

## BANACH ENVELOPES OF NON-LOCALLY CONVEX SPACES

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**1. Introduction.** Let  $X$  be a quasi-Banach space whose dual  $X^*$  separates the points of  $X$ . Then  $X^*$  is a Banach space under the norm

$$\|x^*\| = \sup_{\|x\| \leq 1} |x^*(x)|.$$

From  $X$  we can construct the *Banach envelope*  $X_c$  of  $X$  by defining for  $x \in X$ , the norm

$$\|x\|_c = \sup_{\|x^*\| \leq 1} |x^*(x)|.$$

Then  $X_c$  is the completion of  $(X, \|\cdot\|_c)$ . Alternatively  $\|\cdot\|_c$  is the Minkowski functional of the convex hull of the unit ball.  $X_c$  has the property that any bounded linear operator  $L: X \rightarrow Z$  into a Banach space extends with preservation of norm to an operator  $\tilde{L}: X_c \rightarrow Z$ .

The Banach envelope of  $l_p$  ( $0 < p < 1$ ) is, of course,  $l_1$ . In 1969, Duren, Romberg and Shields [3] identified the dual space of  $H_p$  ( $0 < p < 1$ ) and thus its Banach envelope (cf. [14]). The Banach envelope of  $H_p$  is a Bergman space which turns out to be isomorphic again to  $l_1$  (see [18] for a recent direct proof of this). These examples and others prompted Joel Shapiro to ask what special properties a Banach envelope of a non-locally convex space (with separating dual) must have.

An example of Pelczynski, the space  $l_2(l_p^n)$  ( $0 < p < 1$ ) has a reflexive Banach envelope  $(l_2(l_1^n))$ . This suggests that the answer to Shapiro's question lies with the finite-dimensional structure of the space.

We say that a Banach space  $Y$  contains  $l_1^n$ 's uniformly (or  $l_1$  is finitely representable in  $Y$ ) if for every  $n \in \mathbb{N}$  and  $\epsilon > 0$  there is a subspace  $F$  of  $Y$  with  $\dim F = n$  and a linear isomorphism  $T: l_1^n \rightarrow F$  with

$$\|T\| \|T^{-1}\| < 1 + \epsilon.$$

We say that  $Y$  contains uniformly complemented  $l_1^n$ 's if in addition  $F$  can be chosen so that there is a projection  $P: Y \rightarrow F$  with  $\|P\| < 1 + \epsilon$ .

$Y$  fails to contain  $l_1^n$ 's uniformly if and only if  $Y$  is of type  $p$  for some  $p > 1$  i.e., for some  $C < \infty$  and all  $y_1, \dots, y_n \in Y$

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$$\mathcal{E} \left( \left\| \sum_{i=1}^n \epsilon_i y_i \right\|^p \right) \leq C \sum_{i=1}^n \|y_i\|^p$$

where  $\epsilon_1, \dots, \epsilon_n$  is a sequence of independent Bernoulli random variables on some probability space (with  $P(\epsilon_i = 1) = P(\epsilon_i = -1) = 1/2$ ).  $Y$  fails to contain uniformly complemented  $l_1^n$ 's if and only if  $Y^*$  fails to contain  $l_\infty^n$ 's uniformly; this in turn is equivalent to  $Y^*$  having *finite cotype*  $p$  for some  $p < \infty$ , i.e., for some  $c < 0$  and all  $y_1^*, \dots, y_n^* \in Y^*$

$$\mathcal{E} \left( \left\| \sum_{i=1}^n \epsilon_i y_i^* \right\|^p \right) \geq c \sum_{i=1}^n \|y_i^*\|^p.$$

See [10].

The author [5] has shown:

**THEOREM 1.1.** *Let  $X$  be a non-locally convex quasi-Banach space with separating dual. Then  $X_c$  contains  $l_1^n$ 's uniformly.*

Thus  $l_p$  ( $1 < p < \infty$ ) and  $L_p$  ( $1 < p < \infty$ ) cannot be Banach envelopes of such spaces. However the case of  $c_0$  (or  $l_\infty$ ) was left unresolved. Several examples have been suggested as possible spaces with Banach envelopes isomorphic to  $c_0$ . For example, Shapiro has recently studied the harmonic Hardy spaces  $h_p$  ( $0 < p < 1$ ), and these have subspaces  $h_p^0$  which are hereditarily  $c_0$  [15]. Similar examples have been studied by A. Matheson. The Banach space envelopes of these spaces have not been determined, however.

In this paper, we shall show first that these and similar examples can never have  $c_0$  as a Banach envelope. Precisely, we introduce the concept of a natural quasi-Banach space. The idea is that all non-locally convex quasi-Banach spaces which commonly arise in analysis are natural (e.g. any subspace of an Orlicz function space or of a Lorentz function space). We then show that any non-locally convex subspace of a natural space with a basis (or the bounded approximation property) has a Banach envelope which contains uniformly complemented  $l_1^n$ 's. The same conclusion applies to any subspace of the nonseparable space  $h_p$  (Theorems 3.4 and 3.5). The proofs of these facts depend on a theorem essentially due to Maurey [8] (cf. [9]) on the factorization of operators into  $L_p$  where  $0 < p < 1$ . As we require a slight modification of Maurey's theorem and, in any case, his proof lies deeply embedded in [8], we shall present an exposition of the result we need in Section 2, using ideas of Bennett ([1], [2]).

In Section 4, however, we construct a non-locally convex space whose Banach envelope is isomorphic to  $c_0$ ; although this space has an unconditional basis, it cannot be natural. In Section 5 we show further that for any separable Banach space  $Z$  we can find a subspace of  $l_p$  ( $0 < p < 1$ ) whose Banach envelope is isomorphic to  $l_1 \oplus Z$ . Thus although  $c_0$  is not

the Banach envelope of any subspace of  $l_p, l_1 \oplus c_0$  is such a Banach envelope. Note, of course, that  $l_p$  has no infinite-dimensional locally convex subspaces.

Finally in Section 6 we prove some special results for the Banach envelopes of spaces with unconditional bases.

All our results are stated for real quasi-Banach spaces with the understanding that they generalize to the complex case without difficulty.

We conclude the introduction by making a few remarks which we hope will assist the non-specialist reader.

First we remark that Banach spaces which do not contain  $l_1^n$ 's uniformly have been studied in the past both from the point of view of the geometry of Banach spaces and from the point of view of probability in Banach spaces; in the latter context such spaces are called  $B$ -convex, and this terminology was used by the author in [5]. The reader is referred to [12] for several formulations of this notion. We note that every uniformly convex Banach space fails to contain  $l_1^n$ 's uniformly, but, in the converse direction there do exist non-reflexive  $B$ -convex spaces ([4]). The most important observation for the reader is that  $c_0$  does contain  $l_1^n$ 's; indeed  $l_1^n$  embeds in  $l_\infty^m$  if  $m = 2^n$ , and  $c_0$  contains  $l_\infty^n$ 's. However  $c_0$  does not contain uniformly complemented  $l_1^n$ 's since  $c_0^* = l_1$  does not contain  $l_\infty^n$ 's.

The second observation is that in Section 3 we introduce the notion of a natural space, building on ideas in [6]. However the reader who does not wish to delve into [6] may instead take the conclusion of Theorem 3.1 for the definition of a natural space. Using this it is easy enough to show that  $L_p, 0 < p < 1$ , is natural since it is transitive (see [13]).

**2. A theorem of Maurey.** In this section we present a slight modification of a theorem due to Maurey [8]. This theorem is very difficult to trace in [8] however (cf. Theorem 93 and the remarks in [1] or [2]) and so we present a direct proof based on ideas of Bennett ([1], [2]).

Let  $(\Omega, \Sigma, \mu)$  be a measure space of total mass one. On  $L_p(\Omega, \Sigma, \mu)$  we define for  $E \in \Sigma$  the projection  $P_E$  by  $P_E f = 1_E f$  where  $1_E$  is the indicator function of  $E$ .

**LEMMA 2.1.** *Let  $X$  be a quasi-Banach space whose dual  $X^*$  has finite cotype, and suppose  $0 < p < 1$ . Then there is a constant  $C$  so that whenever*

$$T: X \rightarrow L_1(\Omega, \Sigma, \mu)$$

*is an operator and  $\epsilon > 0$  then there is  $E \in \Sigma$  with  $\mu(E) \geq 1 - \epsilon$  and*

$$\|P_E T\|_1 \leq C \epsilon^{1-1/p} \|T\|_p.$$

*Here  $\|S\|_r$  denotes the norm of the operator  $S: X \rightarrow L_r$ .*

*Proof.* Since  $X^*$  has finite cotype there exists an integer  $m$  so that whenever  $x_1^*, \dots, x_m^* \in X^*$  then

$$\max_{\eta_i = \pm 1} \|\sum \eta_i x_i^*\| \geq 2^{1/p+1} \min_{1 \leq i \leq m} \|x_i^*\|.$$

Now suppose  $S: X \rightarrow L_1(\Omega, \Sigma, \mu)$  is any operator and fix  $\epsilon > 0$ . We shall prove the existence of a set  $E \in \Sigma$  with  $\lambda(E) \geq 1 - \epsilon$  so that for any  $x \in X$

$$(*) \quad \|P_E Sx\|_1 \leq 2^{-1/p} \|S\|_1 \|x\| + \frac{p}{1-p} \left(\frac{m}{\epsilon}\right)^{1/p-1} \|Sx\|_p.$$

To prove this select inductively  $x_1, \dots, x_m \in X$  and disjoint sets  $E_1, \dots, E_m \in \Sigma$  so that

- (a)  $\|x_i\| = 1 \quad 1 \leq i \leq m$
- (b)  $\mu(E_i) = m^{-1}\epsilon \quad 1 \leq i \leq m$
- (c)  $\int_{E_i} |Tx| d\mu \geq \frac{1}{2} \sup \int_F |Sy| d\mu$

where the supremum is taken over all  $F \in \Sigma$  with

$$\mu(F) = m^{-1}\epsilon \quad \text{and} \quad F \subset \Omega \setminus \bigcup_{j < i} E_j$$

and all  $y \in X$  with  $\|y\| = 1$ . Then let

$$E = \Omega \setminus \bigcup_{i=1}^m E_i;$$

clearly  $\mu(E) = 1 - \epsilon$ .

Now pick  $u_i \in L_\infty$  with  $\|u_i\|_\infty = 1$  and  $\text{supp } u_i \subset E_i$  so that

$$\int_{E_i} u_i(\omega) Sx_i(\omega) d\mu = \int_{E_i} |Sx_i| d\mu.$$

Then if  $S^*: L_\infty \rightarrow X^*$  is the adjoint of  $S$ ,

$$\|S^* u_i\| \geq \int_{E_i} |Sx_i| d\mu.$$

Now for  $\eta_i = \pm 1$

$$\|\sum \eta_i S^* u_i\| \leq \|S\|_1$$

so that

$$\min_{i \leq i \leq m} \|S^* u_i\| \leq 2^{-1/p-1} \|S\|_1$$

and hence

$$\min_{1 \leq i \leq m} \int_{E_i} |Sx_i| d\mu \leq 2^{-1/p-1} \|S\|_1.$$

Now suppose  $x \in X$  and let  $h = 1_E Sx$ . Let  $f:(0, 1) \rightarrow \mathbf{R}$  be the decreasing rearrangement of  $h$ , i.e.,

$$f(t) = \inf_{\mu(A)=t} \sup_{\omega \notin A} |h(\omega)|.$$

Hence

$$f(t) \leq t^{-1/p} \|Sx\|_p.$$

It follows that there exists  $F \in \Sigma$  with  $F \subset E$ ,  $\mu(F) = m^{-1}\epsilon$  and

$$\int_{\Omega \setminus F} |h| d\mu \leq \|Sx\|_p \int_{\epsilon/m}^{\infty} t^{-1/p} = \frac{p}{1-p} \left(\frac{m}{\epsilon}\right)^{1/p-1} \|Sx\|_p.$$

However

$$\begin{aligned} \int_F |h| d\mu &\leq \left(2 \min_{1 \leq i \leq m} \int_{E_i} |Sx_i| d\mu\right) \|x\| \\ &\leq 2^{-1/p} \|S\|_1 \|x\|. \end{aligned}$$

Equation (\*) now follows immediately.

To complete the proof of the lemma fix an operator  $T:X \rightarrow L_1$  and define for  $\epsilon > 0$

$$\phi(\epsilon) = \inf\{ \|P_E T\|_1 : \mu(E) \geq 1 - \epsilon \}.$$

For convenience  $\phi(\epsilon) = 0$  if  $\epsilon \geq 1$ . Let

$$A = \sup_{\epsilon > 0} \epsilon^{1/p-1} \phi(\epsilon).$$

Clearly  $A \leq \|T\|_1$ .

Now apply (\*) to  $S = P_E T$  for any  $E$  with  $\mu(E) = 1 - \epsilon$ . We decide that

$$\phi(2\epsilon) \leq 2^{-1/p} \|S\|_1 + \frac{p}{1-p} m^{1/p-1} \epsilon^{1-1/p} \|T\|_p$$

and hence that

$$\phi(2\epsilon) \leq 2^{-1/p} \phi(\epsilon) + \frac{p}{1-p} m^{1/p-1} \epsilon^{1-1/p} \|T\|_p.$$

We deduce that

$$(2\epsilon)^{1/p-1} \phi(2\epsilon) \leq \frac{1}{2} \epsilon^{1/p-1} \phi(\epsilon) + \frac{p}{1-p} (2m)^{1/p-1} \|T\|_p$$

so that

$$A \leq \frac{1}{2}A + \frac{p}{1-p}(2m)^{1/p-1}\|T\|_p.$$

Hence

$$A \leq \frac{2^{1/p}}{1-p}m^{1/p-1}\|T\|_p$$

and the lemma is proved with

$$C = \frac{2^{1/p}}{1-p}m^{1/p-1}\|T\|_p.$$

If  $X$  and  $Y$  are two quasi-Banach spaces we shall define  $\mathcal{A}(X, Y)$  to be the smallest linear subspace of  $\mathcal{L}(X, Y)$  containing the finite-rank operators and closed under pointwise convergence of uniformly bounded nets. If  $T \in \mathcal{A}(X, Y)$  we shall say that  $T$  is approximable.

**THEOREM 2.2.** *Let  $X$  be a quasi-Banach space such that  $X^*$  has finite cotype and suppose  $0 < p < 1$ . Then there is a constant  $C = C(p, X)$  so that if  $T: X \rightarrow L_p(\Omega, \Sigma, \mu)$  is a bounded operator then the following are equivalent:*

- (i)  $T$  is approximable
- (ii) For  $0 < \epsilon < 1$  there exists  $E \in \Sigma$  with  $\mu(E) \geq 1 - \epsilon$  so that  $P_E T \in \mathcal{L}(X, L_1)$  and

$$\|P_E T\|_1 \leq C\epsilon^{1-1/p}\|T\|_p.$$

- (iii) For  $0 < \epsilon < 1$  there exists  $E \in \Sigma$  with  $\mu(E) \geq 1 - \epsilon$  so that  $P_E T$  is a bounded operator of  $X$  into  $L_1$ .

*Proof.* (ii)  $\Rightarrow$  (iii). This is immediate.

(iii)  $\Rightarrow$  (ii). For  $\epsilon > 0$  select  $E_0$  so that

$$\mu(E) \geq 1 - \frac{1}{2}\epsilon$$

and  $P_{E_0} T$  maps  $X$  into  $L_1$ . Now apply Lemma 2.1 to  $P_{E_0} T$  with  $\epsilon$  replaced by  $\frac{1}{2}\epsilon$ . Condition (ii) follows with  $C$  replaced by  $2^{1/p-1}C$ .

(iii)  $\Rightarrow$  (i). Select a sequence  $E_n \in \Sigma$  so that

$$\mu(E_n) \geq 1 - \frac{1}{n} \quad \text{and} \quad P_{E_n} T \in \mathcal{L}(X, L_1).$$

Now  $L_1$  has the metric approximation property so that each  $P_{E_n} T$  is approximable in  $\mathcal{L}(X, L_1)$  and hence also in  $\mathcal{L}(X, L_p)$ . Thus  $T$  is approximable in  $\mathcal{L}(X, L_p)$ .

(i)  $\Rightarrow$  (iii). Let  $\mathcal{J}(X, L_p)$  be the set of operators  $T: X \rightarrow L_p$  so that for every  $\epsilon > 0$  we can find  $E \in \Sigma$  with

$$P_E T \in \mathcal{L}(X, L_1) \quad \text{and} \quad \mu(E) \geq 1 - \epsilon.$$

Hence  $\mathcal{J}(X, L_p)$  is a linear subspace of  $\mathcal{L}(X, L_p)$  and contains the finite-rank operators. Suppose  $T \in \mathcal{A}(X, L_p)$ ; we will show that  $T \in \mathcal{J}(X, L_p)$ . Suppose  $T_\alpha$  is a bounded net in  $\mathcal{J}(X, L_p)$  so that

$$\lim_\alpha T_\alpha x = Tx \quad \text{for } x \in X.$$

Fix  $\epsilon > 0$  and  $E_\alpha \in \Sigma$  so that  $\mu(E_\alpha) \geq 1 - \epsilon$  and

$$\|P_{E_\alpha} T_\alpha\|_1 \leq C\epsilon^{1-1/p} \|T_\alpha\|_p.$$

(using (iii) = (ii) proved above). The net  $1_{E_\alpha}$  in the unit ball of  $L_\infty(\Omega, \Sigma, \mu)$  has a weak\*-cluster point  $h$  with  $0 \leq h \leq 1$  and

$$\int h d\mu \geq 1 - \epsilon.$$

Now for each  $x \in X$

$$\int_\Omega h |Tx| d\mu \leq C\epsilon^{1-1/p} \sup_\alpha \|T_\alpha\|_p \|x\|.$$

Let  $E = \{\omega : h(\omega) \geq \frac{1}{2}\}$ . Then

$$\mu(E) \geq 1 - 2\epsilon \quad \text{and} \quad P_E T \in \mathcal{L}(X, L_1)$$

so that  $T \in \mathcal{J}(X, L_p)$ . It follows that

$$\mathcal{J}(X, L_p) \supset \mathcal{A}(X, L_p)$$

and hence (i)  $\Rightarrow$  (iii).

**3. Natural spaces and Banach envelopes.** Let  $Y$  be a quasi-Banach lattice.  $Y$  is said to be *L-convex* (or *lattice-convex*) [6] if there exists  $0 < \delta < 1$  so that if  $u \geq 0$  and  $0 \leq x_i \leq u$  ( $1 \leq i \leq n$ ) satisfy

$$\frac{1}{n}(x_1 + \dots + x_n) \geq (1 - \delta)u$$

then

$$\max_{1 \leq i \leq n} \|x_i\| \geq \delta \|u\|.$$

As pointed out in [5] those function spaces commonly studied in analysis (e.g. Orlicz spaces, Lorentz spaces, etc.) are automatically *L-convex*; one needs an ‘artificial’ construction to make non *L-convex* lattices. Motivated by this idea we shall define a (real) quasi-Banach space  $X$  to be *natural* if it

is linearly isomorphic to a closed linear subspace of an  $L$ -convex lattice. This definition is rendered consistent by Theorem 4.2 of [6] which asserts that a natural quasi-Banach lattice is always  $L$ -convex.

We now characterize natural quasi-Banach spaces. As noted in the introduction, this theorem can be used to define natural spaces.

**THEOREM 3.1.** *A quasi-Banach space  $X$  is natural if and only if there exists  $p > 0$  and  $C < \infty$  so that if  $x \in X$  with  $x \neq 0$  there exists a measure space  $(\Omega_x, \Sigma_x, \mu_x)$  with  $\mu(\Omega_x) = 1$  and an operator*

$$T_x: X \rightarrow L_p(\Omega_x, \Sigma_x, \mu_x)$$

such that

- (i)  $\|T_x\| \leq C\|x\|^{-1}$
- (ii)  $T_x x = 1_{\Omega_x}$ .

*Proof.* For the “if” direction simply note that the map

$$x \mapsto \{ \|y\| T_{y,x} \}_{y \in X, y \neq 0}$$

embeds  $X$  into an  $l^\infty$ -product of spaces  $L_p(\Omega_x, \Sigma_x, \mu_x)$  which is  $L$ -convex.

The “only if” direction is essentially proved in [6] Theorem 3.3. We can suppose that  $X$  is a closed linear subspace of an  $L$ -convex quasi-Banach lattice  $Y$ . By Theorem 2.2 of [5]  $Y$  is a  $p$ -convex lattice for some  $p > 0$  i.e., for some constant  $B$  and  $y_1, \dots, y_n \in Y$

$$\left\| \left( \sum_{i=1}^n |y_i|^p \right)^{1/p} \right\| \leq B \left( \sum_{i=1}^n \|y_i\|^p \right)^{1/p}.$$

Fix  $x \in X \subset Y$  with  $x \neq 0$ . Let  $v = |x|$  and let  $V$  be the linear span of  $[-v, v]$ . Taking  $[-v, v]$  as the unit ball,  $V$  is an abstract  $M$ -space and then may be isometrically and lattice isomorphically identified with a space  $C(\Omega_x)$ , where  $\Omega_x$  is compact Hausdorff. Let  $J: C(\Omega_x) \rightarrow V$  be the associated lattice isomorphism. Let  $\Sigma_x$  be the Borel subsets of  $\Omega_x$ .

If  $f_1, \dots, f_m \in C(\Omega_x)$  then

$$\|J(\sum |f_i|^p)^{1/p}\|^p \leq B \sum_{i=1}^n \|f_i\|^p.$$

Hence it is impossible that

$$\sum_{i=1}^m |f_i(s)|^p > B^p \|v\|^{-p} \sum_{i=1}^n \|Jf_i\|^p$$

for all  $s \in \Omega_x$ , since this would imply that

$$J(\sum |f_i|^p)^{1/p} \geq (1 + \epsilon) B \|v\|^{-1} \left( \sum_{i=1}^n \|Jf_i\|^p \right)^{1/p} v$$



for some  $\epsilon > 0$ .

Now by a Hahn-Banach separation argument there is a probability measure  $\mu_x$  on  $\Omega_x$  so that

$$\int_{\Omega_x} |f|^p d\mu_x \leq B^p \|v\|^{-p} \|Jf\|^p$$

for  $f \in C(\Omega_x)$ .

For  $y \in Y_+$  define  $Sy \in L_p(\Omega_x, \mu_x)$  by

$$Sy = \sup_n J^{-1}(y \wedge nv)$$

and extend by linearity. Then  $S$  is a bounded linear operator of  $Y$  into  $L_p(\Omega_x, \mu_x)$  with  $\|S\| \leq B\|v\|^{-1}$ .

Now define  $T_x: X \rightarrow L_p(\Omega_x, \mu_x)$  by

$$T_x z = (Sx) \cdot Sz.$$

Note that  $|Sx| = 1$ , since  $S$  is a lattice isomorphism and  $Sv = |x| = 1$ . Thus

$$\|T_x\| \leq B\|v\|^{-1} = B\|x\|^{-1} \quad \text{and} \quad T_x x = 1_{\Omega_x}.$$

We now define a complex quasi-Banach space  $X$  to be *natural* if the underlying real space is natural. Then we observe that Theorem 3.1 holds also for complex spaces. One constructs the real-linear map  $T_x$  as in Theorem 3.1 and then complexifies by setting

$$\tilde{T}_x z = T_x(z) - iT_x(iz)$$

to obtain a complex linear map

$$\tilde{T}_x: X \rightarrow L_p(\Omega_x, \Sigma_x, \mu_x).$$

Finally one sets

$$S_x(z) = (\tilde{T}_x(x))^{-1} \tilde{T}_x(z)$$

(noting that  $\text{Re } \tilde{T}_x(x) = 1_{\Omega_x}$ ).

We shall need an observation before proceeding.

LEMMA 3.2. *Suppose  $X, Y, Z$  are quasi-Banach spaces and  $S: X \rightarrow Y, T: Y \rightarrow Z$  are linear operators. If either  $S$  or  $T$  is approximable then so is  $TS$ .*

*Proof.* For example if  $S$  is approximable set  $\mathcal{J} \subset \mathcal{L}(X, Y)$  to be the set of all  $R$  so that  $TR$  is approximable in  $\mathcal{L}(X, Z)$ . Then  $\mathcal{J}$  contains the finite-rank operators and is closed under pointwise convergence of uniformly bounded nets. Thus  $S \in \mathcal{J}$ . The other case is similar.

THEOREM 3.3. *Let  $X$  and  $Y$  be quasi-Banach spaces and suppose  $X^*$  has finite cotype and  $Y$  is natural. Then there is a constant  $C$  so that if  $T: X \rightarrow Y$*

is an approximable linear operator, then

$$\|Tx\| \leq C\|T\| \|x\|_c \quad x \in X.$$

*Proof.* There exists  $B$  and  $p > 0$  so that if  $y \in Y$  and  $y \neq 0$  there is a bounded linear operator

$$S_y: Y \rightarrow L_p(\Omega, \Sigma, \mu)$$

where

$$\mu(\Omega) = 1, \quad \|S_y\| \leq B\|y\|^{-1} \quad \text{and} \quad S_y y = 1_\Omega.$$

We can use Theorem 2.2 to determine  $A$  (depending only on  $p$ ) so that if

$$R: X \rightarrow L_p(\Omega, \Sigma, \mu)$$

then there exists  $E \in \Sigma$  with  $\mu(E) \geq 1/2$  and

$$P_E R: X \rightarrow L_1(\Omega, \Sigma, \mu) \quad \text{with} \quad \|P_E R\|_1 \leq A\|R\|_p.$$

Now fix  $x \in X$ . If  $Tx = 0$  there is nothing to prove. If  $Tx \neq 0$ , there exists

$$S: Y \rightarrow L_p(\Omega, \Sigma, \mu)$$

so that

$$\|S\| \leq B\|Tx\|^{-1} \quad \text{and} \quad STx = 1_\Omega.$$

Choose  $E \in \Sigma$  so that  $P_E ST$  maps  $X$  into  $L_1(\Omega, \Sigma, \mu)$ ,

$$\|P_E ST\|_1 \leq A\|ST\|_p \quad \text{and} \quad \mu(E) \geq 1/2.$$

Now

$$\begin{aligned} \frac{1}{2} &\leq \|P_E STx\|_1 \leq \|P_E ST\|_1 \|x\|_c \\ &\leq A\|ST\|_p \|x\|_c \\ &\leq AB\|Tx\|^{-1} \|T\| \|x\|_c \end{aligned}$$

so that

$$\|Tx\| \leq 2AB\|T\| \|x\|_c.$$

**THEOREM 3.4.** *Let  $Y$  be a natural quasi-Banach space with a basis (or more generally the identity on  $Y$  is approximable). Let  $X$  be any non-locally convex subspace of  $Y$ . Then the Banach envelope of  $X$  contains uniformly complemented  $l_1^n$ 's (or equivalently  $X^*$  does not have finite cotype).*

*Proof.* The embedding  $J: X \rightarrow Y$  is approximable and the result follows from Theorem 3.3.

*Examples.* If  $X$  is a subspace of  $l_p$  or of  $H_p$  ( $0 < p < 1$ ) then  $X_c$  contains uniformly complemented  $l_1^n$ 's.

As a further example consider the space  $h_p$  ( $0 < p < 1$ ) studied by Shapiro [12]. This consists of all complex harmonic functions  $u$  on the open disc  $\Delta$  in  $\mathbb{C}$  so that

$$\|u\|_{h_p} = \sup_{0 < r < 1} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})|^p d\theta \right\}^{1/p} < \infty.$$

Shapiro shows that  $h_p$  contains many copies of  $c_0$ , and one might suspect that  $h_p$  has a non-locally convex subspace with a containing Banach space isomorphic to  $c_0$ . This is impossible however; in this case a slightly different proof is needed, since the identity on  $h_p$  is not known to be approximable.

**THEOREM 3.5.** *Let  $X$  be a closed linear subspace of  $h_p$ . If  $X$  is non-locally convex then  $X_c$  contains uniformly complemented  $l_1^n$ 's.*

*Proof.* Suppose  $X_c$  does not contain uniformly complemented  $l_1^n$ 's, and consider the map  $T_r: X \rightarrow L_p(\mathbb{T})$  given by

$$T_r u(e^{i\theta}) = u(re^{i\theta}).$$

If  $0 < r < 1$ ,  $T_r$  is approximable. In fact

$$T_r u = \lim_{n \rightarrow \infty} T_r^{[n]} u$$

where

$$T_r^{[n]} u = \sum_{|k| \leq n} \hat{u}(k) r^{|k|} e^{ik\theta}$$

where  $\{\hat{u}(k)\}_{k=-\infty}^{\infty}$  are the Fourier coefficients of  $u$ .

Thus for some constant  $C$  independent of  $r$

$$\|T_r u\|_p \leq C \|u\|_c$$

where  $\|\cdot\|_c$  is the Banach envelope norm on  $X$ . Hence

$$\|u\|_{h_p} \leq C \|u\|_c$$

i.e.,  $X$  is locally convex.

**4. A Banach envelope isomorphic to  $c_0$ .** In this section we construct a non-locally convex quasi-Banach space with an unconditional basis, whose Banach envelope is isomorphic to  $c_0$ . Of course such a space cannot be natural by Theorem 3.4.

We start with an observation used by Talagrand [16] to construct a pathological submeasure. For each  $n \in \mathbb{N}$  we can find a finite set  $\Omega$  ( $= \Omega_n$ ) and a collection of subsets  $\{A_1, \dots, A_{2n}\}$  of  $\Omega$  so that

$$(i) \quad \bigcup_{i \in J} A_i \neq \Omega$$

whenever  $J \subset \{1, 2, \dots, 2n\}$  and  $|J| < n$ , and

$$(ii) \quad (1_{A_1} + \dots + 1_{A_{2n}}) = n1_{\Omega}.$$

Indeed let  $\Omega$  consist of all subsets of  $\{1, 2, \dots, 2n\}$  of cardinality  $n$ . Let

$$A_i = \{\omega \in \Omega : i \in \omega\}.$$

Fix  $p$  with  $0 < p < 1$  and quasi-norm the finite-dimensional space  $V_n = l^\infty(\Omega)$  by taking

$$\|f\|^p = \inf \left\{ \sum |c_i|^p : |f| \leq \sum_{i=1}^{2n} c_i 1_{A_i} \right\}.$$

Let

$$\|f\|_\infty = \max_{\omega \in \Omega} |f(\omega)|.$$

Then clearly  $\|f\|_\infty \leq \|f\|$ . Suppose  $c_i \geq 0$  and

$$\sum c_i 1_{A_i} \geq 1_{\Omega}.$$

Then let  $J = \left\{ i : c_i \geq \frac{1}{2} \right\}$ . If  $|J| < n$  then we may pick

$$\omega \in \Omega \setminus \bigcup_{i \in J} A_i$$

and then

$$\sum c_i 1_{A_i}(\omega) < 1.$$

Hence  $|J| \geq n$  and so

$$\|1_{\Omega}\|^p \geq n \cdot \left(\frac{1}{2}\right)^p = \left(\frac{1}{2}\right)^p n^{1-p}$$

i.e.,

$$\|1_{\Omega}\| \geq \frac{1}{2} n^{1/p-1}.$$

Next we note that for the Banach envelope norm  $\|\cdot\|_{c,n}$

$$\|1_{\Omega}\|_{c,n} \leq \frac{1}{n} \sum_{i=1}^{2n} \|1_{A_i}\| \leq 2.$$

Clearly  $\|\cdot\|_{c,n}$  is a lattice norm on  $l^\infty(\Omega)$  and  $\|\cdot\|_{c,n} \geq \|\cdot\|_\infty$ . Hence for  $f \in l^\infty(\Omega)$

$$\|f\|_\infty \leq \|f\|_c \leq 2\|f\|_\infty.$$

We repeat this construction for each  $n$  and then form the  $c_0$ -product  $c_0(V_n)$  i.e., the space of all sequences  $f_n$  with  $f_n \in V_n, \|f_n\| \rightarrow 0$  and

$$\|(f_n)\| = \max \|f_n\|.$$

The Banach envelope norm on  $c_0(V_n)$  is clearly seen to be

$$\|(f_n)\|_c = \max \|f_n\|_{c,n}.$$

Thus the Banach envelope of  $c_0(V_n)$  is isomorphic to  $c_0$ . However the space is non-locally convex since

$$\|(0, 0, \dots, 1_{\Omega_n}, 0, \dots)\| \cong \frac{1}{2}n^{1/p-1}$$

but

$$\|(0, 0, \dots, 1_{\Omega_n}, 0, \dots)\|_c \cong 2.$$

**5. Subspaces of  $l_p$ .** We shall need the following simple lemma.

LEMMA 5.1. *Let  $(u_n:n \in \mathbf{N})$  be a normalized sequence in  $l_p$  ( $0 < p < 1$ ) with disjoint supports. Let  $M = [u_n]$  be its closed linear span. Then*

- (i)  $l_p/M$  is linearly isomorphic to a subspace of  $l_p$
- (ii) The Banach envelope of  $l_p/M$  is isomorphic to  $l_1$ .

*Proof.* We first note that (ii) is essentially trivial. The Banach envelope of  $l_p/M$  is isomorphic to  $l_1/\hat{M}$  where  $\hat{M}$  is the closure of  $M$  in  $l_1$ . However  $\hat{M}$  is a complemented subspace of  $l_1$  (by a projection of norm one) and  $l_1/\hat{M} \cong l_1$ .

To prove (i) first let  $(e_i)$  be the standard basis of  $l_p$  and

$$u_n = \sum_{i \in A_n} a_i e_i$$

where  $|a_i| > 0$  for all  $i \in A_n$  and

$$\sum_{i \in A_n} |a_i|^p = 1 \quad n = 1, 2, \dots$$

We shall define a linear map

$$S:l_p \rightarrow l_p(\mathbf{N} \times \mathbf{N}).$$

Let  $f_{ij}$  be the standard basis of  $l_p(\mathbf{N} \times \mathbf{N})$  and define  $S$  so that

$$S e_i = f_{ii} \quad \text{if } i \notin \bigcup_{n \in \mathbf{N}} A_n$$

$$S e_i = \sum_{j \in A_n} a_j f_{ij} - \sum_{j \in A_n} a_j f_{ji} \quad i \in A_n.$$

Then  $S$  extends to a bounded linear operator with  $\|S\|^p \cong 2$ . Note that  $Su_n = 0$  for each  $n$ . We shall show that

$$\inf_{v \in M} \|x - v\| = d(x, M) \leq \|Sx\|$$

for all  $x \in l_p$ .

Indeed if

$$x = \sum_{i=1}^{\infty} \xi_i e_i$$

in  $l_p$  then we may select for each  $n \in \mathbf{N}$ ,  $\eta_n$  so that

$$\left\| \sum_{i \in A_n} (\xi_i - \eta_n a_i) e_i \right\| \leq \left\| \sum_{i \in A_n} \xi_i e_i - \lambda u_n \right\|$$

for all  $\lambda \in \mathbf{R}$ .

Thus

$$\sum_{i \in A_n} |\xi_i a_j - \eta_n a_i a_j|^p \leq \sum_{i \in A_n} |\xi_i a_j - \xi_j a_i|^p.$$

Hence

$$\sum_{n=1}^{\infty} \sum_{j \in A_n} \sum_{i \in A_n} |\xi_i a_j - \eta_n a_i a_j|^p \leq \sum_{n=1}^{\infty} \sum_{j \in A_n} \sum_{i \in A_n} |\xi_i a_j - \xi_j a_i|^p.$$

Since the right-hand side is at most  $\|Sx\|^p$  we conclude that  $\sum |\eta_n|^p < \infty$  and if we set  $y = \sum \eta_n u_n$  then

$$\begin{aligned} \|x - y\|^p &\leq \sum_{i \notin \cup A_n} |\xi_i|^p + \sum_{n=1}^{\infty} \sum_{j \in A_n} \sum_{i \in A_n} |\xi_i a_j - \xi_j a_i|^p \\ &\leq \|Sx\|^p. \end{aligned}$$

Thus  $d(x, M) \leq \|Sx\|$  and  $S$  factors to an embedding of  $l_p/M$  into

$$l_p(\mathbf{N} \times \mathbf{N}) \cong l_p.$$

**THEOREM 5.2.** *Let  $Z$  be a separable Banach space and suppose  $0 < p < 1$ . Then there is a closed subspace  $X$  of  $l_p$  whose Banach envelope is linearly isomorphic to  $l_1 \oplus Z$ .*

*Proof.* Let  $(e_i)$  be the standard basis of  $l_p = l_p(\mathbf{N})$  and let  $(f_{ij})$  be the standard basis of  $l_p(\mathbf{N} \times \mathbf{N})$  and, for convenience, we take the quasi-norm on the direct sum to be

$$\|(x, y)\| = \max(\|x\|, \|y\|).$$

Choose an integer  $\alpha$  so that

$$\alpha \left( \frac{1}{p} - 1 \right) > 3 \quad \text{and} \quad d = 2^\alpha.$$

Write  $\mathbf{N}$  as a disjoint union  $\bigcup_{n \in \mathbf{N}} A_n$  where  $|A_n| = d^n$ , and define the normalized sequence

$$u_n = d^{-n/p} \sum_{i \in A_n} e_i.$$

Select a total fundamental biorthogonal system  $(z_n, z_n^*)$  for  $Z$  so that  $\|z_n\| = 1$  ( $n \in \mathbf{N}$ ) and  $\|z_n^*\| \leq 2$  for  $n \in \mathbf{N}$  (see [11]). We define  $K: [u_n] \rightarrow Z$  to be a bounded linear map such that

$$Ku_n = 2^{-n} z_n.$$

Note that  $\|K\| \leq 1$  and that  $K$  is one-one. In fact if  $x \in [u_n]$ , say

$$x = \sum_{n=1}^{\infty} \xi_n u_n$$

then

$$|z_n^*(Kx)| = 2^{-n} |\xi_n| \leq 2 \|Kx\|$$

so that

$$|\xi_n| \leq 2^{n+1} \|Kx\|.$$

Now let  $\|\cdot\|_1$  be the  $l_1$ -norm on  $l_p(\mathbf{N})$ . Then

$$\begin{aligned} \|x\|_1 &= \sum_{n=1}^{\infty} |\xi_n| d^{n(1-1/p)} \\ &\leq 2 \|Kx\| \sum_{n=1}^{\infty} 2^n \cdot 2^{an(1-(1/p))} \\ &\leq 2 \|Kx\| \sum_{n=1}^{\infty} 4^{-n} \\ &\leq \frac{2}{3} \|Kx\|. \end{aligned}$$

Now let  $(y_i: i \in \mathbf{N})$  be any sequence dense in the unit sphere  $\{x: \|x\| = 1\}$  of  $[u_n]$ . Then

$$y_i = \sum_{j=1}^{\infty} a_{ij} e_j$$

where

$$\sum_i |a_{ij}|^p = 1.$$

Let

$$v_i = \sum_{j=1}^{\infty} a_{ij}f_{ij}$$

in  $l_p(\mathbf{N} \times \mathbf{N})$  and define for  $(i, j) \in \mathbf{N} \times \mathbf{N}$

$$w_{ij} = f_{ij} + \beta_i e_j$$

where

$$\beta_i = \|y_i\|_1 / \|Ky_i\|.$$

Note  $0 < \beta_i \leq 2/3$  for all  $i \in \mathbf{N}$  and hence that  $[w_{ij}]$  is isometric to  $l_p$ .

Let  $M = [v_n]$ , and consider the quotient  $l_p \oplus l_p(\mathbf{N} \times \mathbf{N})/M$ . By Lemma 5.1 this quotient is isomorphic to a subspace of  $l_p$ . Let  $q$  be the quotient mapping and  $X$  be the closed linear span of

$$\{q(w_{ij}): i \in \mathbf{N}, j \in \mathbf{N}\}.$$

$X$  will be our example; clearly  $X$  embeds into  $l_p$ .

Note first that for fixed  $i$

$$\sum_j a_{ij}w_{ij} = \sum_j a_{ij}(f_{ij} + \beta_i e_j) = v_i + \beta_i y_i.$$

Thus  $X$  contains a subspace

$$X_0 = [q(y_n)] = [q(u_n)].$$

We first identify the quotient space  $X/X_0$ . To do this note that

$$q^{-1}(X_0) = [u_n] \oplus [y_n] \subset l_p \oplus l_p(\mathbf{N}^2).$$

Now suppose

$$x = \sum_{i,j} \xi_{ij}w_{ij} \in [w_{ij}]$$

and that for some  $c_n, \sum |c_n|^p < \infty$

$$\left| \left| \sum_{i,j} \xi_{ij}f_{ij} - \sum_n c_n y_n \right| \right| < 1.$$

We shall show that

$$\left| \left| \sum_{i,j} \xi_{ij}w_{ij} - \sum_n c_n (y_n + \beta_n y_n) \right| \right| < 1.$$

Indeed we have

$$\sum_{i,j} |\xi_{ij} - c_i a_{ij}|^p < 1$$



and hence

$$\left\| \sum_{i,j} (\xi_{ij} - c_i a_{ij}) \beta_i e_j \right\| < 1$$

i.e.,

$$\left\| \sum_{i,j} \xi_{ij} \beta_i e_j - \sum_n c_n \beta_n y_n \right\| < 1$$

as required.

Now suppose  $x \in [w_{ij}]$  and  $d(x, [u_n] \oplus [v_n]) < 1$ . Then by the above

$$d(x, [v_n + \beta_n y_n]) < 1.$$

However  $v_n + \beta_n y_n$  is a sequence with disjoint support with respect to  $(w_{ij})$ . Hence  $[w_{ij}]/[v_n + \beta_n y_n]$  has a Banach envelope isomorphic to  $l_1$ . It follows that  $X/X_0$  is isomorphic to  $[w_{ij}]/[v_n + \beta_n y_n]$  and hence embeds into  $l_p$  and has a Banach envelope isomorphic to  $l_1$ .  $X_0$  is then weakly closed in  $X$  ( $X/X_0$  has a separating dual).

Let  $\tilde{X}_0$  be the weak closure of  $X_0$  in  $X_c$ . Then

$$X_c/\tilde{X}_0 \cong (X/X_0)_c \cong l_1,$$

and hence

$$X_c \cong l_1 \oplus \tilde{X}_0.$$

It remains to show that  $\tilde{X}_0 \cong Z$ .

To do this we define a bounded linear map  $L_0: [w_{ij}] \rightarrow Z$  so that

$$L_0(w_{ij}) = (\text{sgn } a_{ij}) \frac{\beta_i}{\|y_i\|_1} Ky_i.$$

To see  $L$  is bounded we need only note that

$$\|L_0(w_{ij})\| = \frac{\beta_i}{\|y_i\|_1} \|Ky_i\| = 1$$

for all  $i, j$ .

Suppose

$$x = \sum \xi_{ij} w_{ij} \in [w_{ij}] \quad \text{and} \quad d(x, M) < 1.$$

Then there exist  $c_i, \sum |c_i|^p < 1$  so that

$$\|x - \sum c_i v_i\| < 1.$$

Hence

$$\left\| \sum_{i,j} \xi_{ij} f_{ij} - \sum c_i v_i \right\| < 1$$

and, as before,

$$\|x - \sum c_i(v_i + \beta_i y_i)\| < 1.$$

Thus

$$\|\sum c_i \beta_i K y_i\| < 2^{1/p}.$$

However

$$\begin{aligned} L_0(v_i + \beta_i y_i) &= \sum_j a_{ij} L_0(w_{ij}) \\ &= \left( \sum_j |a_{ij}| \right) \frac{\beta_i}{\|y_i\|_1} K y_i \\ &= \beta_i K y_i \end{aligned}$$

$$\text{since } \|y_i\|_1 = \sum_j |a_{ij}|.$$

Thus

$$\|L_0(\sum c_i(v_i + \beta_i y_i))\| < 2^{1/p}$$

and hence

$$\|L_0 x\| < 3^{1/p}.$$

In general for  $x \in [w_{ij}]$ ,

$$\|L_0(x)\| \leq 3^{1/p} \cdot d(x, M).$$

Hence  $L_0$  factors to a linear operator  $L: X \rightarrow Z$  with  $\|L\| \leq 3^{1/p}$ . As  $Z$  is a Banach space

$$\|Lx\| \leq 3^{1/p} \|x\|_c.$$

Now

$$\begin{aligned} Lq(y_i) &= Lq(y_i + \beta_i^{-1} v_i) \\ &= L_0(y_i + \beta_i^{-1} v_i) = K y_i. \end{aligned}$$

Hence

$$\|Kqy_i\| \leq 3^{1/p} \|q(y_i)\|_c.$$

Extending by continuity, if  $x \in [u_n]$

$$\|Kx\| \leq 3^{1/p} \|q(x)\|_c.$$

Conversely

$$\begin{aligned} \|q(y_i)\|_c &= \beta_i^{-1} \|q(v_i + \beta_i y_i)\|_c \\ &\leq \beta_i^{-1} \sum_j |a_{ij}| \|q(w_{ij})\|_c \\ &\leq \beta_i^{-1} \|y_i\|_1 = \|K y_i\|. \end{aligned}$$

Hence for  $x \in [u_n]$

$$\|q(x)\|_c \leq \|Kx\|.$$

Thus

$$3^{-1/p}\|Kx\| \leq \|q(x)\|_c \leq \|Kx\| \quad \text{for } x \in X_0$$

so that  $\tilde{X}_0 \cong Z$  (since the range of  $K$  is dense).

**6. Spaces with unconditional bases.** In this section we give some special results for spaces with unconditional bases.

**THEOREM 6.1.** *Let  $X$  be a quasi-Banach space with an unconditional basis. Then  $X$  contains a complemented subspace isomorphic to  $c_0$  if and only if  $X_c$  contains a copy of  $c_0$ .*

*Proof.* Suppose  $(e_n)$  is a normalized unconditional basis of  $X$  with unconditional basis constant  $K$  so that if  $N \in \mathbf{N}$ ,  $a_1, \dots, a_n \in \mathbf{R}$  and  $|c_i| \leq 1$  ( $1 \leq i \leq N$ ) then

$$\left\| \sum_{i=1}^N c_i a_i e_i \right\| \leq K \left\| \sum_{i=1}^N a_i e_i \right\|.$$

Let  $e_n^*$  be the biorthogonal functionals in  $X^*$ . Then

$$\|c_n^*\| \leq K \quad \text{and} \quad \|e_n\|_c \geq K^{-1}.$$

Suppose  $X_c$  contains a copy of  $c_0$ . Then by a gliding hump argument we can find a block basic sequence  $(u_n)$  of the form

$$u_n = \sum_{k_{n-1}+1}^{k_n} b_i e_i$$

(where  $k_0 = 0 < k_1 < k_2 \dots$ ) and so that for some  $0 < C < \infty$  and all  $a_1, \dots, a_n \in \mathbf{R}$ ,  $N \in \mathbf{N}$  we have

$$C^{-1} \max_{1 \leq i \leq N} |a_i| \leq \left\| \sum_{i=1}^n a_i u_i \right\|_c \leq C \max_{1 \leq i \leq N} |a_i|.$$

We may also determine

$$u_n^* = \sum_{k_{n-1}+1}^{k_n} c_i e_i^*$$

so that

$$\sup \|u_n^*\| = \beta < \infty \quad \text{and} \quad u_n^*(u_n) = 1.$$

Let  $\Delta = [-2CK, 2CK]^N$ ,  $\Delta$  is a compact metrizable space. For  $\xi = (\xi_i) \in \Delta$  define  $\phi_n(\xi) \in X$  by

$$\phi_n(\xi) = \sum_{i=k_{n-1}+1}^{k_n} \xi_i e_i.$$

Let

$$\Delta_0 = \{\xi \in \Delta : \|\phi_1(\xi) + \dots + \phi_n(\xi)\| \leq 2CK, n \in \mathbf{N}\}.$$

$\Delta_0$  is a closed subset of  $\Delta$ .

For each  $n \in \mathbf{N}$ ,

$$\|u_1 + \dots + u_n\|_c \leq C$$

and so we can express  $u_1 + \dots + u_n$  as a convex combination of elements of the unit ball of  $X$ . Identify  $x \in X$  such that  $\|x\| \leq 1$  with  $\xi \in \Delta$  so that  $x = \sum \xi_i e_i$ . Thus we can find a probability measure  $\mu_n$  on  $\Delta$  whose support is finite so that if  $1 \leq m \leq n$

$$\int \phi_m(\xi) d\mu_n(\xi) = u_m.$$

(The integral is well-defined in the finite-dimensional space  $[e_{k_{m-1}+1}, \dots, e_{k_m}]$ ). Furthermore  $\mu_n$  is supported on  $\Delta_0$ .

Let  $\mu$  be any weak\*-cluster point of  $(\mu_n : n \in \mathbf{N})$  in  $C(\Delta)^*$ .  $\mu$  is also supported on  $\Delta_0$  and

$$\int \phi_m(\xi) d\mu(\xi) = u_m \quad m \in \mathbf{N}.$$

Thus

$$\int u_m^*(\phi_m(\xi)) d\mu(\xi) = 1$$

and as

$$|u_m^*(\phi_m(\xi))| \leq 2BCK,$$

we can conclude by the Bounded Convergence Theorem that there exists  $m_k \uparrow \infty$  and  $\xi \in \Delta_0$  so that

$$u_{m_k}^*(\phi_{m_k}(\xi)) = \beta_k > \gamma > 0.$$

Let

$$v_k = \beta_k^{-1} \phi_{m_k}(\xi).$$

Then  $v_k$  is an unconditional basic sequence in  $X$ .

We also have

$$\begin{aligned} \|a_1 v_1 + \dots + a_k v_k\| &\leq \gamma^{-1} K \max |a_k| \left\| \sum_{i=1}^{m_k} \phi_i(\xi) \right\| \\ &\leq 2CK^2 \gamma^{-1} \max |a_k|. \end{aligned}$$

Now  $u_{m_k}^*(v_k) = 1$  so that  $\|v_k\| \cong B^{-1}$  so that  $(v_k)$  is equivalent to the usual  $c_0$ -basis and is complemented by the projection

$$Px = \sum_{k=1}^{\infty} u_{m_k}^*(x)v_k.$$

In fact

$$\|P\| \leq 2BCK^2\gamma^{-1}.$$

The converse is trivial; if  $X \cong c_0 \oplus Y$  then  $X_c \cong c_0 \oplus Y_c$ .

If we further assume  $X_c \cong c_0$  then since  $(e_n)$  is also an unconditional basis for  $X_c$  with

$$K^{-1} \leq \|e_n\|_c \leq 1,$$

then by a theorem of Lindenstrauss and Pelczynski [7]  $(e_n)$  is actually equivalent to the usual  $c_0$ -basis in  $X_c$ . Thus in the preceding proof we can take  $u_n = e_n$ , and each  $v_k$  is a multiple of  $e_{m_k}$ . Thus we have also proved:

**THEOREM 6.2.** *Let  $X$  be a quasi-Banach space with normalized unconditional basis  $(e_n)$ . If  $X_c \cong c_0$ , then some subsequence of  $(e_n)$  is equivalent (in  $X$ ) to the standard basis of  $c_0$ .*

**COROLLARY 6.3.** *Let  $X$  be a quasi-Banach space with a symmetric basis. If  $X_c \cong c_0$ , then  $X \cong c_0$ .*

We remark that the example constructed in Section 4 has an unconditional basis and  $X_c \cong c_0$ , but is not locally convex. We also remark that it is easy to find spaces  $X$  with unconditional basis which contain  $c_0$ , but so that  $X_c$  does not contain  $c_0$ . For example let  $X$  be the “weak”  $l^p$ -sequence space ( $0 < p < 1$ ) of all sequences  $x_n$  so that the decreasing rearrangement  $x_n^*$  satisfies  $n^{1/p}x_n^* \rightarrow 0$ . This is quasi-normed by

$$\|x\| = \sup_n n^{1/p}x_n^*.$$

Then  $X$  has an unconditional basis and contains  $c_0$ , but  $X_c \cong l_1$ .

Finally we conclude with a theorem suggested by the example in Section 5.

**THEOREM 6.4.** *Let  $X$  be a quasi-Banach space with an unconditional basis. Suppose  $X_c \cong l_1 \oplus Y$ . Then*

(i) *If  $Y$  does not contain  $l_1^n$ 's uniformly, then  $X$  contains a complemented subspace isomorphic to  $Y$ .*

(ii) If  $Y$  does not contain uniformly complemented  $l_1^n$ 's and  $X$  is natural, then  $X$  contains a complemented copy of  $Y$ .

*Proof.* In either case every operator from  $Y$  into  $l_1$  is compact. Let  $(e_n)$  be the unconditional basis of  $X$ . Then  $(e_n)$  is also an unconditional basis of  $X_c \cong l_1 \oplus Y$  and also by a theorem of Wojtaszyk [17] there is an infinite subset  $[e_n : n \in M]$  which is an unconditional basis of  $Y$ . Then  $[e_n : n \in M]$  spans a subspace  $X_0$  of  $X$  which is complemented and such that  $(X_0)_c \cong Y$ . Now apply either Theorem 1.1. or Theorem 3.4, to conclude that  $X_0 \cong Y$ .

Thus the examples in Theorem 6.2 do not have an unconditional basis in the case when  $Z$  does not contain uniformly complemented  $l_1^n$ 's.

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