The Locus of the Straight Lines which intersect Three Given Lines.

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(*Read and Received 11th March 1910.*)

The following note indicates how the equation to the locus of the straight lines which intersect three given lines may be obtained by the use of the conditions that three planes should have a line of intersection.

Three given planes

$$a_r x + b_r y + c_r z + d_r = 0, \ r = 1, 2, 3,$$

have a line of intersection if any two of the determinants in the scheme

$$\| \ a_r, b_r, c_r, d_r \ \|$$

are equal to zero.

Let the equations to the three given lines be

$$u_1 = 0 = v_1, \ u_2 = 0 = v_2, \ u_3 = 0 = v_3.$$

Then if the three planes

$$u_1 - \lambda_1 v_1 = 0, \qquad (1)$$

$$u_2 - \lambda_2 v_2 = 0, \qquad (2)$$

$$u_3 - \lambda_3 v_3 = 0, \qquad (3)$$

pass through one line, that line is coplanar with the three given lines, and therefore intersects all three. The conditions that the planes may have a line of intersection may be written,

$$f_1(\lambda_1, \lambda_2, \lambda_3) = 0, \qquad (4)$$

$$f_2(\lambda_1, \lambda_2, \lambda_3) = 0, \qquad (5)$$

If they are satisfied any two of the equations (1), (2), (3), may be taken to represent the line of intersection. Suppose that we take (1) and (2). Eliminate λ_3 between equations (4) and (5) and we obtain an equation

$$\phi(\lambda_1, \lambda_2) = 0.$$

Thus the equations $u_1 - \lambda_1 v_1 = 0$, $u_2 - \lambda_2 v_2 = 0$, where λ_1 and λ_2 are parameters connected by the equation $\phi(\lambda_1, \lambda_2) = 0$, represent any line that intersects the three given lines. The elimination of λ_1 and λ_2 leads to the equation to the locus of such lines, viz.,

$$\phi\left(\frac{u_1}{v_1}, \frac{u_2}{v_2}\right) = 0.$$

It is to be noted that if λ_1, λ_2, λ_3, satisfy equations (4) and (5), the three equations (1), (2), (3), reduce to two. The finding of the equation to the locus therefore depends on the elimination of λ_1, λ_2, λ_3, between *four* independent equations, and any process of elimination leads to the same result. The locus is therefore also given by

$$f_1\left(\frac{u_1}{v_1}, \frac{u_2}{v_2}, \frac{u_3}{v_3}\right) = 0, \text{ or } f_2\left(\frac{u_1}{v_1}, \frac{u_2}{v_2}, \frac{u_3}{v_3}\right) = 0.$$

Application. The equations to any three non-intersecting straight lines can be put in the form

$$y = b,\ z = -c\ ;\ z = c,\ x = -a\ ;\ x = a,\ y = -b.$$

The three planes

$$y - b - \lambda_1(z + c) = 0,\ z - c - \lambda_2(x + a) = 0,\ x - a - \lambda_3(y + b) = 0,$$

have a line of intersection of any two of the determinants

$$\left\| \begin{matrix} 0, & 1, & -\lambda_1, & -b - \lambda_1 c \\ -\lambda_2, & 0, & 1, & -c - \lambda_2 a \\ 1, & -\lambda_3, & 0, & -a - \lambda_2 b \end{matrix} \right\|$$

are equal to zero; *i.e.* if

$$\lambda_1 \lambda_2 \lambda_3 - 1 = 0, \text{ etc.}$$

The equation to the locus is therefore

$$\frac{y-b}{z+c} \cdot \frac{z-c}{x+a} \cdot \frac{x-a}{y+b} = 1, \text{ or}$$

$$ayz + bzx + cxy + abc = 0.$$

https://doi.org/10.1017/S0013091500034799 Published online by Cambridge University Press