There exists a constant $C_\ell > 0$ such that*

$$\|(I - L^{2\ell})^{1/2}L^n f\|_2 \le C_\ell n^{-1/2}\|f\|_2.$$

As an easy consequence of (3), we have

Lemma 3.2 (Cf. [5, Lemma 5.2]) *Set $\omega_s(x,y) = \exp(sd_X(x,y))$ $(x, y \in V)$. Then*

$$\|k_n(x,\cdot)\omega_s(x,\cdot)\|_q \le Cn^{-\frac{D}{2}(1-1/q)}\exp(C's^2n). \tag{12}$$

Proof of Theorem 2

By the same argument of [5], it is easy to show that

$$\nabla^y k_n(x,y) \le C \sup_{d_X(y,z)\le 1} \nabla_2^y k_n(x,z). \tag{13}$$

Hence we will consider $\nabla_2^y k_n(x,y)$. Fix $s > 0$, $\nu = n + m$, and note that $\omega_s(x,y) \le \omega_s(x,z)\omega_s(z,y)$. This implies

$$\omega_s(x,y)\nabla_2^y k_\nu(x,y) \le \|k_m(x,\,\cdot\,)\omega_s(x,\,\cdot\,)\|_2\|\nabla_2^y k_n(\,\cdot\,,y)\omega_s(\,\cdot\,,y)\|_2.$$

Lemma 3.2 yields a good bound for $\|k_m(x,\cdot)\omega_s(x,\cdot)\|_2$. The second factor can be estimated by

$$\begin{aligned}\left\|\omega_s(\,\cdot\,,y)\nabla_2^y k_n(\,\cdot\,,y)\right\|_2^2 &\le C\sum_{z_3\in R_y}\|\omega_s(\,\cdot\,,z_3)\nabla_2^{z_3}k_n(\,\cdot\,,z_3)\|_2^2\,m(z_3)\\ &= C\sum_{z_3\in R_y}\sum_{z\in V}\omega_{2s}(z,z_3)\sum_{d(z_3,z')\le 2}|k_n(z,z_3)-k_n(z,z')|^2\,m(z')m(z)m(z_3).\end{aligned}$$

Since X is a non-bipartite graph, there exists $n_0\in\mathbb{N}$ such that

$$\inf\{k_{2n_0}(z',z_3)\mid d_X(z_3,z')\le 2,\ z_3\in R\}>0.$$

Hence

$$\begin{aligned}&\|\omega_s(\,\cdot\,,y)\nabla_2^y k_n(\,\cdot\,,y)\|_2^2\\ &\quad\le C'\sum_{z_3\in R_y}\sum_{z\in V}\omega_{2s}(z,z_3)\sum_{d(z_3,z')\le 2}|k_n(z,z_3)-k_n(z,z')|^2\\ &\qquad\quad\times k_{2n_0}(z',z_3)m(z')m(z)m(z_3)\\ &\quad\le C'\sum_{z_3\in R_y}\sum_{z,z'\in V}\omega_{2s}(z,z_3)\Big(k_n(z,z_3)^2-2k_n(z,z_3)k_n(z,z')+k_n(z,z')^2\Big)\\ &\qquad\quad\times k_{2n_0}(z',z_3)m(z')m(z)m(z_3)\\ &\quad=2C'\sum_{z_3\in R_y}\sum_{z,z'\in V}\omega_{2s}(z,z_3)k_n(z,z_3)\,\big(k_n(z,z_3)-k_n(z,z')\big)\\ &\qquad\quad\times k_{2n_0}(z',z_3)m(z')m(z)m(z_3)\\ &\qquad+C'\Big(\sum_{z_3\in R_y}\sum_{z,z'\in V}\omega_{2s}(z,z_3)k_n(z,z')^2k_{2n_0}(z',z_3)m(z')m(z)m(z_3)\\ &\qquad\quad-\sum_{z_3\in R_y}\sum_{z,z'\in V}\omega_{2s}(z,z_3)k_n(z,z_3)^2k_{2n_0}(z',z_3)m(z')m(z)m(z_3)\Big)\\ &\quad=B_1+B_2.\end{aligned}$$

By using Lemma 3.1 and Lemma 3.2, B_1 is estimated by

$$\begin{aligned}B_1&=2C'\sum_{z_3\in R_y}\omega_{2s}(z,z_3)k_n(z,z_3)(I-L^{2n_0})k_n(z,z_3)m(z)m(z_3)\\ &\le 2C'\|\omega_{2s}(\,\cdot\,,z_3)k_n(\,\cdot\,,z_3)\|_2\cdot\|(I-L^{2n_0})k_n(\,\cdot\,,z_3)\|_2 m(z_3)\\ &\le Cn^{-\frac{D}{4}}\exp(C's^2n)\cdot n^{-1}\cdot n^{-\frac{D}{4}}=Cn^{-1-\frac{D}{2}}\exp(C's^2n).\end{aligned}$$

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Because every $z \in V$ can be written as $z = \gamma z_0$ ($\gamma \in \Gamma, z_0 \in R_y$), and the weight m is Γ-invariant, we have

$$B_2 = C'\Big(\sum_{z_3 \in R_y} \sum_{\substack{z_1, z_2 \in R_y, \\ \gamma_1, \gamma_2 \in \Gamma}} \omega_{2s}(\gamma_1 z_1, z_3) k_n(\gamma_1 z_1, \gamma_2 z_2)^2 k_{2n_0}(\gamma_2 z_2, z_3) m(z_2) m(z_1) m(z_3)$$
$$- \sum_{z_3 \in R_y} \sum_{\substack{z_1, z_2 \in R_y, \\ \gamma_1, \gamma_2 \in \Gamma}} \omega_{2s}(\gamma_1 z_1, z_2) k_n(\gamma_1 z_1, z_2)^2 k_{2n_0}(z_2, \gamma_2^{-1} z_3) m(z_3) m(z_1) m(z_2) \Big).$$

By replacing γ_1 with $\gamma_2^{-1}\gamma_1$ in the second term,

$$B_2 = C'\Big(\sum_{\substack{z_1, z_2, z_3 \in R_y, \\ \gamma_1, \gamma_2 \in \Gamma}} \omega_{2s}(\gamma_1 z_1, z_3) k_n(\gamma_1 z_1, \gamma_2 z_2)^2 k_{2n_0}(\gamma_2 z_2, z_3) m(z_3) m(z_2) m(z_1)$$
$$- \sum_{\substack{z_1, z_2, z_3 \in R_y, \\ \gamma_1, \gamma_2 \in \Gamma}} \omega_{2s}(\gamma_2^{-1}\gamma_1 z_1, z_2) k_n(\gamma_2^{-1}\gamma_1 z_1, z_2)^2 k_{2n_0}(\gamma_2 z_2, z_3) m(z_3) m(z_2) m(z_1) \Big)$$
$$= C' \sum_{\substack{z_1, z_2, z_3 \in R_y, \\ \gamma_1, \gamma_2 \in \Gamma}} \big(\omega_{2s}(\gamma_1 z_1, z_3) - \omega_{2s}(\gamma_1 z_1, \gamma_2 z_2) \big) k_n(\gamma_1 z_1, \gamma_2 z_2)^2$$
$$\times k_{2n_0}(\gamma_2 z_2, z_3) m(z_3) m(z_2) m(z_1).$$

By inverting z_2 and z_3, replacing $\gamma_2^{-1}\gamma_1$ with γ_1 and γ_2 with γ_2^{-1}, B_2 is

$$B_2 = C' \sum_{\substack{z_1, z_2, z_3 \in R_y, \\ \gamma_1, \gamma_2 \in \Gamma}} \big(\omega_{2s}(\gamma_1 z_1, \gamma_2 z_2) - \omega_{2s}(\gamma_1 z_1, z_3) \big) k_n(\gamma_1 z_1, z_3)^2$$
$$\times k_{2n_0}(\gamma_2 z_2, z_3) m(z_3) m(z_2) m(z_1).$$

Since $|\omega_s(x, y) - \omega_s(x, z)| \leq r_0 |s| (\omega_s(x, y) + \omega(x, z))$ for $d_X(y, z) \leq r_0$ (see [5, Lemma 2.3]), we have

$$B_2 = \frac{C'}{2} \sum_{\substack{z_1, z_2, z_3 \in R_y, \\ \gamma_1, \gamma_2 \in \Gamma}} (\omega_{2s}(\gamma_1 z_1, z_3) - \omega_{2s}(\gamma_1 z_1, \gamma_2 z_2))$$
$$\times \big(k_n(\gamma_1 z_1, \gamma_2 z_2)^2 - k_n(\gamma_1 z_1, z_3)^2 \big) k_{2n_0}(\gamma_2 z_2, z_3) m(z_3) m(z_2) m(z_1)$$
$$\leq C|s| \sum_{\substack{z_1, z_2, z_3 \in R_y, \\ \gamma_1, \gamma_2 \in \Gamma}} (\omega_{2s}(\gamma_1 z_1, z_3) + \omega_{2s}(\gamma_1 z_1, \gamma_2 z_2))$$
$$\times \big| k_n(\gamma_1 z_1, \gamma_2 z_2)^2 - k_n(\gamma_1 z_1, z_3)^2 \big| k_{2n_0}(\gamma_2 z_2, z_3) m(z_3) m(z_2) m(z_1).$$

By using the Cauchy-Schwarz inequality and Lemma 3.2,

$$\begin{aligned} B_2 \le C|s|\Big(\sum_{\substack{z_1,z_2,z_3\in R_y,\\ \gamma_1,\gamma_2\in\Gamma}} &\Big\{ k_n(\gamma_1 z_1, z_2)\big(k_n(\gamma_1 z_1, z_2) - k_n(\gamma_2\gamma_1 z_1, z_3)\big)\, k_{2n_0}(\gamma_2 z_2, z_3) \\ &+ k_n(\gamma_1 z_1, z_3)\big(k_n(\gamma_1 z_1, z_3) - k_n(\gamma_1 z_1, \gamma_2 z_2)\big)\, k_{2n_0}(\gamma_2 z_2, z_3)\Big\} \\ &\times m(z_3)m(z_2)m(z_1)\Big)^{1/2} \\ \times \Big[\Big(\sum_{z_2\in R_y z'\in V} &\|\omega_{2s}(\cdot, z_2)k_n(\cdot, z_2)\|_2^2 \omega_{4s}(z_2, z')k_{2n_0}(z_2, z')m(z')m(z_2)\Big)^{1/2} \\ &+ n^{-\frac{D}{4}} \exp(C's^2 n) + n^{-\frac{D}{4}} \exp(C's^2 n) \\ &+ \Big(\sum_{z_3\in R_y z'\in V} \|\omega_{2s}(\cdot, z_3)k_n(\cdot, z_3)\|_2^2 \omega_{4s}(z_3, z')k_{2n_0}(z_3, z')m(z')m(z_3)\Big)^{1/2}\Big]. \end{aligned}$$

Lemma 3.1 implies

$$\begin{aligned} B_2 &\le C|s|\Big(\sum_{z_3\in R_y} \|\big(I - L^{2n_0}\big)^{1/2} k_n(\cdot, z_3)\|_2^2 m(z_3)\Big)^{1/2} n^{-\frac{D}{4}} \exp(C's^2 n) \\ &\le C|s| n^{-\frac{1}{2}-\frac{D}{2}} \exp(C's^2 n). \end{aligned}$$

By choosing $n = m$ or $n = m + 1$ depending on whether ν is even or odd, we obtain

$$\omega_s(x, y)\nabla_2^y k_\nu(x, y) \le C(1 + s\sqrt{\nu})^{1/2} \nu^{-D/2-1/2} \exp(C's^2\nu).$$

Choosing $s = d_X(x, y)/2C'\nu$ in this last inequality yields the estimate

$$\nabla_2^y k_\nu(x, y) \le C\nu^{-1/2-D/2} \exp(-d_X(x, y)^2/C'\nu).$$

Hence we conclude Theorem 2.

Finally, we consider a Gaussian bound for ∇k_n when X is a bipartite graph. By the same argument of the last of Section 2, we have

$$\begin{aligned} \nabla^y k_{2n}(x, y) &= \sup_{d_X(y,z)=2} |k_{2n}(x, y) - k_{2n}(x, z)| \\ &= \sup_{d_{X_A}(y,z)=1} |k_n^A(x, y) - k_n^A(x, z)| \\ &\le Cn^{-\frac{D+1}{2}} \exp(-d_X(x, y)^2/C'n) \end{aligned}$$

for $x, y \in A$. If $x \in A$, $y \in B$ or $x \in B$, $y \in A$, we conclude

$$\begin{aligned} \nabla^y k_{2n+1}(x, y) &= \sup_{d_X(y,z)=2} \Big| \sum_{\omega\in V} k(x, \omega)(k_{2n}(\omega, y) - k_{2n}(\omega, z))m(z)\Big| \\ &\le \sup_{d_X(x,\omega)\le 1} Cn^{-\frac{D+1}{2}} \exp(-d_X(\omega, y)^2/C'n) \\ &\le Cn^{-\frac{D+1}{2}} \exp(-d_X(x, y)^2/C'n). \end{aligned}$$

Hence we complete the proof of Theorem 2. ■

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