ADIABATIC OSCILLATIONS OF A DIFFERENTIALLY-ROTATING STAR*

Second-Order Perturbation Theory

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Abstract. Perturbation theory is developed for calculating the influence of slow differential rotation on the adiabatic nonradial modes of stellar oscillations. The effects of Coriolis forces and ellipticity are analysed simultaneously using the perturbation technique for Hermitian operators which is developed up to second order in eigenvalues and to first order in eigenvectors.

1. Introduction

The effect of slow differential rotation on linear adiabatic oscillations of a star was first analysed by Hansen *et al.* (1977) using a variational-type perturbation technique. A general expression for first-order corrections to eigenfrequencies was obtained and reduced to a form convenient for computation of a particular type of differential rotation distributions. For a wider class of angular velocity distributions, the reduction was done by Cuypers (1980).

In the present paper the perturbation theory is developed up to second order in the eigenfrequency and first-order corrections to the eigenfunctions are also determined. In addition to the Coriolis effects, the rotational distortion of the stellar configuration and centrifugal forces are taken into account.

2. Perturbation Theory

Linear adiabatic oscillations of a differentially-rotating star are analysed in an inertial frame. The distribution of the velocity of rotation is assumed to be stationary and φ-independent in a spherical coordinate system (r, θ, φ). General vector equations – the moment equation, the Poisson equation for gravitational perturbations and the continuity equation – may be written as

$$\rho_0[-\omega^2\mathbf{u} + \Omega(2i\omega\hat{z} \times \mathbf{u} - 2m\omega\mathbf{u}) + \Omega^2(2im\hat{z} \times \mathbf{u} - m^2\mathbf{u})] =$$
$$= \nabla(K\nabla\cdot\mathbf{u}) - \nabla[\mathbf{u}\cdot\rho_0(\nabla\psi_0 - \hat{r}_c r_c\Omega^2)] - \rho'(\nabla\psi_0 - \hat{r}_c r_c\Omega^2) -$$
$$- \rho_0\nabla\psi' - \rho_0\hat{r}_c r_c(\mathbf{u}\cdot\nabla\Omega^2)\,, \tag{1}$$

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$$\nabla^2\psi' = 4\pi G\rho' , \quad (2)$$

$$\rho' = -\nabla\cdot(\rho_0\mathbf{u}) . \quad (3)$$

Here ω is the angular frequency of oscillations, $\mathbf{u}$ is the vector displacement field, K is the adiabatic compressibility ($K = \Gamma_1 p_0$, where p_0 is the unperturbed pressure), ρ_0 and ψ_0 are the unperturbed density and gravitational potential, ρ' and ψ' are their Eulerian perturbations, r_c is the distance from the axis of rotation (z), $\hat{r}_c$ and $\hat{z}$ are the unit vectors in r_c, z-directions. The solutions of Equations (1)–(3) must satisfy the usual free-boundary conditions.

Equation (1) may be written in operator form

$$-\omega^2\mathbf{u} + \Omega(2i\omega\hat{z}\times\mathbf{u} - 2m\omega\mathbf{u}) + \Omega^2(2im\hat{z}\times\mathbf{u} - m^2\mathbf{u}) = = -H_0\mathbf{u} - \Omega_0^2(\Psi + E)\mathbf{u} . \quad (4)$$

Here H_0 is an integro-differential Hermitian operator corresponding to the zero-order boundary-value problem (a nonrotating, spherically symmetric star):

$$H_0\mathbf{u}_0 = \omega_0^2\mathbf{u}_0 . \quad (5)$$

Operators Ψ and E correspond to the influence of centrifugal forces and ellipticity. They are determined by equivalency of Equations (1), (4); an explicit form of these operators is not necessary because only their matrix elements, calculated by a variational technique, will be needed.

In Equation (4), Ω_0 denotes the average angular velocity of rotation, so that

$$\Omega(r,\theta) = \Omega_0\Omega_d(r,\theta) . \quad (6)$$

A small parameter $\lambda = \Omega_0/\omega_0$ is then introduced and the solutions of Equation (4) are found in the form

$$\omega = \omega_0(1 + \sigma_1\lambda + \sigma_2\lambda^2 + \ldots) , \quad (7)$$

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1\lambda + \mathbf{u}_2\lambda^2 + \ldots . \quad (8)$$

Substitution of expansions (7), (8) into Equation (4) leads to the system of equations of perturbation theory

$$[I - \omega_0^{-2}H_0]\mathbf{u}_0 = 0 , \quad (9)$$

$$[I - \omega_0^{-2}H_0]\mathbf{u}_1 = -2\sigma_1\mathbf{u}_0 - \Omega_d[2m - 2i\hat{z}\times]\mathbf{u}_0 , \quad (10)$$

$$[I - \omega_0^{-2}H_0]\mathbf{u}_2 = -2\sigma_1\mathbf{u}_1 - (\sigma_1^2 + 2\sigma_2)\mathbf{u}_0 - \Omega_d[2m - 2i\hat{z}\times]\mathbf{u}_1 - -\sigma_1\Omega_d[2m - 2i\hat{z}\times]\mathbf{u}_0 - - \Omega_d^2[m^2 - 2im\hat{z}\times]\mathbf{u}_0 + [\Psi + E]\mathbf{u}_0 , \quad (11)$$

where I is the identity operator. Equation (9) represents the zero-order problem; its solutions are orthogonal and assumed to be normalized in the sense of a scalar product defined by

$$(\mathbf{u}, \mathbf{u}') = \int_V \rho_0 \mathbf{u}^* \cdot \mathbf{u}' \, dv \,, \tag{12}$$

where V is the volume occupied by the star. Scalar multiplication of Equation (10) onto $\mathbf{u}_0$ gives the first-order correction to eigenfrequency

$$\sigma_1 = -(\mathbf{u}_0, \Omega_d[m - i\hat{z} \times]\mathbf{u}_0) \,, \tag{13}$$

which is the result obtained by Hansen *et al.* (1977). The first-order correction to the eigenfunctions is also found from Equation (10):

$$\mathbf{u}_1 = \sum_{l',n' \neq l,n} a_{l,m,n,1}^{l',m,n'} \mathbf{u}_0' + \{-\Omega_d[2m - 2i\hat{z} \times]\mathbf{u}_0\}_T \,, \tag{14}$$

$$\frac{\omega_0^2 - (\omega_0')^2}{\omega_0^2} a_{l,m,n,1}^{l',m,n'} = (-\mathbf{u}_0', \Omega_d[2m - 2i\hat{z} \times]\mathbf{u}_0) \,, \tag{15}$$

where index T denotes orthogonal projection onto the field of toroidal vector spherical harmonics and the corresponding term in Equation (14) represents a torsional-type correction to the eigenfunctions. The second-order correction to the eigenfrequency is found from Equation (11):

$$2\sigma_2 - \sigma_1^2 = -(\Omega_d[2m - 2i\hat{z} \times]\mathbf{u}_0, \mathbf{u}_1) - m(\mathbf{u}_0, \Omega_d^2[m - 2i\hat{z} \times]\mathbf{u}_0) + + (\mathbf{u}_0, [\Psi + E]\mathbf{u}_0) \,. \tag{16}$$

The reduction of the scalar products in (13)–(16) to a form convenient for computation (unidimensional integrals containing radial eigenfunctions) is given by Vorontsov (1981). The angular velocity distribution is represented by a finite set $\Omega = \Omega_0 \sum_l \Omega_{dl}(r) Y_{l_0}(\theta, \varphi)$ and the angular dependences are reduced to a computation of angular integrals containing three vector spherical harmonics, which is simply performed in terms of Wigner's 3-j symbols.

Computationally-convenient formulae (Vorontsov, 1981) are given for the case $\partial\Omega/\partial z = 0$. The simplifying assumption was used that the effect of differentiality of rotation on the stellar configuration is relatively small, i.e., the theory of rotational distortion of a star for the case of rigid rotation is appropriate. A corresponding expression for the matrix elements of the operator $\Psi + E$ is given by Vorontsov and Zharkov (1981), taking into account possible discontinuities in density distribution in a star.

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