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A GENERAL FORM OF THE FUNCTIONAL LIL FOR BANACH-VALUED BROWNIAN MOTION

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1. Introduction. In a recent paper [**12**], C. Mueller proved a general version of the functional LIL which unifies Strassen's LIL and the Lévy modulus of continuity for Brownian motion $W(t)$. His theorem also contains other known forms of the LIL.

For each $t \geqq 0$, let $\mathscr{P}_t$ be a family of points in the first quadrant of the plane. Let $r > 0$; to each point (s_0, l_0), we associate a rectangle

$$R_r(s_0, l_0) = \{ (s, l) \,|l_0e^{-r} \leqq l \leqq l_0e^r, |s - s_0| \leqq l_0r\}.$$

Define $A_r(t)$ to be the area of the union of these rectangles up to time t under the measure $\dfrac{dsdl}{l^2}$. Then, Theorem 1 [**12**, p. 166] states that for an increasing function h such that

$$\inf \{a > 0| \int_0^\infty e^{-ah(t)}dA_1(t) < \infty\} = 1;$$

the set of limit points of

$$C(t) = \left\{ f_{s,l}(x) = \frac{W(s + xl) - W(s)}{\sqrt{l}} \,\middle|\, (s, l) \in \mathscr{P}_t \right\}$$

in $C[0, 1]$ is the closed unit ball of the reproducing kernel Hilbert space (rkhs) associated with Wiener measure.

The proof given in [**12**] does not generalize easily to Banach space-valued Brownian motion. Furthermore, the above function $A_r(t)$ is not easy to compute even in the simplest cases. In this paper, we prove the above result for the Banach space-valued Brownian motion in the form first studied by Bulinskii [**1**] (also used by the author in [**17**]). He proved that for an increasing function h and if

$$R = \inf \left\{ a > 0 \,\middle|\, \sum_k e^{-ah(c^k)} < \infty, c > 1 \right\}.$$

then the set of limit points of the sequence

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$$g_n(\cdot) = \frac{W(n\cdot)}{\sqrt{2nh(n)}}$$

in $C[0, 1]$ is the closed ball $K_{\sqrt{R}}$ of radius $\sqrt{R}$ $(= C[0, 1]$ if $R = \infty)$ in the rkhs for Wiener measure.

Our proof uses a rectangular exponential grid in the first quadrant of the plane and a sequence $(s_k, l_k) \in \mathscr{P}_{t_k}$; t_k being the first t when $\mathscr{P}_t$ first exits the k^{th} rectangle. If W is a B-valued Brownian motion, then we prove that the sequence

$$\left\{ f_k(x) = \frac{W(s_k + xl_k) - W(s_k)}{\sqrt{l_k}} \right\}$$

is "asymptotically independent" in the sense of Nisio [**13**]. Consequently, if

$$R = \limsup \frac{\log k}{h(t_k)} < \infty,$$

by a theorem analogous to Theorem 4.2 of Carmona-Kôno [**2**] the sequence

$$\left\{ \frac{f_k}{\sqrt{2h(t_k)}} \right\}$$

satisfies the LIL, with the set of limit points being the closed ball $K_{\sqrt{R}}$ in the corresponding rkhs.

To complete the proof, we show that the other

$$\{f_{sl} | (s, l) \in \mathscr{P}_t\}$$

can be controlled. The criterion

$$R = \limsup \frac{\log k}{h(t_k)}$$

proves to be easier to work with as we shall see in some examples. When $R = \infty$, this proof has to be modified because we use the fact that $K_{\sqrt{R}}$ is compact for finite R.

Section 2 introduces the machinery we need for "asymptotically independent" Gaussian sequences. We prove a generalized form of Nisio's lemma [**13**]. As a corollary, we get an improvement of a lemma of Lai [**10**] and Pathak-Qualls [**14**].

Section 3 contains the main result for a B-valued Brownian motion when $R < \infty$.

Section 4 takes up the case $R = \infty$. Here our proof follows the same lines as that of Bulinskii and uses the Haar basis for Wiener measure.

2. Preliminaries. Let $\{\alpha_n | n \geqq 1\}$ be a positive non-decreasing sequence. Set

$$R(\alpha_n) = \inf \{a > 0 | \sum_{n=1}^{\infty} e^{-a\alpha_n} < \infty\}.$$

The first lemma gives another characterization of $R(\alpha_n)$.

LEMMA 1. $R(\alpha_n) = \limsup_j \dfrac{\log j}{\alpha_j}$.

Proof. Let

$$r = \limsup_j \frac{\log j}{\alpha_j}.$$

First suppose $r < \infty$, and $a > r$. There exists $\epsilon > 0$ for which $a > r(1 + \epsilon)$ and this shows that

$$\sum e^{-a\alpha_n} < \infty.$$

Therefore $R(\alpha_n) \leqq r$.

Conversely, suppose

$$\sum_n e^{-a\alpha_n} < \infty$$

for some finite positive a. Since $e^{-a\alpha_n}$ decreases, and $\sum e^{-a\alpha_n}$ converges it is easy to show that $ne^{-\alpha_n} \rightarrow 0$. Suppose further that $a < r$, then there exists a sequence $\{n_j\}$ such that

$$a\alpha_{n_j} < \log n_j \quad \text{or} \quad e^{a\alpha_{n_j}} < n_j.$$

This is a contradiction. Therefore either

$$r = \infty \quad \text{and} \quad \sum e^{-a\alpha_n} = \infty \text{ for all } a$$

or

$$r < \infty \quad \text{then} \quad \sum e^{-a\alpha_n} < \infty \text{ implies } a \geqq r.$$

This completes the proof.

In [**13**], M. Nisio studied what was described later as "asymptotically independent" Gaussian sequences $\{\xi_n\}$, that is, for which

$$\limsup_{\substack{m-n\rightarrow\infty \\ n\rightarrow\infty}} E(\xi_n\xi_m) \leqq 0.$$

For the proof of our main theorem in the next section, we need an extension of Theorem 2 in [**13**] to sequences satisfying a slightly weaker condition. The proof uses a lemma of Slepian and the Borel-Cantelli lemma.

LEMMA 2. *Let* $\{\xi_n\}$ *be a mean-zero Gaussian sequence with* $E(\xi_n^2) = \sigma^2$; *and let* $\{\alpha_n\}$ *be a positive non-decreasing sequence with* $R(\alpha_n) = R < \infty$. *Suppose further that the following condition* (N) *is satisfied:*

For every $\epsilon > 0$, *there exists a subsequence* $\{\xi_{n_j}\}$ *of* $\{\xi_n\}$ *such that*

$$E(\xi_{n_j}\xi_{n_k}) \leqq \epsilon \quad \text{whenever } j \neq k$$

and

$$R(\alpha_{n_j}) = R.$$

Then

$$\limsup \frac{\xi_n}{\sqrt{2\alpha_n}} = \sigma\sqrt{R} \quad \text{a.s.}$$

Proof. Without loss of generality we are going to suppose $\sigma = 1$. If $R = 0$ then for any $\epsilon > 0$.

$$\sum_{n=1}^{\infty} P\left[\frac{|\xi_n|}{\sqrt{2\alpha_n}} > \epsilon\right] \leqq \sum_{n=1}^{\infty} \frac{e^{-\epsilon^2\alpha_n}}{\epsilon\sqrt{2\alpha_n}} < \infty.$$

By the Borel-Cantelli lemma

$$\limsup \frac{\xi_n}{\sqrt{2\alpha_n}} = 0 \quad \text{a.s.}$$

Let $0 < R < \infty$, for $\delta > 0$ the Borel-Cantelli lemma again implies that

$$\limsup \frac{\xi_n}{\sqrt{2\alpha_n}} \leqq \sqrt{R}(1 + \delta) \quad \text{a.s.}$$

For the reverse inequality, choose $0 < \epsilon < 1/2$ and $\beta > 1$ such that

$$\frac{1-\epsilon}{\epsilon}\frac{\delta^2}{4} > 1 \quad \text{and} \quad \beta\left(1 - \frac{\delta}{4}\right)^2 < 1.$$

Condition (N) implies the existence of a sequence $\{n_j\}$ such that

$$E(\xi_{n_j}\xi_{n_k}) \leqq \epsilon \quad \text{for } j \neq k$$

and

$$R(\alpha_{n_j}) = \limsup \frac{\log j}{\alpha_{n_j}} = R.$$

Choose a further subsequence $\{n_{j_k}\}$ of $\{n_j\}$ such that

$$j_k > 2j_{k-1} \quad \text{and} \quad \log j_k \geqq R\alpha_{n_{j_k}}\beta\left(1 - \frac{\delta}{4}\right)^2.$$

We now follow Nisio's proof using Slepian's lemma.

Let $\{X_0, X_1, X_2, \dots\}$ be an independent mean-zero Gaussian sequence with

$$E(X_0^2) = \epsilon \quad \text{and} \quad E(X_i^2) = 1 - \epsilon, i > 1.$$

Set $Y_i = X_0 + X_i$, $i > 1$. Since $EY_k^2 = 1$,

$$E(Y_k Y_j) \geqq E(\xi_{n_k} \xi_{n_j}).$$

Therefore, by a lemma of Slepian [**15**]

$$P[\max_{j_k \leqq j \leqq j_{k+1}} \xi_{n_j} \leqq c] \leqq P[\max_{j_k \leqq j \leqq j_{k+1}} Y_j \leqq c]$$

for every $c > 0$.

$$\begin{aligned}
P[\max_{j_k \leqq j \leqq j_{k+1}} \xi_{n_j} &\leqq (1 - \delta)\sqrt{R(1 - \epsilon)2\alpha_{n_{j_{k+1}}}}] \\
&\leqq P[\max_{j_k \leqq j \leqq j_{k+1}} Y_j \leqq (1 - \delta)\sqrt{R(1 - \epsilon)2\alpha_{n_{j_{k+1}}}}] \\
&\leqq P[X_0 \leqq -\frac{\delta}{2} \sqrt{R(1 - \epsilon)2\alpha_{n_{j_{k+1}}}}] \\
&\quad + P[\max_{j_k \leqq j \leqq j_{k+1}} X_j \leqq (1 - \frac{\delta}{2}) \sqrt{R(1 - \epsilon)2\alpha_{n_{j_{k+1}}}}] \\
&= (\mathrm{I}) + (\mathrm{II}).
\end{aligned}$$

(I) is bounded by the general term of a convergent series because

$$\frac{\delta^2}{4} \cdot \frac{1 - \epsilon}{\epsilon} > 1.$$

$(\mathrm{II}) \leqq (1 - p_k)^{j_{k+1} - j_k}$ *where*

$$\begin{aligned}
p_k &= P\left[\frac{X_k}{\sqrt{1 - \epsilon}} > \left(1 - \frac{\delta}{2}\right)\sqrt{2R\alpha_{n_{j_{k+1}}}}\right] \\
&\leqq e^{-(j_{k+1} - j_k)p_k} \\
&\leqq \exp\left(-e^{\log(j_{k+1} - j_k) - (1 - (\delta/4))^2 R\alpha_{n_{j_{k+1}}}}\right)
\end{aligned}$$

using standard estimates

$$\leqq \exp\left(-\frac{(j_{k+1})^c}{2}\right)$$

for some positive c, because of the choice of the subsequence $\{j_k\}$.

The latter being the general term of a convergent series; by the Borel-Cantelli lemma, the proof is complete.

COROLLARY 1. (Nisio) *For a mean-zero Gaussian sequence* $\{\xi_n\}$ *if*

$$\limsup_{\substack{m\to\infty \\ n-m\to\infty}} E(\xi_n\xi_m) \leqq 0,$$

then condition (N) *is satisfied.*

Therefore

$$\limsup \frac{\xi_n}{\sqrt{2\alpha_n}} = \sigma\sqrt{R} \quad \text{a.s.}$$

if $E(\xi_n^2) = \sigma^2$.

Proof. By hypothesis, if $\epsilon > 0$ is given, then there exist integers m_0 and k_0 such that

$$E(\xi_{m+jk_0}\xi_{m+ik_0}) \leqq \epsilon \quad \text{for } m \geqq m_0.$$

It is clear that for some m_1 with $m_0 \leqq m_1 \leqq m_0 + k_0$

$$\limsup \frac{\log j}{\alpha_{m_1+jk_0}} = R.$$

If, in the above, $R(\alpha_n) = \infty$, N. Kôno has proved (an unpublished result) that Nisio's lemma is still true. The proof of the following corollary follows his idea.

COROLLARY 2. (N. Kôno) *If in Lemma* 2, $R(\alpha_n) = \infty$, *then*

$$\limsup \frac{\xi_n}{\sqrt{2\alpha_n}} = \infty \quad \text{a.s.}$$

Proof. Since $R(\alpha_n) = \infty$, for $0 < \delta < 1$ if

$$\overline{\alpha_n} = \max \{\delta \log n, \alpha_n\}$$

then

$$\limsup_n \frac{\log n}{\bar{\alpha}_n} = \frac{1}{\delta}.$$

Consequently, by Lemma 2,

$$\limsup \frac{\xi_n}{\sqrt{2\bar{\alpha}_n}} = \frac{\sigma}{\sqrt{\delta}} \quad \text{a.s.}$$

Thus, for almost all ω,

$$\limsup_n \frac{\xi_n(\omega)}{\sqrt{2\alpha_n}} \geqq \limsup \frac{\xi_n(\omega)}{\sqrt{2\bar{\alpha}_n}} = \frac{\sigma}{\sqrt{\delta}}.$$

Since δ is arbitrary, the proof is complete.

An easy application of the above gives the following.

COROLLARY 3. *Let* $\{\xi_k\}$ *be a stationary zero-mean Gaussian sequence such that*

$$E(\xi_k^2) = 1 \text{ and } E(\xi_1\xi_k) \to 0.$$

Let $\{\alpha_k\}$ *be a positive non-decreasing sequence. Then*

$$P[\xi_k > \sqrt{2\alpha_k} \text{ i.o.}] = 0 \text{ or } 1$$

according as

$$\sum_1^\infty \alpha_k^{-\frac{1}{2}} e^{-\alpha_k} < \infty \text{ or } = \infty.$$

Corollary 3 improves the result of Lai [**10**, Corollary 2, p. 835] and Pathak and Qualls [**14**]. They proved Corollary 3 with the stronger assumption that

$$E(\xi_1\xi_k) = O\left(\frac{1}{\log k}\right).$$

3. A general LIL in Banach spaces. Throughout this section, let B denote a real separable Banach space, B^* its dual. $X:\Omega \to B$ is a mean-zero Gaussian random variable if $x^*(X)$ is a mean-zero Gaussian for each $x^* \in B^*$. We will suppose that the support of $\mathscr{L}(X)$ is B itself.

We list a few well-known results concerning these B-valued random variables. For details, see [**2**], [**6**], [**8**] and [**9**].

(i) For a semi-norm $|||\cdot|||$ on B, using Fernique's theorem [**6**] we get

$$P[\,|||X||| > t(E|||X|||^2)^{\frac{1}{2}}] \leqq \exp\left(-\frac{t^2}{96}\log 3\right)$$

if $t \geqq 2$. (See [**2**].)

(ii) The formula $Sx^* = E(x^*(X)\cdot X)$, $x^* \in B^*$ defines a continuous linear map from B^* into B, and the completion of SB^*, equipped with the inner product:

$$\langle Sx^*, Sy^*\rangle = E(x^*(X)y^*(X))$$

is a Hilbert space. It is the reproducing kernel Hilbert space (rkhs) determined by X and can be identified with a dense subspace of B. S will be called the canonical embedding of B^* into B associated to X or $\mathscr{L}(X)$.

(iii) The closed ball K_r of radius r in H is compact when considered as a subset of B.

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(iv) There exist biorthornormal sequences $\{e_j^*\} \subset B^*$ and $\{e_j = Se_j^*\} \subset H$ such that

(a) $\{e_j\}$ is an orthonormal basis of H.

(b) $\{e_j^*(X)\}$ form an independent standard Gaussian sequence.

(c) If $x \in B$, set

$$P_n x = \sum_{j=1}^{n} \langle e_j^*, x \rangle e_j \quad \text{and} \quad Q_n = \text{Id} - P_n$$

then

$$P_n X \to X \quad \text{a.s.} \quad \text{and} \quad E(\|Q_n X\|^2) \to 0.$$

(v) For a given Gaussian measure μ on B, there exists a B-valued stochastic process $\{W(t), t \geqq 0\}$ such that $W(0) = 0$, the distribution of $W(1)$ is μ, W has stationary independent increments and the distribution of $t^{-\frac{1}{2}}W(t)$ is μ. Furthermore the sample paths of W are continuous. W is called μ-Brownian motion.

(vi) W defines a new mean-zero Gaussian random variable $\underline{W}$ on

$$C_B[0, 1] = \{\phi: [0, 1] \to B, \phi \text{ continuous}, \phi(0) = 0\};$$

namely $\underline{W}(\omega)(t) = W(t)(\omega)$. For $\underline{W}$ the corresponding rkhs H is given by $H_0 \otimes H_\mu$ where

H_μ = rkhs determined by μ.

H_0 = rkhs determined by Wiener measure in $\mathscr{C}[0, 1]$.

As an application of (i) we establish the following lemma which will be useful later. It is the analogue of Lemma 1 in [**12**, p. 166].

LEMMA 3. *If* $0 < \epsilon < 1$, *then for L sufficiently large*

$$P[\sup_{\substack{1 \leqq a \leqq 2 \\ -\epsilon \leqq \Delta \leqq \epsilon}} \|W(a + \Delta) - W(a)\|_B > L]$$

$$\leqq 3\epsilon^{-1} \exp\left(-\frac{L^2}{8\epsilon} \cdot \frac{\log 3}{96d}\right)$$

where

$$d = E(\|\underline{W}\|_{C_B}^2).$$

Proof. Break [0, 1] into intervals of length $1/N$ where

$$\frac{1}{N} \leqq \epsilon \leqq \frac{1}{N - 1}.$$

Then

$$P[\sup_{\substack{1 \leqq a \leqq 2 \\ -\epsilon \leqq \Delta \leqq \epsilon}} \|W(a + \Delta) - W(a)\|_B > L]$$

$$\leqq P\left[\sup_{\substack{0 \leqq n \leqq N \\ -\epsilon \leqq \Delta \leqq \epsilon}} \left|\left| W\left(1 + \frac{n}{N} + \Delta\right) - W\left(1 + \frac{n}{N}\right) \right|\right|_B > \frac{L}{2}\right]$$

$$\leqq (N + 1) P\left[\sup_{0 \leqq \Delta \leqq 2\epsilon} ||W(\Delta)||_B > \frac{L}{2}\right]$$

$$\leqq 3\epsilon^{-1} P\left[||Y||_{C_B[0,2\epsilon]} > \frac{L}{2}\right]$$

(where

$$Y:\Omega \to C_B[0, 2\epsilon]$$
$$\omega \to \{t \to W(t)\})$$

$$\leqq 3\epsilon^{-1}\exp\left[-\frac{L^2}{4}\,\frac{\log 3}{E||Y||^2\ \ 96}\right]$$

(using (i) for sufficiently large L)

$$\leqq 3\epsilon^{-1}\left(-\frac{L^2}{8\epsilon}\,\frac{\log 3}{E(||\underline{W}||^2_{C_B})\ \ 96}\right)$$

because

$$E(\sup_{0 \leqq \Delta \leqq 2\epsilon} ||W(\Delta)||)^2$$

$$\leqq E(2\epsilon\left(\sup_{0 \leqq \Delta \leqq 2\epsilon} ||\frac{1}{\sqrt{2\epsilon}}\, W(\Delta)||\right)^2$$

$$= 2\epsilon E\left(||\underline{W}||^2_{C_B}\right).$$

As a preliminary step towards the proof of the main result of this section; we prove a theorem analogous to a theorem of Carmona-Kôno [**2**, Theorem 4.1] which itself uses a theorem of Kuelbs [**8**, Theorem 3.1].

THEOREM 1. *Suppose that* $\{\alpha_k\}$ *is a positive non-decreasing sequence with* $R = R(\alpha_n) < \infty$. *Let* $c > 1$ *and* (s_k, l_k) *a sequence in* $\mathbf{R}^2$ *where*

$$s_k = n_k c^{m_k} \log c \quad \textit{and} \quad l_k = c^{m_k},$$

$m_k \in \mathbf{Z}$ *and* n_k *is a non-negative integer. We suppose that* $(s_k, l_k) \neq (s_j, l_j)$ *for* $j \neq k$.

If

$$f_k(x) = \frac{W(s_k + xl_k) - W(s_k)}{\sqrt{l_k}} \quad x \in [0, 1]$$

then

$$P\left[\lim_{k\to\infty} d\left(\frac{f_k}{\sqrt{2\alpha_k}}, K_{\sqrt{R}}\right) = 0\right] = 1$$

$$P[\mathscr{C}\left(\frac{f_k}{\sqrt{2\alpha_k}}\right) = K_{\sqrt{R}}] = 1$$

where $\mathscr{C}(y_k)$ *stands for the set of limit points of a sequence* $\{y_k\}$ *and* $K_{\sqrt{R}}$ *is the closed ball of radius* $\sqrt{R}$ *in the* rkhs *of* μ*-Brownian motion in* C_B.

Proof. We will consider the case $n_k = 0$ for each k, that is $s_k = 0$. The proof of the general case is similar though technically more involved.

We first prove that for each $x^* \in C_B^*$ the sequence $\{x^*f_k\}$ satisfies condition (N) of Lemma 2.

If $x^* \in C_B^*$, then there exists a bounded mapping of $G:[0, 1] \to B^*$ and a finite Borel measure ν on [0, 1] such that

$$x^*(\phi) = \int_0^1 < G(s), \phi(s) > d\nu(s) \quad \text{for each } \phi \in C_B.$$

(For details see [**5**, p. 389].)

Suppose that $l_k < l_j$

$$E(x^*f_k x^*f_j) = \frac{1}{\sqrt{l_k l_j}} E\left(\int_{[0,1]x[0,1]} \langle G(x), W(xl_k)\rangle\langle G(y), W(yl_j\rangle d\nu(x)d\nu(y)\right)$$

G being bounded, an easy computation yields a constant M such that

$$E(x^*f_k x^*f_j) \leqq M\left(\frac{l_k}{l_j}\right)^{1/2} = Mc^{-(m_j - m_k)}.$$

Given $\epsilon > 0$, let q be an integer such that $Mc^{-q} < \epsilon$ and choose a subsequence $\{f_{k_j}\}$ as follows: $k_1 = 1$, if $k_1, \ldots, k_j$ have been chosen, let k_{j+1} be the first k after k_j for which

$$l_k \notin \bigcup_{i=1}^{j} (l_{k_i}c^{-q}, l_{k_i}c^q).$$

The above shows that

$$E(x^*f_{k_i}x^*f_{k_j}) \leqq \epsilon$$

and since $k_j \leqq 2jq$,

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$$R(\alpha_{k_j}) = \limsup \frac{\log j}{\alpha_{k_j}} = R(\alpha_k).$$

Now, if in the proof of Theorem 4.1 of [**2**] we replace Nisio's lemma by Lemma 2, we get the proof of Theorem 1. (See also [**16**], Lemma 3.1.)

The following setting was first introduced by C. Mueller in [**12**] to formulate his LIL.

Let $c > 1$ be fixed (c to be conveniently chosen later) and consider the following grid of rectangles in the first quadrant of the plane: $m \in \mathbf{Z}$, $n \geqq 0$

$$R_{m,n} = \{ (s, l) \,|\, c^m \leqq l \leqq c^{m+1}, nc^m \log c \leqq s \leqq (n+1)c^m \log c \}.$$

For each $t \geqq 0$, we associate a subset $\mathscr{P}_t$ of the first quadrant in the plane such that

(i) for each t,

$$\mathscr{A}(t) = \bigcup_{s \leqq t} \mathscr{P}_s$$

is contained in a finite union of rectangles $R_{m,n}$ of the grid.

(ii) $\cup_{s \geqq 0} \mathscr{P}_s$ is not contained in any finite union of these rectangles.

These two conditions are independent of $c > 1$ chosen. Now, set $A_c(t)$ to be the minimum number of rectangles such that $\mathscr{A}(t)$ is contained in their union. For an increasing function

$$h:\mathbf{R}_+ \to \mathbf{R}_+ \quad \text{with } h(t) \to \infty \text{ as } t \to \infty,$$

define

$$R_c^{(h)} = \limsup_t \frac{\log A_c(t)}{h(t)}.$$

$R_c(h)$ is in fact independent of the chosen $c > 1$, because if $1 < c_0 < c$ then there exists an integer p such that $c_0^p > c$. This in turn implies that there exists an integer M such that every rectangle in the grid determined by c is contained in at most M rectangles of the grid determined by c_0. Thus

$$A_{c_0}(t) \leqq MA_c(t) \quad \text{for each } t$$

and since $h(t) \to \infty$;

$$R_{c_0}^{(h)} = \limsup \frac{\log A_{c_0}(t)}{h(t)}$$

$$\leqq \limsup \frac{\log A_c(t)}{h(t)} = R_c^{(h)}.$$

A similar argument proves the reverse inequality.

Finally, we choose a sequence $t_0 \leqq t_1 \leqq t_2 \dots$ such that $\mathscr{A}(t)$ enters a new rectangle R_{m_k,n_k} at time t_k. Let $t_0 \neq 0$ and R_{m_0,n_0} a rectangle such that

$$\mathscr{P}_{t_0} \cap R_{m_0 n_0} \neq \emptyset.$$

Suppose $t_0 \leqq \dots \leqq t_k$ have been chosen, let

$$t_{k+1} = \inf \{t \geqq t_k | \mathscr{A}(t) \not\subset R_{m_0 n_0} \cup \dots \cup R_{m_k n_k}\}.$$

Choose $R_{m_{k+1} n_{k+1}}$ such that for every $\epsilon > 0$, there exists t, $t_{k+1} < t < t_{k+1} + \epsilon$ such that

$$R_{m_{k+1} n_{k+1}} \cap \mathscr{A}(t) \neq \emptyset.$$

All this is possible because of the hypothesis on $\mathscr{A}(t)$. Then

$$R(h) = R_c(h) = \inf \{a > 0 \Big| \sum_{k=0}^{\infty} e^{-ah(t_k)} < \infty\}$$

and this is independent of the $c > 1$ chosen.

To simplify the statement of the following theorem, we introduce the following notation:

(i) if $\Phi_t \subset C_B$ for each $t \in \mathbf{R}_+$ and $h:\mathbf{R}_+ \to \mathbf{R}_+$ and $A \subset C_B$; we write

$$\lim_{t\to\infty} d\left(\frac{\Phi t}{h(t)}, A\right) = 0$$

if for every $\epsilon > 0$,

$$d\left(\frac{\phi}{h(t)}, A\right) < \epsilon$$

for all large t and $\phi \in \Phi_t$.

(ii) $\mathscr{C}\left(\frac{\Phi_t}{h(t)}\right)$ will denote the set of limit points of subsequences

$$\left\{\frac{\Phi_{t_n}}{h(t_n)} | \phi_{t_n} \in \Phi_{t_n}, t_n \to \infty\right\} \text{ in } C_B.$$

THEOREM 2. *Let $\{W(t) \,|\, t \geqq 0\}$ be μ-Brownian motion in B and $\{\mathscr{P}_t\}$ be given as above. Let*

$$R = \limsup \frac{\log A_e(t)}{h(t)} < \infty.$$

For each $t \geqq 0$, set

$$\Phi_t = \{f_{s,l} | (s, l) \in \mathscr{P}_t\} \subset C_B$$

where

$$f_{sl}(x) = \frac{W(s + xl) - W(s)}{\sqrt{l}}.$$

Then,

$$P[\lim_{t \to \infty} d\left(\frac{\Phi t}{\sqrt{2h(t)}}, K_{\sqrt{R}}\right) = 0] = 1$$

$$P[\mathscr{C}\left(\frac{\Phi_t}{\sqrt{2h(t)}}\right) = K_{\sqrt{R}}] = 1$$

where $K_{\sqrt{R}}$ *is the closed ball of radius* $\sqrt{R}$ *in the* rkhs *for* μ*-Brownian motion.*

Proof. From Theorem 1 above: for any $c > 1$ and (s_k, l_k) the bottom left hand vertex of R_{m_k,n_k}, we find that

$$P[\lim_{k \to \infty} d\left(\frac{f_{s_k l_k}}{\sqrt{2h(t_k)}}, K_{\sqrt{R}}\right) = 0] = 1$$

$$P[\mathscr{C}\left(\frac{f_{s_k l_k}}{\sqrt{2h(t_k)}}\right) = K_{\sqrt{R}}] = 1.$$

The proof will be complete, if we prove that for any $\delta > 0$ and $c > 1$ chosen sufficiently close to 1

$$(*) \quad \sum_{k=1}^{\infty} P\left[\sup_{\substack{(s,l) \in \mathscr{P}_t \\ (s,l) \in R_{m_k n_k}}} \|f_{s,l} - f_{s_k l_k}\|_{C_B} > \delta\sqrt{2h(t_k)}\right] < \infty$$

(note that if $(s, l) \in \mathscr{P}_t \cap R_{m_k n_k}$ then $t \geqq t_k$).

(*) is proved using the following lemma (cf. [**12**, Lemma 2]).

LEMMA 4. *In the setting established above; with a grid determined by some* $c > 1$. *If* $\epsilon = \log c + (c - 1)$ *and* (s_0, l_0) *is the left hand bottom vertex of a fixed rectangle* $R_{m,n}$, *then*

$$P\left[\sup_{(s,l) \in R_{m,n}} \|f_{s,l} - f_{s_0 l_0}\|_{C_B} > \delta\right] \leqq \frac{C}{\epsilon} \exp\left(-\frac{\delta^2}{2^7 \epsilon}\rho\right)$$

where

$$\rho = (\log 3)/(96E(\|\underline{W}\|^2)).$$

Proof. The proof follows along the same lines as Mueller's Lemma 2 [**12**, p. 167]:

$$\sup_{(s,l)\in R_{m,n}} \|f_{sl} - f_{s_0l_0}\|_{C_B}$$

$$\leqq \sup_{(s,l)} \frac{\|W(s) - W(s_0)\|_B}{\sqrt{l}}$$

$$+ \sup_{(s,l)} \sup_{x\in[0,1]} \frac{\|W(s + xl) - W(s_0 + xl_0)\|_B}{\sqrt{l}}$$

$$+ \sup_{(s,l)} \left| \frac{\sqrt{l_0}}{\sqrt{l}} - 1 \right| \sup_{x\in[0,1]} \frac{\|W(s_0 + xl_0) - W(s_0)\|_B}{\sqrt{l_0}}$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

Therefore,

$$P\left[\sup_{(s,l)\in R_{m,n}} \|f_{s,l} - f_{s_0l_0}\|_{C_B} > \delta\right]$$

$$\leqq P[\mathrm{I} + \mathrm{II} + \mathrm{III} > \delta]$$

$$P\left[\mathrm{III} > \frac{\delta}{2}\right] \leqq \exp\left(-\frac{\delta^2}{4\epsilon^2}\,\rho\right)$$

because of Fernique's inequality and

$$\left|\frac{l_0}{l} - 1\right| \leqq c - 1 < \epsilon.$$

Furthermore, since

$$\frac{|(s + xl) - (s_0 + xl_0)|}{l_0} \leqq \frac{|s - s_0|}{l_0} + \frac{|l - l_0|}{l}$$

$$\leqq \log c + (c - 1) \leqq \epsilon.$$

By Lemma 3

$$P\left[2(\mathrm{II}) > \frac{\delta}{2}\right]$$

$$\leqq P\left[\sup_{\substack{1\leqq a\leqq 2\\ -\epsilon\leqq\Delta\leqq\epsilon}} \|W(a + \Delta) - W(a)\|_{C_B} > \frac{\delta}{4}\right]$$

$$\leqq \frac{3}{\epsilon} \exp\left(-\frac{\delta^2}{2^7\epsilon}\rho\right).$$

Combining these two inequalities, we get Lemma 4.

Remark. If h, $A:\mathbf{R}_+ \to \mathbf{R}_+$ are increasing, unbounded and have no common discontinuities then

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$$\inf \{a > 0 | \int_0^\infty e^{-ah(t)} dA(t) < \infty\}$$
$$= \limsup_t \frac{\log A(t)}{h(t)}.$$

The proof of this equality uses integration by parts following the same lines of the Laplace-Stieltjes transform when $h(t) = t$ in [7, Section 19.4]. Therefore, for real-valued Brownian motion, Theorem 2 implies Mueller's Theorem 1 in [**12**].

Examples. We give three applications of our Theorem 2.

a) *Strassen's theorem.* If $\mathcal{P}_t = \{ (0, t) \}, t \geqq 0\}$. Then for any grid with $c > 1$, $t_k = c^k$; and if $h(t) = \log \log t$

$$\limsup \frac{\log k}{h(t_k)} = 1.$$

Consequently the set of limit points of $\left\{ \frac{W(xt)}{\sqrt{2t \log \log t}} \right\}$ is the unit ball of the rkhs in C_B.

b) *Lévy's modulus of continuity.* If

$$\mathcal{P}_t = \left\{ (s, l) \,|\, l = \frac{1}{t}, 0 \leqq s \leqq 1 - l \right\} \quad \text{for each } t \geqq 1:$$

In this case, if we use the grid with $c = 2$ and $h(t) = \log t$, it is fairly simple to check that

$$\limsup_k \frac{\log k}{h(t_k)} = \limsup \frac{\log \sum_{j=1}^{m} (2^j + j)}{h(t_m)} = 1.$$

Therefore, when $t \downarrow 0$ the set of limit points of

$$\left\{ f(x) = \frac{W(s + xt) - W(s)}{\sqrt{(2t \log 1/t)}} \,\middle|\, 0 \leqq s \leqq 1 - t \right\}.$$

is the unit ball in C_B.

c) *Moving averages* ([**3**], [**4**]). If $\{a_n\}$ is a sequence satisfying $a_n \leqq n$, $a_n \uparrow \infty$, a_n/n decreasing: Set

$$b_n = \log \frac{n}{a_n} + \log \log n = \log \left(\frac{n}{a_n} \log n \right).$$

(I) If $a_n/n \downarrow \alpha > 0$, set

$$n_k(\epsilon) = (1 + \epsilon)^k.$$

Then for a grid with $c = 1 + \epsilon$, it is easy to see that the exit times $t_k \sim n_k$. Therefore $b_{n_k} \sim \log k$.

(II) If $a_n/n \downarrow 0$, Deo in [**4**, p. 104] introduces two sequences $m_k(\epsilon)$ and $n_k(\epsilon)$ for each $\epsilon > 0$ for which

$$b_{n_k} \sim \log k \quad \text{and} \quad b_{m_k} \sim (1 + \epsilon)\log k.$$

From the properties of n_k and m_k we find

$$\limsup \frac{\log k}{b_{m_k}} \leqq \limsup \frac{\log k}{b_{t_k}} \leqq \limsup \frac{\log k}{b_{n_k}}.$$

Consequently,

$$\limsup \frac{\log k}{b_{m_k}} = 1.$$

By Theorem 2, we get that if $n \to \infty$ the set of limit points of

$$\left\{ \frac{W(n - a_n + xa_n) - W(n - a_n)}{\sqrt{2a_n b_n}} \right\}$$

is the unit ball K of the rkhs in C_B.

4. The case $R = \infty$. When $R = \infty$ the proofs given above are not valid because they assume that K_R is compact. However we are going to prove that in case $\mathscr{P}_t = \{ (0, t) \}$ then the set of limit points include all of H, therefore all of C_B because H is dense in C_B and the set of limit points is closed. The theorem is still true for more general sets $\mathscr{P}_t$; the proof uses the same idea but is more involved.

THEOREM 3. *Let* $\{W(t) | t \geqq 0\}$ *be* μ*-Brownian motion in* B *and* h *a non-decreasing function with* $h(t) \to \infty$ *as* $t \to \infty$. If

$$\sum_{k=1}^{\infty} e^{-ah(c^k)} = \infty$$

for any $c > 1$ *and any* $a > 0$ *then*

$$P\left[\mathscr{C}\left(\frac{W(t)}{\sqrt{2th(t)}}\right) = C_B\right] = 1.$$

Proof. As stated above the rkhs determined by μ-Brownian motion is

$$H = H_0 \otimes H_\mu \subset C_B.$$

Let $\{e_j^*\} \subset B^*$ and $\{e_j\} \subset H_\mu$ be the bi-orthonormal bases for the Gaussian measure μ. For H_0, let $\{g_i\}$ be the Haar basis with $\{g_i^*\} \subset C[0, 1]^*$ the corresponding sequence. Consequently $\{g_i^* \otimes e_j^*\}$ and $\{g_i \otimes e_j\}$ give a pair of bases for the Gaussian measure on C_B induced by μ-Brownian motion W.

Let $\phi \in H$ with $\|\phi\|_H^2 = r > 0$. It is known [9] that $\phi(t) \in H_\mu$ for all $t \in [0, 1]$ and

$$\|\phi\|_H^2 = \sum_j \int_0^1 \left[\frac{d}{dt}(e_j^*(\phi)(t)\right]^2 dt.$$

We are going to prove that ϕ is a limit point of some sequence

$$\left\{\frac{W(t_k)}{\sqrt{2t_k h(t_k)}}\right\}.$$

Let $\epsilon > 0$ be such that $\epsilon < r/2$. Choose m and m_0 (large enough) such that:

(a) $m = 2^p$ for some integer p,

(b) For $x \in C_B$, if

$$P_0(x) = \sum_{i=1}^{m} \sum_{j=1}^{m_0} < g_i^* \otimes e_j^*, x > g_i \otimes e_j$$

and

$$Q_0 = \text{Id} - P_0;$$

then

$$\|Q_0 \phi\|_H^2 \leqq \frac{\epsilon}{4}.$$

(c) $$\sum_{j=1}^{m_0} \int_0^{1/m} [(e_j^*\phi)'(t)]^2 dt < \frac{\epsilon}{4}$$

and

$$\sup_{0 \leqq t \leqq \frac{1}{m}} \|\phi(t)\|_B < \frac{\epsilon}{8}.$$

Let H_m be the subspace of H_0 generated by $g_1, g_2, \ldots, g_m$ the first $m = 2^p$ elements of the Haar basis. If δ_t denotes as usual unit mass at t then $\delta_t \in C[0, 1]^*$ and if S denotes the canonical map

$$S: C[0, 1]^* \to C[0, 1]$$

induced by Wiener measure in $C[0, 1]$, then $\{S\delta_{1/m}, S\delta_{2/m}, \ldots, S\delta_1\}$ also generate H_m; therefore by the Gram-Schmidt orthogonalisation process we get an orthonormal basis $\{d_1, d_2, \ldots, d_m\}$ and the corresponding $\{d_1^*, \ldots, d_m^*\}$ such that $Sd_i^* = di$. For example

$$d_1^* = m^{\frac{1}{2}}\delta_{1/m} \quad \text{and}$$

$$d_1(t) = \begin{cases} \sqrt{m}t & 0 \leqq t \leqq \frac{1}{m} \\ \frac{1}{\sqrt{m}} & \frac{1}{m} \leqq t \leqq 1 \end{cases}$$

and

$$P_0(x) = \sum_{i=1}^{m} \sum_{j=1}^{m_0} < d_i^* \otimes e_j^*, \quad x > d_i \otimes e_j.$$

Set

$$f_k(x) = \frac{W(x \cdot m^k)}{\sqrt{m^k}} \quad x \in [0, 1].$$

We are going to prove that a.s. ϕ is a limit point of $f_k/\sqrt{2h(m^k)}$; that is

$$P\left[\left|\left|\frac{f_k}{\sqrt{2h(m^k)}} - \phi\right|\right|_{C_B} < \epsilon \text{ i.o.}\right] = 1.$$

Since

$$\left[\left|\left|\frac{f_k}{\sqrt{2h(m^k)}} - \phi\right|\right|_{C_B} < \epsilon\right]$$

$$\supseteq \left[\left|\left|P_0\left(\frac{f_k}{\sqrt{2h(m^k)}}\right) - \phi\right|\right|_{C_B} < \frac{\epsilon}{2}\right]$$

$$\cup \left[\left|\left|Q_0\left(\frac{f_k}{\sqrt{2h(m^k)}}\right) - \phi\right|\right|_{C_B} < \frac{\epsilon}{2}\right]$$

and because on any finite dimensional subspace all norms are equivalent we can replace the C_B-norm by the equivalent H-norm.

$$\left[\left|\left|\frac{f_k}{\sqrt{2h(m^k)}} - \phi\right|\right|_{C_B} < \epsilon\right] \supseteq U_k$$

where

$$U_k = V_k' \cap V_k'' \cap V_k'''$$

and

$$V_k' = \bigcap_{\substack{2 \leqq i \leqq m \\ 1 \leqq j \leqq m_0}} \left[\left| (d_i^* \otimes e_j^*)\left(\frac{f_k}{\sqrt{2h(m^k)}} - \phi \right) \right| < \sqrt{\frac{\epsilon}{4mm_0}} \right]$$

$$V_k'' = \left[\sup_{0 \leqq x \leqq \frac{1}{m}} \left\| \frac{f_k(x)}{\sqrt{2h(m^k)}} - \phi \right\|_B < \frac{\epsilon}{4} \right]$$

$$V_k''' = \sup_{\frac{1}{m} \leqq x \leqq 1} \left\| Q_0\left(\frac{f_k(x)}{\sqrt{2h(m^k)}} \right) \right\|_B < \frac{\epsilon}{4} \Bigg].$$

We will prove that $P(U_k \text{ i.o.}) = 1$.

(I) Using standard estimates for the independent $N(0, 1)$: $d_i^* \otimes e_j^*$ and the choice of m and m_0.

$$P(V_k') \geqq \exp(-r'h(m^k)) \quad \text{for some } \frac{r}{2} < r' < r.$$

Thus

$$P(\bar{V}_k') \leqq \exp(-\exp(-r'h(m^k))).$$

($\bar{V}_k'$ denotes the complement of V_k'.)

(II) Because

$$|||g||| = \sup_{\frac{1}{m} \leqq t \leqq 1} \|g(t)\|_B$$

is a semi norm on C_B with $|||g||| \leqq \|g\|_{C_B}$ we get

$$P(\bar{V}_k''') \leqq \exp(-\alpha h(m^k))$$

for some positive α given by Fernique's inequality.

(III) m was chosen such that

$$\sup_{0 \leqq x \leqq \frac{1}{m}} \|\phi(x)\|_B \leqq \frac{\epsilon}{8}$$

$$P(\bar{V}_k'') \leqq P\left[\sup_{0 \leqq x \leqq \frac{1}{m}} \left\| \frac{W(x \cdot m^k)}{\sqrt{m^k}} \right\|_B > \frac{\epsilon}{8}\sqrt{2h(m^k)} \right]$$

$$\leqq \exp(-\alpha' h(m^k))$$

for some positive α'.

From (I) to (III), we get that by a suitable choice of k_0,

$$P(\bar{U}_k) \leqq \exp\left(-\tfrac{1}{2}\exp(-r'h(m^k))\right)$$

if $k \geqq k_0$.

It is clear that $\bar{U}_k \cap \ldots \cap \bar{U}_q$ and $\bar{V}'_{q+1}$ are independent; the same for $\bar{U}_k \cap \ldots \cap \bar{U}_q$ and $\bar{V}'''_{q+1}$; using an induction proof as in [**1**, Lemma 4] it can be shown that

$$P(\bar{U}_{k_0} \cap \ldots \cap \bar{U}_q) \leqq \exp\left(-\tfrac{1}{2} \sum_{k=k_0}^{q} \exp(-r'h(m^k))\right).$$

The series

$$\sum_{q=k_0}^{\infty} \exp(-r'h(m^k))$$

diverging, we conclude that $P(U_k \text{ i.o.}) = 1$, and therefore w.p.1 ϕ is a limit point of $f_k/\sqrt{2h(m^k)}$. Since the set of limit points is a closed set and H is dense in C_B; the proof is complete.

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