Canad. J. Math. Vol. **56** (4), 2004 pp. 843–870

Type Decomposition and the Rectangular AFD Property for W^*-TRO's

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Abstract. We study the type decomposition and the rectangular AFD property for W^*-TRO's. Like von Neumann algebras, every W^*-TRO can be uniquely decomposed into the direct sum of W^*-TRO's of type *I*, type *II*, and type *III*. We may further consider W^*-TRO's of type $I_{m,n}$ with cardinal numbers m and n, and consider W^*-TRO's of type $II_{\lambda,\mu}$ with $\lambda, \mu = 1$ or ∞. It is shown that every separable stable W^*-TRO (which includes type $I_{\infty,\infty}$, type $II_{\infty,\infty}$ and type *III*) is TRO-isomorphic to a von Neumann algebra. We also introduce the rectangular version of the approximately finite dimensional property for W^*-TRO's. One of our major results is to show that a separable W^*-TRO is injective if and only if it is rectangularly approximately finite dimensional. As a consequence of this result, we show that a dual operator space is injective if and only if its operator predual is a rigid rectangular $\mathcal{OL}_{1,1^+}$ space (equivalently, a rectangular $\mathcal{OL}_{1,1^+}$ space).

1 Introduction

In the recent development of operator space theory, there is an increasing interest in the study of *ternary rings of operators*. A ternary ring of operators (or simply, TRO) can be identified with the off-diagonal corner of a C^*-algebra and thus can be equipped with a canonical operator space matrix norm. TRO's are natural non-self-adjoint generalizations of C^*-algebras and come out very naturally from operator space theory. For instance, it is known from Youngson [38] that TRO's are closed under completely contractive projections, but C^*-algebras are only closed under completely positive and completely contractive projections (see Choi-Effros [4]). It is also known from Ruan [33] that every injective operator space has a canonical TRO structure, but need not be a C^*-algebra.

It is interesting to note that TRO's actually coincide with certain objects familiar to operator algebraists. They are nothing but *full Hilbert modules* over C^*-algebras. On the other hand, W^*-TRO's, which can be identified with the off-diagonal corners of von Neumann algebras, are exactly *self-dual and weakly full Hilbert modules* over von Neumann algebras. These important connections were first observed by Zettl [39].

Hilbert modules were first investigated by Kaplansky [19] over commutative C^*-algebras in the early 1950s, and were further generalized to C^*-algebras and von Neumann algebras by Paschke [26] and Rieffel [30] in the early 1970s. The theory has become to a very important topic in operator algebras and has been a very important tool in the study of KK-theory and non-commutative geometry. There is a different emphasis in the study of TRO's and W^*-TRO's, which has been more focused on their algebraic structure and the properties analogous to C^*-algebras and

Received by the editors March 31, 2002; revised November 5, 2002.
The author was partially supported by the National Science Foundation Grant DMS-9877157.
AMS subject classification: Primary 46L07; Secondary 46L08, 46L89.

von Neumann algebras. Some interesting operator space properties have also been studied recently for TRO's and W^*-TRO's. The readers are referred to Harris [16], Zettl [39], Hamana [14, 15], Kirchberg [21], Exel [11], Effros-Ozawa-Ruan [7] and Kaur-Ruan [20] for more details.

The purpose of this paper is to further investigate the type decomposition and the equivalence between injectivity and the rectangular approximately finite dimensional property for W^*-TRO's. In our approach, we will need to use results from both TRO's and Hilbert modules. So we recall some necessary notations and carefully discuss the connection between TRO's and Hilbert modules in §2.

We discuss the structure of stable W^*-TRO's in §3. A W^*-TRO V is said to be *stable* if it is TRO-isomorphic to $M_\infty \overline{\otimes} V$, where we let $M_\infty = B(\ell_2(\mathbb{N}))$. Our main result (Theorem 3.2) in §3 shows that if V is a separable stable W^*-TRO, then V is TRO-isomorphic to a von Neumann algebra. To prove Theorem 3.2 we need to develop a *normal version* of *Kasparov's stabilization theorem* (Theorem 3.1). We provide a detailed argument for this result since it is not (to the author's knowledge) available in the literature.

In §4 we discuss the type decomposition for W^*-TRO's via their linking von Neumann algebras. We prove in Theorem 4.1 that every W^*-TRO of type I has the form

$$V = \sum_\alpha {}^\oplus M_{I_\alpha, J_\alpha} \overline{\otimes} L_\infty(X_\alpha, \mu_\alpha),$$

where we let $M_{I_\alpha,J_\alpha} = B(\ell_2(J_\alpha), \ell_2(I_\alpha))$. We prove in Theorem 4.4 that if V is a W^*-TRO of type $II_{1,\infty}$ (respectively, a W^*-TRO of type $II_{\infty,1}$), then V is TRO-isomorphic to the row space $R_I^w(M(V)) = M_{1,I}(M(V))$ (respectively, TRO-isomorphic to the column space $C_I^w(N(V)) = M_{I,1}(N(V))$) for some index set I. In §5 we study the *rectangular approximately finite dimensional* property for W^*-TRO's, which is the W^*-TRO analogue of *hyperfiniteness* for von Neumann algebras. It is known (by Connes [5], Haagerup [13], and Elliott-Woods [10]) that a separable von Neumann algebra is injective if and only if it is hyperfinite (see details in Takesaki's book [36]). We show in Theorem 5.5 that this is also true for separable W^*-TRO's, *i.e.*, a separable W^*-TRO is injective if and only if it is rectangularly approximately finite dimensional.

It was shown in [8] that the operator predual of a separable injective von Neumann algebra has a very nice local structure, *i.e.*, it is a rigid $\mathcal{OL}_{1,1^+}$ space (equivalently, an $\mathcal{OL}_{1,1^+}$ space). Using a technique developed by Haagerup we can extend this result to the non-separable case (see [17]). On the other hand, it was shown by Ng-Ozawa [25] that if a separable operator space X is an $\mathcal{OL}_{1,1^+}$ space, then its operator dual $V = X^*$ is completely isometric to an injective von Neumann algebra. This shows that the local $\mathcal{OL}_{1,1^+}$ structure on separable operator spaces somehow reflects the *square* structure of the whole space and this completely characterizes the operator preduals of separable injective von Neumann algebras. However, this is no longer true if the separability is removed. It was noticed by Ng-Ozawa [25] that if I is an uncountable index set, then $T_{\infty,I}$ is a rigid $\mathcal{OL}_{1,1^+}$ space. Its operator dual $M_{\infty,I} = B(\ell_2(I), \ell_2(\mathbb{N}))$ is an injective operator space (and thus is an injective W^*-TRO), but it is not completely isometric to any von Neumann algebra (see more details in §6). Motivated by this example, Ng and Ozawa suggested that it is worthy to study the rectangular version of $\mathcal{OL}_{1,1^+}$ spaces, and they asked in [25] whether the operator predual of a dual injective operator space always has a rectangular $\mathcal{OL}_{1,1^+}$ space structure. As a consequence of Theorem 5.5, we can give an affirmative answer to Ng-Ozawa's question in §6. We show in Theorem 6.1 that a dual operator space is injective if and only if its operator predual is a rigid rectangular $\mathcal{OL}_{1,1^+}$ space (equivalently, a rectangular $\mathcal{OL}_{1,1^+}$ space).

2 TRO's and Hilbert Modules

Let us first recall that a *ternary ring of operators* (or simply, *TRO*) between Hilbert spaces K and H is a norm closed subspace V of $B(K,H)$, which is closed under the triple product

$$(x, y, z) \in V \times V \times V \to xy^*z \in V.$$

A TRO $V \subseteq B(K,H)$ is called a W^*-*TRO* if it is strong operator closed (equivalently, weak* closed) in $B(K,H)$. A W^*-TRO is said to be *separable* if it can be represented on some separable Hilbert spaces H and K.

It is important to note that given Hilbert spaces H and K, there is a canonical operator norm $\|\cdot\|_n$ on the TRO $M_n(B(K,H)) \cong B(K^n, H^n)$ for every $n \in \mathbb{N}$. We call this family of operator norms $\{\|\cdot\|_n\}$ the canonical *TRO matrix norm* on $B(K,H)$. In general if V is a TRO contained in $B(K,H)$, then we may obtain a canonical TRO matrix norm on V by identifying $M_n(V)$ with a TRO contained in $M_n(B(K,H))$ for every $n \in \mathbb{N}$. This canonical TRO matrix norm determines a distinguished operator space structure on V, which will play a very important role in our study. The readers are referred to [9, 28], and [27] for details on operator spaces, and are referred to [39, 15, 7] and [20] for the details on TRO's.

Given a TRO $V \subseteq B(K,H)$, we let $V^\sharp = \{x^* \in B(H,K) : x \in V\}$ denote the *adjoint space* of V. Then $V^\sharp$ is again a TRO. Its canonical TRO matrix norm satisfies

$$\|[x_{ij}^*]\| = \|[x_{ji}]\| \tag{2.1}$$

for all $[x_{ij}^*] \in M_n(V^\sharp)$. We let $VV^\sharp$ and $V^\sharp V$ denote the linear spans of vw^* and v^*w for all $v, w \in V$, respectively. Then $VV^\sharp$ and $V^\sharp V$ are $*$-subalgebras of $B(H)$ and $B(K)$, and we let

$$C(V) = \overline{VV^\sharp}^{\|\cdot\|} \text{ and } D(V) = \overline{V^\sharp V}^{\|\cdot\|}$$

denote the C^*-algebras generated by $VV^\sharp$ and $V^\sharp V$, respectively. Without loss of generality, we may always assume that $C(V)$ and $D(V)$ are *non-degenerately* represented on H and K. If V is a W^*-TRO, then we let

$$M(V) = \overline{VV^\sharp}^{s.o.t} \text{ and } N(V) = \overline{V^\sharp V}^{s.o.t}$$

denote the von Neumann algebras generated by $VV^\sharp$ and $V^\sharp V$, respectively.

Given TRO's V and W, a linear map $\theta\colon V \to W$ is called a *TRO-homomorphism* if it preserves the ternary product

$$\theta(xy^*z) = \theta(x)\theta(y)^*\theta(z)$$

for all $x, y, z \in V$. If, in addition, θ is an isomorphism from V onto W, we call θ a *TRO-isomorphism* from V onto W. It is known from Harris [16] and Hamana [15] that if $\theta\colon V \to W$ is a TRO-homomorphism, then it is a *complete contraction*, *i.e.*,

$$\theta_n\colon [x_{ij}] \in M_n(V) \to [\theta(x_{ij})] \in M_n(W)$$

is a contraction for every $n \in \mathbb{N}$. Moreover, it was proved independently by the author [32, Corollary 2.3.5] and by Hamana [15, Proposition 2.1] that a linear map $\theta\colon V \to W$ between TRO's V and W is a TRO-isomorphism if and only if it is a *completely isometric* linear isomorphism from V onto W. If, in addition, V and W are W^*-TRO's, then we can conclude (by the uniqueness of operator preduals) that every TRO-isomorphism (equivalently, every completely isometric linear isomorphism) $\theta\colon V \to W$ between V and W is automatically weak* continuous. We also note that if $V = A$ and $W = B$ are unital C^*-algebras, then every completely isometric linear isomorphism φ from A onto B (in this case φ is automatically a TRO-isomorphism) must have the form $\varphi = u\pi$, where u is a unitary element in B and π is a unital $*$-isomorphism from A onto B. Therefore, TRO-isomorphic unital C^*-algebras (respectively, von Neumann algebras) must be $*$-isomorphic.

If V is a TRO contained in $B(K, H)$, then

$$A(V) = \begin{bmatrix} C(V) & V \\ V^\sharp & D(V) \end{bmatrix} \tag{2.2}$$

is the C^*-subalgebra of $B(H \oplus K)$ generated by V via the canonical TRO-inclusion

$$\iota_V\colon v \in V \to \iota_V(v) = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \in B(H \oplus K). \tag{2.3}$$

It is known from [15] and [20] that $A(V)$ is uniquely determined by V (up to TRO-isomorphisms) and is just the *C^*-envelope* of V. We call $A(V)$ the *linking C^*-algebra* of V.

If V is a W^*-TRO (which is usually assumed to be non-degenerately) contained in $B(K, H)$, then it is known from [7] and [20] that $M(V)$ and $N(V)$ are exactly the multiplier algebras of $C(V)$ and $D(V)$, and we call

$$R(V) = \begin{bmatrix} M(V) & V \\ V^\sharp & N(V) \end{bmatrix} = A(V)'' \tag{2.4}$$

the *linking von Neumann algebra* of V. If we let

$$e = \begin{bmatrix} 1_H & 0 \\ 0 & 0 \end{bmatrix} \text{ and } e^\perp = \begin{bmatrix} 0 & 0 \\ 0 & 1_K \end{bmatrix} \tag{2.5}$$

denote the corresponding projections on H and K respectively, then we may identify V with the off-diagonal corner $\iota_V(V)$ of $R(V)$ and write

$$V = eR(V)e^\perp. \tag{2.6}$$

https://doi.org/10.4153/CJM-2004-038-7 Published online by Cambridge University Press

We can also identify von Neumann algebras $M(V)$ and $N(V)$ with $eR(V)e$ and $e^\perp R(V)e^\perp$ and identify $V^\sharp$ with $e^\perp R(V)e$.

The *central cover* C_f of a projection f in $R(V)$ is the smallest central projection p in $R(V)$ such that $pf = f$.

Lemma 2.1 *If we let C_e and $C_{e^\perp}$ denote the central covers of e and $e^\perp$ in $R(V)$ respectively, then we have $C_e = C_{e^\perp} = 1$.*

Proof Let x be an arbitrary element in $V = eR(V)e^\perp$. Since $C_e x = x$ and C_e is a central projection in $R(V)$, we obtain

$$C_e x^* = (xC_e)^* = (C_e x)^* = x^*.$$

It follows that $C_e xy^* = xy^*$ and $C_e x^* y = x^* y$ for all $x, y \in V$. Then we must have $C_e = 1$ in $R(V)$ since $R(V)$ is the von Neumann algebra generated by V. We can prove $C_{e^\perp} = 1$ by a similar argument. ■

If V is a TRO (respectively, a W^*-TRO) contained in $B(K, H)$, then there is a natural left-$C(V)$ and right-$D(V)$ (respectively, left-$M(V)$ and right-$N(V)$) bimodule structure on V. More precisely, there exists a left-$C(V)$ (respectively, a left-$M(V)$) inner product on V given by

$$\prec x \,|\, y \succ = xy^* \tag{2.7}$$

and there exists a right-$D(V)$ (respectively, a right-$N(V)$) inner product on V given by

$$\langle x \mid y \rangle = x^* y \tag{2.8}$$

for all $x, y \in V$. With these two inner products, V is a Hilbert bimodule whose left and right module operations satisfy

$$\prec x \,|\, y \succ \cdot z = xy^* z = x\langle y \mid z \rangle,$$

and V is *full* (respectively, *weakly full*) in the sense that the spans of the inner products are norm dense (respectively, weak* dense) in corresponding C^*-algebras (respectively, von Neumann algebras). In this case, we can obtain C^*-isomorphisms

$$C(V) = K(V_D) \text{ and } D(V)^{\mathrm{op}} = K({}_C V),$$

where we let $K(V_D)$ denote the space of all compact right-$D(V)$ module homomorphisms on V and let $K({}_C V)$ denote the space of all compact left-$C(V)$ module homomorphisms on V (respectively, we can obtain normal $*$-isomorphisms

$$M(V) = B(V_N) \text{ and } N(V)^{\mathrm{op}} = B({}_M V),$$

where we let $B(V_N)$ denote the space of all bounded adjointable right-$N(V)$ module homomorphisms on V and let $B(_M V)$ denote the space of all bounded adjointable left-$M(V)$ module homomorphisms on V). Futhermore, Zettl proved in [39, Theorem 4.12] that if V is a W^*-TRO, then it is actually a self-dual Hilbert right-$N(V)$ module over the von Neumann algebra $N(V)$, *i.e.*, every bounded right-$N(V)$ module morphism $\tau\colon V \to N(V)$ has the form

$$\tau(x) = \langle y \mid x \rangle = y^* x \tag{2.9}$$

for some $y \in V$. Using a similar argument, one can show that V is also a self-dual Hilbert left-$M(V)$ module over the von Neumann algebra $M(V)$.

On the other hand, if V is a full Hilbert right-D module over a C^*-algebra D (respectively, a self-dual and weakly full Hilbert right-N module over a von Neumann algebra N), then we may obtain a triple product on V given by

$$\langle x, y, z \rangle = x \cdot \langle y \mid z \rangle$$

for all $x, y, z \in V$. Zettl proved in [39, Theorem 2.6 and Theorem 2.8] that there exist Hilbert spaces H and K, a (completely) isometric isomorphism $U\colon V \to U(V) \subseteq B(K, H)$ and a faithful $*$-representation $\pi\colon D \to B(K)$ (respectively, a normal faithful $*$-representation $\pi\colon N \to B(K)$) such that

$$U(x)\pi(a) = U(x \cdot a) \ \text{ and } \ U(x)^* U(y) = \pi(\langle x \mid y \rangle) \tag{2.10}$$

for all $x \in V$ and $a \in D$ (respectively, $a \in N$). It is easy to see from (2.10) that $U(V)$ is a TRO (respectively, a W^*-TRO) contained in $B(K, H)$.

Let us briefly describe this construction for the self-dual case. Let Λ denote the set of all normal states on N. Then we can obtain a family of cyclic normal $*$-representations $\{\pi_\varphi, K_\varphi, \xi_\varphi\}_{\varphi \in \Lambda}$ for N, and obtain a family of Hilbert spaces $\{H_\varphi\}_{\varphi \in \Lambda}$ by taking the GNS constructions on V with respect to the semi-inner products $\{\varphi(\langle \cdot \mid \cdot \rangle)\}_{\varphi \in \Lambda}$. For each $\varphi \in \Lambda$, we may define a contraction $U_\varphi\colon V \to B(K_\varphi, H_\varphi)$ given by

$$U_\varphi(x)[a]_{K_\varphi} = [x \cdot a]_{H_\varphi}$$

for all $x \in V$ and $a \in N$. Then $U = \sum_{\varphi \in \Lambda}{}^{\oplus} U_\varphi$ is an isometry from V into $B(K, H)$ with $H = \oplus_{\varphi \in \Lambda} H_\varphi$ and $K = \oplus_{\varphi \in \Lambda} K_\varphi$, and $\pi = \sum_{\varphi \in \Lambda}{}^{\oplus} \pi_\varphi$ is a normal faithful $*$-representation of N into $B(K)$ such that (2.10) is satisfied. By the self-duality, we can prove that $U(V)$ is strong operator closed in $B(K, H)$ and $U\colon V \to U(V)$ is a homeomorphism with respect to the topology on V generated by the semi-inner products $\{\varphi(\langle \cdot \mid \cdot \rangle)\}_{\varphi \in \Lambda}$ and the strong operator topology on $U(V)$. Due to this fact, we will simply call the topology generated by the semi-inner products $\{\varphi(\langle \cdot \mid \cdot \rangle)\}_{\varphi \in \Lambda}$ the *strong operator topology* on V. It is worthy to note that U is also a homeomorphism with respect to the weak* and the σ-weak topologies on V and $U(V)$. Therefore, V can be identified with the W^*-TRO $U(V)$.

Finally we note that if V has a separable predual, then $N = N(V)$ also has a separable predual. In this case, we may choose a faithful countable subset Λ_0 of Λ and thus obtain a normal faithful state φ on N. Then φ induces a standard representation of N on a separable Hilbert space K_φ, for which every normal state of N can be represented as a vector state, and V can be identified with the W^*-TRO $U_\varphi(V)$ contained in $B(K_\varphi, H_\varphi)$ such that the strong operator topology (respectively, weak* topology) on V corresponds to the strong operator topology (respectively, the σ-topology) on $U_\varphi(V)$. This shows that a W^*-TRO is separable (*i.e.*, can be represented on separable Hilbert spaces) if and only if it has a separable predual.

It has been discussed by Wittstock [37] and Blecher [1, 3] that every Hilbert right module (over a C^*-algebra D or a von Neumann algebra N) has a canonical operator space matrix norm given by

$$\|[x_{ij}]\| = \Big\| \Big[\sum_{k=1}^{n} \langle x_{ki} \mid x_{kj} \rangle \Big] \Big\|^{\frac{1}{2}} \tag{2.11}$$

for every $x = [x_{ij}] \in M_n(V)$. If V is a W^*-TRO contained in $B(K, H)$, then it is easy to see that the matrix norm given in (2.11) with respect to the right inner product $\langle x \mid y \rangle = x^* y$ is exactly the same as the canonical TRO matrix norm on V. Similarly since

$$\|[x_{ij}]\| = \big\| \big[x_{ij}\big] \big[x_{ij}\big]^* \big\|^{\frac{1}{2}} = \Big\| \Big[\sum_{k=1}^{n} \prec x_{ik} | x_{jk} \succ \Big] \Big\|^{\frac{1}{2}}, \tag{2.12}$$

the canonical TRO matrix norm on V also coincides with the matrix norm determined by its Hilbert left module structure.

In general, a Hilbert right (respectively, left) module carries a natural *column* (respectively, *row*) Hilbert module structure. For example, if V and W are W^*-TRO's such that $N(V) = N(W) = N$, then the *column direct sum*

$$V \oplus_c W = \Big\{ \begin{bmatrix} v \\ w \end{bmatrix} : v \in V, w \in W \Big\}$$

is again a W^*-TRO and is a self-dual Hilbert right module over N with the inner product given by

$$\Big\langle \begin{bmatrix} v_1 \\ w_1 \end{bmatrix} \Big| \begin{bmatrix} v_2 \\ w_2 \end{bmatrix} \Big\rangle = \langle v_1 \mid v_2 \rangle + \langle w_1 \mid w_2 \rangle.$$

Given an index set I, we let $C_I^w(V)$ (a notion introduced in [3]) denote the column direct sum of I-copies of V such that

$$\sup\{\| \sum_{\alpha \in S} \langle x_\alpha \mid x_\alpha \rangle \| : \text{ all finite subset } S \subseteq I\} < \infty.$$

This is just the weakly direct sum of the Hilbert right module V discussed by Paschke [26]. If I is an infinite set, then the inner product

$$\langle [x_\alpha] \mid [y_\alpha] \rangle = \sum_{\alpha \in I} \langle x_\alpha \mid y_\alpha \rangle$$

converges in the weak* topology on N.

Given index sets I and J, we usually let $M_{I,J} = B(\ell_2(J), \ell_2(I))$ and write M_I for $M_{I,I}$. Then for any W^*-TRO V, we have the complete isometry

$$M_{I,J}(V) = M_{I,J} \overline{\otimes} V = (T_{I,J} \widehat{\otimes} V_*)^*$$

where we let $\overline{\otimes}$ denote the normal spatial tensor product for dual operator spaces, $\widehat{\otimes}$ denote the operator space projective tensor product and $T_{I,J}$ denote the operator predual of $M_{I,J}$. It is easy to see that $C_I^w(V)$ can be identified with the first column $M_{I,1}(V)$ in $M_I(V)$, and it is again a self-dual Hilbert right-N module. But its left module structure is changed to left-$M_I(M(V))$ module structure with the left-$M_I(M(V))$ inner product given by

$$\prec [x_\alpha] \,|\, [y_\beta] \succ_{M_I(M(V))} = \left[\prec x_\alpha \,|\, y_\beta \succ\right] = \left[x_\alpha y_\beta^*\right].$$

Similarly, we can consider the *row direct sum*

$$V \oplus_r W = \{[v, w] : v \in V, w \in W\}$$

with the inner product given by

$$\prec [v_1, w_1] \,|\, [v_2, w_2] \succ = \prec v_1 \,|\, v_2 \succ + \prec w_1 \,|\, w_2 \succ.$$

Given an index set I, we can show that the *row direct sum* $R_I^w(V) = M_{1,I}(V)$ is a W^*-TRO, which preserves the same left-$M(V)$ module structure, but change the right-$N(V)$ module structure to the right-$M_I(N(V))$ module structure.

3 Stable W^*-TRO's

A W^*-TRO V is said to be *stable* if there is a TRO-isomorphism $V \cong M_\infty \overline{\otimes} V$. It is well-known that the von Neumann algebra $M_\infty = B(\ell_2(\mathbb{N}))$ is stable. Then for any W^*-TRO V, $M_\infty \overline{\otimes} V$ is a stable W^*-TRO. The main purpose of this section is to study the structure of stable W^*-TRO's.

Let V and W be two W^*-TRO's such that $N(V) = N(W)$. Then for any $x \in V$ and $y \in W$, we may define a map $\Theta_{x,y} : W \to V$ given by

$$\Theta_{x,y}(z) = x\langle y \mid z\rangle = x(y^*z) \tag{3.1}$$

for all $z \in W$. It is easy to see that $\Theta_{x,y}$ is completely bounded by $\|x\|\|y\|$ and is continuous with respect to the strong operator topologies on V and W. The following result can be regarded as a normal version of Kasparov's stabilization theorem for Hilbert C^*-modules (see Lance [23, Chapter 5]). We state and prove it in terms of W^*-TRO's.

Theorem 3.1 *Let V be a separable W^*-TRO, and let $M = M(V)$ and $N = N(V)$. Then we have the (completely isometric) TRO-isomorphisms*

$$C_\infty^w(N) \cong C_\infty^w(N) \oplus_c V \text{ and } R_\infty^w(M) \cong R_\infty^w(M) \oplus_r V.$$

Proof Let us prove the TRO-isomorphism $C^w_\infty(N) \cong C^w_\infty(N) \oplus_c V$. As we discussed in §2, the column direct sum $C^w_\infty(N)$ is a Hilbert right-N module with inner product given by

$$\langle [a_n] \mid [b_n] \rangle = \sum_{n=1}^{\infty} a_n^* b_n.$$

If we let 1_N denote the unital element of N, then we may obtain a canonical orthonormal basis $\{f_n = E_{n,1} \otimes 1_N\}$ for $C^w_\infty(N)$ such that every element $a \in C^w_\infty(N)$ can be uniquely written as

$$a = \sum_{n=1}^{\infty} f_n \cdot \langle f_n \mid a \rangle = \sum_{n=1}^{\infty} f_n(f_n^* a).$$

Since V is a separable W^*-TRO, it is a self-dual Hilbert left module over M. Then we may apply the left-M module analogue of Paschke's result [26, Theorem 3.12] to obtain a (finite or countable) family of non-zero partial isometries $\{v_k\}$ in V which is maximal under the condition that

$$v_k v_l^* = \prec v_k \,|\, v_l \succ = 0 \tag{3.2}$$

for all $k \neq l$. The condition (3.2) implies that $\{v_k^* v_k\}$ is a family of mutually orthogonal projections in N such that $\sum_k v_k^* v_k = 1_N$. Let us assume that $\{y_n\}$ is a sequence of elements in V, which consists of v_k and each v_k repeatedly appears infinitely many times in $\{y_n\}$. Then $T\colon C^w_\infty(N) \to C^w_\infty(N) \oplus_c V$ defined by

$$T(a) = \sum_{n=1}^{\infty} \begin{bmatrix} \frac{1}{4^n} \Theta_{f_n, f_n}(a) \\ \frac{1}{2^n} \Theta_{y_n, f_n}(a) \end{bmatrix} = \sum_{n=1}^{\infty} \begin{bmatrix} \frac{1}{4^n} f_n(f_n^* a) \\ \frac{1}{2^n} y_n(f_n^* a) \end{bmatrix}$$

is a completely bounded and strong operator continuous map from $C^w_\infty(N)$ into $C^w_\infty(N) \oplus_c V$. Moreover, T is an adjointable map with T^* given by

$$T^*\left(\begin{bmatrix} b \\ w \end{bmatrix} \right) = \sum_{n=1}^{\infty} \frac{1}{4^n} \Theta_{f_n, f_n}(b) + \frac{1}{2^n} \Theta_{f_n, y_n}(w).$$

It is clear that T^* is also completely bounded and strong operator continuous from $C^w_\infty(N) \oplus_c V$ into $C^w_\infty(N)$.

Applying the same calculation as that given in the proof of Kasparov's stable theorem for Hilbert C^*-modules, we can show that for each $m > n$ with $y_m = y_n$,

$$T(2^m f_m) = \begin{bmatrix} \frac{1}{2^m} f_m \\ y_n \end{bmatrix}.$$

Since there are infinitely many such $m > n$, we can conclude that $\left[\begin{smallmatrix} 0 \\ y_n \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} f_n \\ 0 \end{smallmatrix}\right]$ are contained in the norm closure of $T(C^w_\infty(N))$. Therefore, $T(C^w_\infty(N))$ is strong operator dense in $C^w_\infty(N) \oplus_c V$. Using a similar argument, we can prove that $T^*(C^w_\infty(N) \oplus_c V)$ is also strong operator dense in $C^w_\infty(N)$.

Then it is easy to show that T^*T and its positive square root $|T|$ have strong operator dense ranges in $C^w_\infty(N)$. Since

$$\big\langle\, |T|(a) \,\big|\, |T|(b) \big\rangle = \langle a \mid T^*T(b)\rangle = \langle T(a) \mid T(b)\rangle$$

for all $a, b \in C^w_\infty(N)$, we may define an isometric right-N module homomorphism u from $|T|(C^w_\infty(N))$ onto $T(C^w_\infty(N))$ given by

$$u\big(|T|(a)\big) = T(a).$$

We note that u is actually a complete isometry since

$$\begin{aligned} \|[|T|(a_{ij})]\|^2 &= \Big\| \Big[\sum_{k=1}^n \langle |T|(a_{ki}) \mid |T|(a_{kj})\rangle_N \Big] \Big\| \\ &= \Big\| \Big[\sum_{k=1}^n \langle T(a_{ki}) \mid T(a_{kj})\rangle_N \Big] \Big\| \\ &= \big\| [T(a_{ij})] \big\|^2 \end{aligned}$$

for any $[a_{ij}] \in M_n(C^w_\infty(N))$. We may easily extend u to a complete isometry from the norm closure $\overline{|T|(C^w_\infty(N))}^{\|\cdot\|}$ onto the norm closure $\overline{T(C^w_\infty(N))}^{\|\cdot\|}$. However, we need to show that u can be further extended to a complete isometry from the strong operator closure $C^w_\infty(N) = \overline{|T|(C^w_\infty(N))}^{s.o.t}$ onto the strong operator closure $C^w_\infty(N) \oplus_c V = \overline{T(C^w_\infty(N))}^{s.o.t}$.

Given $a \in C^w_\infty(N)$, it follows from the W^*-TRO analogue of Kaplansky density theorem (see [39, Proposition 1.4]) that there exists a net of elements $\{|T|(a_\alpha)\}$ such that $\||T|(a_\alpha)\| \le \|a\|$ and $|T|(a_\alpha) \to a$ in the strong operator topology on $C^w_\infty(N)$. Since $\||T|(a_\alpha)\| = \|T(a_\alpha)\|$ and

$$\big\langle\, T(a_\alpha) - T(a_{\alpha'}) \,\big|\, T(a_\alpha) - T(a_{\alpha'}) \big\rangle = \big\langle\, |T|(a_\alpha) - |T|(a_{\alpha'}) \,\big|\, |T|(a_\alpha) - |T|(a_{\alpha'}) \big\rangle$$

we can conclude that $\{T(a_\alpha)\}$ is a bounded strong operator Cauchy net in $C^w_\infty(N) \oplus_c V$. Then $\{T(a_\alpha)\}$ strong operator converges to an element

$$\begin{bmatrix} b \\ w \end{bmatrix} \in M_{\infty,1}(N) \oplus_c V.$$

We define $\tilde{u}(a) = \left[\begin{smallmatrix} b \\ w \end{smallmatrix}\right]$.

It is routine to verify that $\tilde{u}$ is a well-defined completely isometric right-N module extension of u to the whole space $C^w_\infty(N)$. Similarly, we may obtain a completely isometric right-N module extension $\widetilde{u^{-1}}$ of the map $u^{-1}\colon T(C^w_\infty(N)) \to |T|(C^w_\infty(N))$ to the whole space $C^w_\infty(N) \oplus_c V$. By the construction of $\tilde{u}$ and $\widetilde{u^{-1}}$, we obtain

$$\widetilde{u^{-1}} \circ \tilde{u} = id_{C^w_\infty(N)} \text{ and } \tilde{u} \circ \widetilde{u^{-1}} = id_{C^w_\infty(N) \oplus_c V}.$$

https://doi.org/10.4153/CJM-2004-038-7 Published online by Cambridge University Press

This shows that $\tilde{u}$ is a completely isometric TRO-isomorphism from $C^w_\infty(N)$ onto $C^w_\infty(N) \oplus_c V$. ■

We acknowledge that after the paper was submitted, D. Blecher pointed out to the author at the 2002 GPOTS that one can give another simpler proof for Theorem 3.1 by applying the normal version of "Eilenberg swindle" he used in [2, Theorem 8.3]. We outline this proof in the following for $R^w_\infty(M) \cong R^w_\infty(M) \oplus_r V$. Let us first assume that $\{v_k\}$ is the family of partial isometries obtained in the proof of Theorem 3.1. We can identify V with a completely contractively complemented left-M submodule $\iota(V)$ of $R^w_\infty(M)$ by the completely isometric left-M module inclusion

$$\iota \colon x \in V \hookrightarrow [xv_1^* \, xv_2^* \cdots] \in R^w_\infty(M).$$

In this case, we have a completely contractive left-M module projection π from $R^w_\infty(M)$ onto $\iota(V)$ given by

$$\pi([a_1 \, a_2 \, \cdots]) = [\tilde{v}v_1^* \, \tilde{v}v_2^* \, \cdots]$$

where we let $\tilde{v} = \sum_{k=1} a_k v_k \in V$. Then $W = \ker \pi$ is a left-M submodule of $R^w_\infty(M)$ and every element $[a_1 \, a_2 \, \cdots] \in R^w_\infty(M)$ can be decomposed into the (direct) sum

$$[a_1 \, a_2 \, \cdots] = [\tilde{v}v_1^* \;\; \tilde{v}v_2^* \cdots] \; + \; [a_1 - \tilde{v}v_1^* \;\; a_2 - \tilde{v}v_2^* \; \cdots] \in \iota(V) + W = \iota(V) \oplus W.$$

A routine calculation shows that this determines a completely isometric row decomposition

$$R^w_\infty(M) \cong \iota(V) \oplus_r W \cong V \oplus_r W. \tag{3.3}$$

Since $R^w_\infty(M)$ is completely isometric to $R^w_\infty(R^w_\infty(M))$ and $W \oplus_r V$ is completely isometric to $V \oplus_r W$, we get the complete isometries

$$R^w_\infty(M) \cong R^w_\infty(V \oplus_r W) = V \oplus_r R^w_\infty(W \oplus_r V) \cong V \oplus_r R^w_\infty(M) \cong R^w_\infty(M) \oplus_r V.$$

Therefore, $R^w_\infty(M)$ is completely isometrically TRO-isomorphic to $R^w_\infty(M) \oplus_r V$.

We note that if V is a stable W^*-TRO, then $M(V)$ and $N(V)$ are stable von Neumann algebras. In this case, we may obtain the TRO-isomorphisms

$$V \cong C^w_n(V) \text{ and } V \cong R^w_n(V) \tag{3.4}$$

for arbitrary $1 \leq n \leq \infty$ since V is TRO-isomorphic to $M_\infty \overline{\otimes} V$ and M_∞ is TRO-isomorphic to $C^w_n(M_\infty) = M_{n\times\infty,\infty}$ and $R^w_n(M_\infty) = M_{\infty,n\times\infty}$.

Theorem 3.2 *Let V be a separable stable W^*-TRO. Then we have the (completely isometric) TRO-isomorphisms*

$$V \cong M(V) \textit{ and } V \cong N(V).$$

Proof We will only prove the TRO-isomorphism $V \cong N(V)$. The argument for $V \cong M(V)$ is similar. Let us write $N = N(V)$. We have seen from the proof of Theorem 3.1 that there exists a family of non-zero partial isometries $\{v_k\}$ in V such that $v_k^* v_k$ are mutually orthogonal projections in N and $\sum_{k=1} v_k^* v_k = 1_N$. Since V acts on separable Hilbert spaces, this is either a finite or a countable set. Let us assume that it consists of n elements with $1 \leq n \leq \infty$. Then the column vector $v = [v_k]$ is a partial isometry in $C_n^w(V)$ such that

$$v^* v = \sum_{k=1} v_k^* v_k = 1_N.$$

We may define completely contractive right-N module morphisms $T\colon N \to C_n^w(V)$ and $S\colon C_n^w(V) \to N$ by letting

$$T(a) = v \cdot a \text{ and } S(x) = v^* x$$

for all $a \in N$ and $x = [x_k] \in C_n^w(V)$. Since $S \circ T = id_N$, T is a completely isometric TRO-isomorphism from N onto $T(N) = T \circ S(C_n^w(V)) = vv^* C_n^w(V)$, which is a completely contractive complemented Hilbert right-N submodule of $C_n^w(V)$. Let $e_v = vv^*$ be the corresponding projection in $M_n(M)$ and let $e_v^\perp = 1_{M_n(M)} - e_v$. Then $W = \ker S = e_v^\perp C_n^w(V)$ is a W^*-TRO contained in $C_n^w(V)$ and we can write

$$C_n^w(V) = N \oplus_c W.$$

Since V is stable, then so is N. Therefore, we have the (completely isometric) TRO-isomorphisms

$$V \cong C_n^w(V) = N \oplus_c W \cong C_\infty^w(N) \oplus_c W \cong C_\infty^w(N) \cong N,$$

where we used Theorem 3.1 for the TRO-isomorphism $C_\infty^w(N) \oplus_c W \cong C_\infty^w(N)$. ■

In [31], Rieffel introduced Morita equivalence for von Neumann algebras. We recall by an equivalent definition that two von Neumann algebras M and N are *Morita equivalent* if there exists a W^*-TRO V such that $M = M(V)$ and $N = N(V)$. Using Theorem 3.2, we can easily prove the following corollary, which is known by experts, but has never been proved in the literature.

Corollary 3.3 *Let M and N be two separable von Neumann algebras. Then M is Morita equivalent to N if and only if they are stable $*$-isomorphic,* i.e., *there is a $*$-isomorphism*

$$M_\infty \overline{\otimes} M \cong M_\infty \overline{\otimes} N.$$

Proof Let us first assume that M is Morita equivalent to N via a W^*-TRO V. Then V must be separable and it is easy to see that $M_\infty \overline{\otimes} M$ is Morita equivalent to $M_\infty \overline{\otimes} N$ via the separable W^*-TRO $M_\infty \overline{\otimes} V$. Since $M_\infty \overline{\otimes} V$ is stable, it is known from Theorem 3.2 that $M_\infty \overline{\otimes} V$ is TRO-isomorphic to both $M_\infty \overline{\otimes} M$ and $M_\infty \overline{\otimes} N$. It follows that $M_\infty \overline{\otimes} M$ is TRO-isomorphic and thus is $*$-isomorphic to $M_\infty \overline{\otimes} N$.

https://doi.org/10.4153/CJM-2004-038-7 Published online by Cambridge University Press

On the other hand, it is easy to see that M and N are Morita equivalent to $M_\infty \overline{\otimes} M$ and $M_\infty \overline{\otimes} N$, respectively. Since Morita equivalence is an equivalent condition, the $*$-isomorphism

$$M_\infty \overline{\otimes} M \cong M_\infty \overline{\otimes} N$$

implies that M is Morita equivalent to N. ■

It is interesting to note that if V is a W^*-TRO, then $R(V)$ is also Morita equivalent to $M(V)$ (respectively, to $N(V)$). This can be obtained by considering the W^*-TRO $\tilde{V} = [M(V), V]$ and the linking von Neumann algebra

$$R(\tilde{V}) = \begin{bmatrix} M(V) & \tilde{V} \\ \tilde{V}^\sharp & R(V) \end{bmatrix} = \begin{bmatrix} M(V) & M(V) & V \\ M(V) & M(V) & V \\ V^\sharp & V^\sharp & N(V) \end{bmatrix}. \tag{3.5}$$

Therefore, if V is a stable separable W^*-TRO, then V is also TRO-isomorphic to $R(V)$ by Theorem 3.2 and 3.5.

Using Theorem 3.1, we can prove the following normal version of the Kasparov representation theorem (see Lance [23, Theorem 6.5]).

Theorem 3.4 *Let N and M be separable von Neumann algebras. If $\rho\colon N \to M$ is a normal unital completely positive map, then there exists a normal unital $*$-homomorphism $\pi\colon N \to M_\infty \overline{\otimes} M$ such that*

$$\rho(a) = e_1^* \pi(a) e_1,$$

where $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix}$ *is a unit vector in $M_{\infty,1}$.*

Proof Since ρ is a unital completely positive map from N into M, we can define an M-valued semi-inner product on $N \otimes C_\infty^w(M)$ given by

$$\langle a \otimes [x_n] \mid b \otimes [y_n] \rangle_M = \sum_{n=1}^{\infty} x_n^* \rho(a^* b) y_n \tag{3.6}$$

for all $a, b \in N$ and $[x_n], [y_n] \in C_\infty^w(M)$. If we let

$$N_M = \{z \in N \otimes C_\infty^w(M) : \langle z \mid z \rangle_M = 0\}$$

then $(N \otimes C_\infty^w(M))/N_M$ is a pre-Hilbert right-M module with a natural operator space matrix norm given in (2.11). In the following we show that there exists a completely isometric representation of this pre-Hilbert right-M module on some Hilbert spaces K_φ and H_φ.

Let us assume that φ is a normal faithful state on M. It is well-known from the GNS construction that φ induces a normal faithful unital cyclic representation $(\pi_\varphi, K_\varphi, \xi_\varphi)$ for M. We identify M with $\pi_\varphi(M)$ in $B(K_\varphi)$ and for any $y \in M$ we let

$[y]_{K_\varphi} = \pi_\varphi(y)([1]_\varphi)$ denote the corresponding element in K_φ. The normal faithful state φ also induces a semi-inner product on $N \otimes C^w_\infty(M)$ given by

$$\langle a \otimes [x_n] \mid b \otimes [y_n]\rangle_\varphi = \varphi\Big(\sum_{n=1}^{\infty} x_n^* \rho(a^*b) y_n\Big). \tag{3.7}$$

We let $N_\varphi = \{z \in N \otimes C^w_\infty(M) : \langle z \mid z\rangle_\varphi = 0\}$ and let H_φ denote the norm closure of $(N \otimes C^w_\infty(M))/N_\varphi$. Since φ is faithful, we have $N_M = N_\varphi$. Then we can obtain a completely isometric (right-M module) inclusion $U_\rho \colon (N \otimes C^w_\infty(M))/N_M \to B(K_\varphi, H_\varphi)$ given by

$$U_\rho(a \otimes [x_n])([y]_{K_\varphi}) = \big[a \otimes [x_n y]\big]_{H_\varphi}. \tag{3.8}$$

Let us identify $(N \otimes C^w_\infty(M))/N_M$ with $U_\rho((N \otimes C^w_\infty(M))/N_M)$ and let

$$V_\rho = \overline{(N \otimes C^w_\infty(M))/N_M}^{s.o.t}$$

denote the strong operator closure of $(N \otimes C^w_\infty(M))/N_M$ in $B(K_\varphi, H_\varphi)$. Then V_ρ is a separable W^*-TRO contained in $B(K_\varphi, H_\varphi)$ with the triple product given by

$$\begin{aligned}\langle a \otimes [x_n], b \otimes [y_n], c \otimes [z_n]\rangle &= a \otimes \big[x_n \langle b \otimes [y_n], c \otimes [z_n]\rangle_M\big] \\ &= a \otimes \Big[x_n\Big(\sum_{k=1}^{\infty} y_k^* \rho(b^*c) z_k\Big)\Big].\end{aligned}$$

In this case V_ρ is a self-dual Hilbert right-M module with $M(V_\rho) = M$. The map $v \colon C^w_\infty(M) \to V_\rho$ given by

$$v([x_n]) = 1 \otimes [x_n]$$

is a (strong operator continuous) completely isometric right-M module inclusion and $\pi_\rho \colon N \to B(V_{\rho M})$ given by

$$\pi_\rho(a)(b \otimes [x_n]) = ab \otimes [x_n]$$

is a normal unital $*$-homomorphism from N into $B(V_{\rho M})$ such that

$$\rho^\infty(a) = I_\infty \otimes \rho(a) = v^* \pi_\rho(a) v$$

for all $a \in N$. Then $G = v(C^w_\infty(M))$ and $G^\perp$ are self-dual Hilbert right-M submodules of V_ρ and we can decompose V_ρ into the column direct sum

$$V_\rho = G \oplus_c G^\perp.$$

It follows from Theorem 3.1 that we have the completely isometric Hilbert right-M module isomorphisms (or W^*-TRO-isomorphisms)

$$V_\rho = G \oplus_c G^\perp \cong C^w_\infty(M) \oplus_c G^\perp \cong C^w_\infty(M).$$

Let u be the completely isometric Hilbert right-M module isomorphism from $C^w_\infty(M)$ onto $G^\perp \oplus_c C^w_\infty(M)$. Then $v \oplus u$ is a unitary operator from $C^w_\infty(M) \oplus_c C^w_\infty(M)$ onto $V_\rho \oplus_c C^w_\infty(M)$. If we assume that π_1 is a normal unital $*$-homomorphism from N into $B((G^\perp \oplus_c C^w_\infty(M))_M)$, then we obtain a normal unital $*$-homomophism

$$\pi(a) = \begin{bmatrix} v^* & 0 \\ 0 & u^* \end{bmatrix} \begin{bmatrix} \pi_\rho(a) & 0 \\ 0 & \pi_1(a) \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix}$$

from N into $B((C^w_\infty(M) \oplus_c C^w_\infty(M))_M) = M_{\infty \sqcup \infty} \overline{\otimes} M \cong M_\infty \overline{\otimes} M$ such that

$$\rho(a) = e_1^* \pi(a) e_1. \qquad \blacksquare$$

4 Type Decomposition for W^*-TRO's

Let us first recall the type decomposition for von Neumann algebras. A von Neumann algebra R is said to be of type I if it has an abelian projection with central cover 1. If R has no non-zero abelian projection but has a finite projection with central cover 1, then R is said to be of type II (type II_1 if 1 is finite and type II_∞ if 1 is properly infinite). A von Neumann algebra is of type III if it has no non-zero finite projection. It is known that every von Neumann algebra R has a unique type decomposition

$$R = p_1 R \oplus p_2 R \oplus p_3 R,$$

where p_i are mutually orthogonal central projections in R such that $p_1 R$, $p_2 R$ and $p_2 R$ are von Neumann subalgebras of type I, II, and III (see details in Takesaki [35] and Kadison and Ringrose [18]).

Suppose that we are given a W^*-TRO V, which is (always non-degenerately) contained in $B(K, H)$. We let $R(V)$ be its linking von Neumann algebra and let e and $e^\perp = 1 - e$ be the projections given in (2.5). Then $R(V)$ has a unique type decomposition

$$R(V) = p_1 R(V) \oplus p_2 R(V) \oplus p_3 R(V).$$

It is clear that for $i = 1, 2, 3$, $p_i e$ and $e^\perp p_i$ are projections in $R(V)$ dominated by the central projections p_i. Actually, p_i are the central covers of $p_i e$ and $e^\perp p_i$. Then we can decompose V into the direct sum of

$$\begin{aligned} V &= p_1 V \oplus p_2 V \oplus p_2 V \\ &= p_1 e R(V) e^\perp p_1 \oplus p_2 e R(V) e^\perp p_2 \oplus p_3 e R(V) e^\perp p_3. \end{aligned}$$

For each $i = 1, 2, 3$, $p_i V$ is a W^*-TRO contained in $B(p_i K, p_i H)$. Since

$$(p_i V)(p_i V)^\sharp = p_i V V^\sharp \text{ and } (p_i V)^\sharp (p_i V) = p_i V^\sharp V,$$

we can get

$$M(p_i V) = p_i M(V) \text{ and } N(p_i V) = p_i N(V),$$

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and the linking von Neumann algebra $R(p_iV)$ of p_iV is equal to

$$p_iR(V) = \begin{bmatrix} p_iM(V) & p_iV \\ (p_iV)^\sharp & p_iN(V) \end{bmatrix}. \tag{4.1}$$

In this case, the compression von Neumann subalgebras $p_iM(V)$ and $p_iN(V)$ have the same type as $p_iR(V)$ (see [18, Exercise 6.9.16]).

This suggests that we may define the type of a W^*-TRO according to the type of its linking von Neumann algebra, *i.e.*, a W^*-TRO V is said to be of *type* I, II or III if its linking von Neumann algebra $R(V)$ is of type I, II or III. A type I W^*-TRO V is said to be of *type* $I_{m,n}$ if $M(V)$ is of type I_m and $N(V)$ is of type I_n for some cardinal numbers m and n. A type II W^*-TRO V is said to be of *type* $II_{1,1}$, $II_{1,\infty}$, $II_{\infty,1}$ or $II_{\infty,\infty}$ if correspondingly $M(V)$ is of type II_1 or II_∞ and $N(V)$ is of type II_1 or II_∞.

In the following let us first discuss the structure of type I W^*-TRO's. Since every von Neumann algebra R of type I has the form

$$R = \sum_\alpha{}^{\oplus} B(\ell_2(L_\alpha)) \,\overline{\otimes}\, L_\infty(X_\alpha, \mu_\alpha) = \sum_\alpha{}^{\oplus} M_{L_\alpha} \,\overline{\otimes}\, L_\infty(X_\alpha, \mu_\alpha),$$

it is easy to see that for any subsets $I_\alpha, J_\alpha \subseteq L_\alpha$

$$V = \sum_\alpha{}^{\oplus} M_{I_\alpha,J_\alpha} \,\overline{\otimes}\, L_\infty(X_\alpha, \mu_\alpha)$$

is an off-diagonal corner of R and thus is a W^*-TRO of type I. The following theorem shows that every W^*-TRO of type I can be expressed in a such form.

Theorem 4.1 *If V is a W^*-TRO of type I, then we have the TRO-isomorphism*

$$V \cong \sum_\alpha{}^{\oplus} M_{I_\alpha,J_\alpha} \,\overline{\otimes}\, L_\infty(X_\alpha, \mu_\alpha). \tag{4.2}$$

Proof Let us write $V = eR(V)e^\perp$. Since $R(V)$ is a von Neumann algebra of type I, there exists a non-zero abelian projection f in $R(V)$ such that $C_f = 1$. If there exist index sets I and J such that e can be written as a sum $e = \sum_{i\in I} e_i$ of equivalent abelian projections e_i in $R(V)$ and $e^\perp$ can be written as a sum $e^\perp = \sum_{j\in J} e_j^\perp$ of equivalent abelian projections $e_j^\perp$ in $R(V)$. Then we must have $C_{e_i} = C_{e_j^\perp} = 1$. It follows that e_i and $e_j^\perp$ are mutually orthogonal equivalent abelian projections in $R(V)$. Since the sum

$$\sum_{i\in I} e_i + \sum_{j\in J} e_j^\perp = e + e^\perp = 1,$$

we can find a matrix unit $\{u_{i,j} : i, j \in I \sqcup J\}$ in $R(V)$ such that $e_i = u_{i,i}$ for $i \in I$ and $e_j^\perp = u_{j,j}$ for $j \in J$. This matrix unit determines a $*$-isomorphism

$$R(V) \cong M_{I\sqcup J} \,\overline{\otimes}\, L_\infty(X, \mu),$$

where $L_\infty(X,\mu)$ is $*$-isomorphic to the center $\mathcal{C}$ of $R(V)$. Therefore, we may obtain the TRO-isomorphism

$$V = eR(V)e^\perp \cong M_{I,J} \overline{\otimes} L_\infty(X,\mu).$$

In general, since f is an abelian projection with $C_f = C_e = C_{e^\perp} = 1$, we can apply Corollary 6.5.5 in [18] to e and $e^\perp$ (simultaneously) to obtain a family $\{z_\alpha\}$ of central projections in $R(V)$ with sum $\sum_\alpha z_\alpha = 1$ such that for each α, ez_α is a sum of m_α equivalent abelian projections in $R(V)$ and $e^\perp z_\alpha$ is a sum of n_α equivalent abelian projections in $R(V)$. Then for each α, there exist index sets I_α and J_α with cardinal numbers m_α and n_α respectively and there exist equivalent abelian projections e_i ($i \in I_\alpha$) and $e_j^\perp$ ($j \in J_\alpha$) in $R(V)$ such that $ez_\alpha = \sum_{i\in I_\alpha} e_i$ and $e^\perp z_\alpha = \sum_{j\in J_\alpha} e_j^\perp$. Since $z_\alpha = ez_\alpha + e^\perp z_\alpha$, e_i and $e_j^\perp$ are mutually orthogonal equivalent projections such that $z_\alpha = C_{ez_\alpha} = C_{e^\perp z_\alpha}$. It follows from the above argument that we can obtain the TRO-isomorphism

$$Vz_\alpha = ez_\alpha R(V) z_\alpha e^\perp z_\alpha \cong M_{I_\alpha,J_\alpha} \overline{\otimes} L_\infty(X_\alpha,\mu_\alpha).$$

Then we can conclude that

$$V = \sum_\alpha{}^\oplus V z_\alpha \cong \sum_\alpha{}^\oplus M_{I_\alpha,J_\alpha} \overline{\otimes} L_\infty(X_\alpha,\mu_\alpha).$$

This completes the proof. ∎

We note that every finite dimensional TRO can be identified with a finite direct sum of rectangular matrix algebras, *i.e.*, it has the form

$$V = M_{m(1),n(1)} \oplus \cdots \oplus M_{m(k),n(k)}$$

(see [7] and [34]). Therefore, every finite dimensional TRO is of type I.

Proposition 4.2 *Let $V = eR(V)e^\perp$ be a W^*-TRO. If e and $e^\perp$ are properly infinite projections, then there is a W^*-TRO V_1 contained in V such that V is TRO-isomorphic to $M_\infty \overline{\otimes} V_1$. Therefore, V is a stable W^*-TRO.*

Proof Since e is a properly infinite projection, there exists a sequence of mutually orthogonal projections e_n in $eR(V)e$ such that each e_n is equivalent to e and $e = \sum_{n=1}^\infty e_n$ in $eR(V)e$ (see the Halving lemma in [18] and [35]). If we let v_n be the partial isometries in $eR(V)e$ such that

$$v_n^* v_n = e_n \text{ and } v_n v_n^* = e,$$

then we have $v_m v_n^* = 0$ if $m \neq n$ and thus $e_{m,k} = v_m^* v_k$ form a matrix unit in $eR(V)e$. In this case, it is easy to see that

$$v = \left[e_{1,1},\ldots,e_{n,1},\ldots\right] = \left[v_1^* v_1,\ldots,v_n^* v_1,\ldots\right]$$

is a partial isometry in $R^w_\infty(eR(V)e)$ such that

$$vv^* = \sum_{n=1}^{\infty} v_n^* e v_n = e \text{ and } v^* v = \big[v_1^* v_m v_n^* v_1\big] = I_\infty \otimes e_{1,1}.$$

Similarly, there exists a sequence of mutually orthogonal projections $e_n^\perp$ in $e^\perp R(V) e^\perp$ such that $e^\perp = \sum_{n=1}^\infty e_n^\perp$ and

$$e_n^\perp = w_n^* w_n \text{ and } e^\perp = w_n w_n^*$$

for some partial isometries $w_n \in e^\perp R(V) e^\perp$. Then $e^\perp_{m,n} = w_m^* w_n$ is a matrix unit in $e^\perp R(V) e^\perp$. In this case,

$$w = [e^\perp_{1,1}, \dots, e^\perp_{n,1}, \dots] = [w_1^* w_1, \dots, w_n^* w_1, \dots]$$

is a partial isometry in $R^w_\infty(e^\perp R(V) e^\perp)$ such that

$$ww^* = e^\perp \text{ and } w^* w = I_\infty \otimes e^\perp_{1,1}.$$

Since for every $x \in V$ we can write

$$x = exe^\perp = \sum_{m,n=1}^{\infty} e_m x e_n^\perp \in \sum_{m,n=1}^{\infty} e_m V e_n^\perp,$$

we can decompose V into the direct sum

$$V = \sum_{m,n=1}^{\infty} e_m V e_n^\perp,$$

where each

$$e_m V e_n^\perp = e e_m V e_n^\perp e^\perp \subseteq eR(V) e^\perp = V$$

is a W^*-TRO contained in V and is TRO-isomorphic to $e_1 V e_1^\perp$. Let $V_1 = e_1 V e_1^\perp$. Then we may define a map $\phi\colon V \to M_\infty \overline{\otimes} V_1$ given by

$$\phi(x) = v^* x w = \big[v_1^* v_m x w_n^* w_1\big]$$

for every $x \in V$. It is easy to show that ϕ is a TRO-isomorphism from V onto $M_\infty \overline{\otimes} V_1$, and its inverse $\psi\colon M_\infty \overline{\otimes} V_1 \to V$ is given by

$$\psi([x_{m,n}]) = \sum_{m,n=1}^{\infty} v_m^* v_1 x_{m,n} w_1^* w_n.$$

This shows that V is TRO-isomorphic to $M_\infty \overline{\otimes} V_1$. ■

The following corollary is an immediate consequence of Theorem 3.2 and Proposition 4.2.

Corollary 4.3 *If V is a separable W^*-TRO of type $I_{\infty,\infty}$, type $II_{\infty,\infty}$ or type III, then V is stable and thus is TRO-isomorphic to a von Neumann algebra ($M(V)$, $N(V)$ or $R(V)$).*

Theorem 4.4 *If V is a W^*-TRO of type $II_{1,\infty}$ (respectively, of type $II_{\infty,1}$), then there exists an index set I such that V is TRO-isomorphic to $R^w_I(M(V))$ (respectively, TRO-isomorphic to $C^w_I(N(V))$).*

Proof We will only prove the case when V is a W^*-TRO of type $II_{1,\infty}$. The proof for the type $II_{\infty,1}$ case is similar. Let $V = eR(V)e^\perp$ be a W^*-TRO of type $II_{1,\infty}$. Then e is a finite projection and $e^\perp$ is a properly infinite projection in $R(V)$. Since $C_e = C_{e^\perp} = 1$, there exists a family of mutually orthogonal subprojections $\{e^\perp_\alpha\}_{\alpha\in I}$ of $e^\perp$ such that every $e^\perp_\alpha$ is equivalent to e and $\sum_{\alpha\in I} e^\perp_\alpha = e^\perp$, where the sum converges in the strong operator topology in $R(V)$ (see [18, Theorem 6.3.12]).

Let v_α be the partial isometies in $R(V)$ such that $e = v_\alpha v^*_\alpha$ and $e^\perp_\alpha = v^*_\alpha v_\alpha$ for all $\alpha \in I$. Then we have

$$ev_\alpha e^\perp = (v_\alpha v^*_\alpha)v_\alpha e^\perp = v_\alpha e^\perp_\alpha e^\perp = v_\alpha e^\perp_\alpha = v_\alpha$$

for every $\alpha \in I$. This shows that $v_\alpha \in V$ and $v_\alpha R(V)v^*_\beta \subseteq eR(V)e = M(V)$. We actually have

$$v_\alpha R(V)v^*_\beta = M(V)$$

since

$$M(V) = eR(V)e = v_\alpha(v^*_\alpha R(V)v_\beta)v^*_\beta \subseteq v_\alpha R(V)v^*_\beta.$$

For our convenience, let us assume that $v_0 = e$ and let $J = \{0\} \cup I$. Put all v_α in one column and let $v_J = [v_\alpha]_{\alpha\in J}$ denote this column vector in $C^w_I(R(V))$. Then v_J is an isometry since

$$v^*_J v_J = \sum_{\alpha\in J} v^*_\alpha v_\alpha = e + e^\perp = 1.$$

Let us consider a map $\phi\colon R(V) \to M_J \overline{\otimes} M(V)$ defined by

$$\phi(x) = [v_\alpha x v^*_\beta]$$

for $x \in R(V)$ and $\alpha, \beta \in J$. It is easy to see that ϕ is a unital map since

$$\phi(1) = [v_\alpha v^*_\beta] = 1_J \otimes e.$$

Moreover, ϕ is a $*$-homomorphism since

$$\phi(x)^* = [v_\alpha x v^*_\beta]^* = [v_\beta x^* v^*_\alpha] = \phi(x^*)$$

and

$$\phi(xy) = [v_\alpha xyv^*_\beta] = [v_\alpha xv^*_\beta][v_\beta yv^*_\gamma] = \phi(x)\phi(y)$$

for all $x, y \in R(V)$. It is easy to verify that ϕ is actually a $*$-isomorphism by considering its inverse map $\psi\colon M_J \overline{\otimes} M(V) \to R(V)$ defined by

$$\psi([x_{\alpha,\beta}]) = \sum_{\alpha\in J} v_\alpha^* x_{\alpha,\beta} v_\beta$$

for all $x = [x_{\alpha,\beta}] \in M_J \overline{\otimes} M(V)$. Since an element $x \in V$ if and only if

$$x = exe^\perp = \sum_{\beta\in I} v_0 v_0^* x v_\beta^* v_\beta,$$

ϕ restricted to $V = eR(V)e^\perp$ induced a TRO-isomorphism from V onto $R_I^w(M(V))$. ∎

We note that if V in Theorem 4.4 has a separable predual, then we may choose I to be a countable set. In this case we may obtain a TRO-isomorphism $V \cong R_\infty^w(M(V))$ (respectively, $V = C_\infty^w(N(V))$). The structure of a type $II_{1,1}$ W^*-TRO is a little bit more complicated. We will discuss this in next section.

5 Injectivity and Rectangular AFD Property for W^*-TRO's

The study of approximately finite dimensional (or hyperfinite) factors and von Neumann algebras has been a central topic of operator algebras for several decades. One of the most deep and important results on this topic is that a separable von Neumann algebra is approximately finite dimensional if and only if it is injective. We refer the readers to [24, 5, 10, 13], and especially [36] for details. Our goal in this section is to study the corresponding results for W^*-TRO's.

Let us first recall that a separable von Neumann algebra R is said to be *approximately finite dimensional* (or equivalently, *hyperfinite*) if there exists an increasing sequence of finite dimensional C^*-subalgebras $\{N_n\}$ such that

$$R = \left(\bigcup N_n\right)'' = \overline{\bigcup N_n}^{s.o.t}. \tag{5.1}$$

We note that this property was first studied for II_1 factors by Murray and von Neumann [24], where they used the term "approximate finite". The term "hyperfinite" was introduced by J. Dixmier [6], and the term "approximately finite dimensional" was introduced by Elliott-Woods [10]. We feel that the terminology of Elliott and Woods is more appropriate for us, and we will use it throughout this paper. Motivated by this, we say that a separable W^*-TRO V is *rectangularly approximately finite dimensional* (or simply, *rectangularly AFD*) if there exists an increasing sequence of finite dimensional TRO's $\{V_n\}$ contained in V such that

$$V = \overline{\bigcup V_n}^{s.o.t}.$$

An operator space $V \subseteq B(K, H)$ is said to be *injective* if it is completely contractively complemented in $B(K, H)$, *i.e.*, there exists a completely contractive projection $P\colon B(K, H) \to V$ from $B(K, H)$ onto V. It is known from [7] and [20] that a W^*-TRO is injective if and only if its linking von Neumann algebra $R(V)$ is injective.

Theorem 5.1 *If V is a separable rectangular AFD W^*-TRO, then its linking von Neumann algebra $R(V)$ is AFD and thus is injective. Therefore, the W^*-TRO V is injective.*

Proof Let us first note that V can be represented on separable Hilbert spaces H and K (by considering its GNS-construction discussed in §2) so that the strong operator topology on V coincides with the strong operator topology on $B(H \oplus K)$. Let $V_1 \subseteq \cdots \subseteq V_n \subseteq \cdots$ be an increasing sequence of finite dimensional TRO's contained in V such that $\bigcup V_n$ is strong operator dense in V. For each $n \in \mathbb{N}$ we may identify V_n with the finite dimensional TRO $\iota_V(V_n)$ contained in $R(V)$. By the uniqueness of the linking C^*-algebras (see details in [15] and [20]), we may identify the linking C^*-algebra $A(V_n)$ with the finite dimensional C^*-subalgebras of $R(V)$ generated by $\iota_V(V_n)$. With this identification, it is easy to see that $\{A(V_n)\}$ is an increasing sequence of (not necessarily unital) finite dimensional C^*-subalgebras of $R(V)$.

Set $\tilde{R} = \overline{\bigcup A(V_n)}^{s.o.t}$ to be the von Neumann algebra generated by the $*$-subalgebra $\bigcup A(V_n)$ of $R(V)$. It is clear that $\tilde{R} \subseteq R(V)$. On the other hand, we can conclude that $R(V)$ is contained in $\tilde{R}$ since the W^*-TRO $\iota_V(V) = \overline{\bigcup \iota_V(V_n)}^{s.o.t}$ is contained in $\tilde{R}$ and thus the $*$-subalgebra spanned by $\iota_V(V)$ is contained in $\tilde{R}$. This shows that $\tilde{R} = R(V)$ and thus $R(V)$ is an AFD von Neumann algebra. Therefore, $R(V)$ is injective. Since $V = eR(V)e^{\perp}$ is an off-diagonal corner of $R(V)$, it is completely contractively complemented in $R(V)$ and thus is also injective. ∎

Using a similar argument to that given in Theorem 5.1, we may easily obtain the following result.

Corollary 5.2 *Let V be a separable W^*-TRO. If there exists an increasing sequence of rectangular AFD W^*-TRO's $\{V_n\}$ contained in V such that $\bigcup V_n$ is strong operator dense in V, then V is rectangularly AFD.*

In the rest of this section, we are going to show that every separable injective W^*-TRO is rectangularly AFD. First if R is an injective von Neumann algebra, then it is AFD and thus is rectangularly AFD.

Proposition 5.3 *Let R be a separable injective von Neumann algebra. Then for any $1 \leq m, k \leq \infty$, $M_{m,k}(R)$ is a rectangular AFD W^*-TRO.*

Proof Since R is injective, it is an AFD von Neumann algebra. There exists an increasing sequence of finite dimensional C^*-subalgebras $\{N_n\}$ of R such that $R = \overline{\bigcup N_n}^{s.o.t}$. Given any positive integers $m, k \in \mathbb{N}$, the rectangular matrices $\{M_{m,k}(N_n)\}$ are an increasing sequence of finite dimensional TRO's contained in $M_{m,k}(R)$. It is clear that $\bigcup M_{m,k}(N_n)$ is strong operator dense in $M_{m,k}(R)$. Therefore, $M_{m,k}(R)$ is a rectangular AFD W^*-TRO.

If m is a positive integer and $k = \infty$, then $\{M_{m,n}(N_n)\}$ is an increasing sequence of finite dimensional TRO's contained in $M_{m,\infty}(R)$ and the union $\bigcup M_{m,n}(N_n)$ is strong

https://doi.org/10.4153/CJM-2004-038-7 Published online by Cambridge University Press

operator dense in $M_{m,\infty}(R)$. Therefore, $M_{m,\infty}(R)$ is a rectangular AFD W^*-TRO. The other cases can be proved by a similar argument. ■

As a consequence of Proposition 5.3 and results discussed in §3, we can conclude that a separable W^*-TRO is rectangularly AFD if it is: (1) of type I, (2) injective and of type $II_{1,\infty}$, $II_{\infty,1}$, or $II_{\infty,\infty}$, (3) injective and of type III. So we only need to discuss separable injective W^*-TRO of type $II_{1,1}$. In this case, $R(V)$ is a separable injective von Neumann algebra of type II_1. Then $R(V)$ is an AFD von Neumann algebra and thus there exists an increasing sequence of finite dimensional C^*-subalgebras $\{N_n\}$ of $R(V)$ such that $R(V) = \overline{\bigcup N_n}^{s.o.t}$. An immediate thought is to consider the finite dimensional subspaces $eN_ne^{\perp}$ contained in $V = eR(V)e^{\perp}$. However, $eN_ne^{\perp}$ need not be TRO's unless e and $e^{\perp}$ are contained in N_n. Therefore, the proof for the $II_{1,1}$ case is not that obvious and requires some careful discussion. In our approach, we need to use some techniques developed for AFD von Neumann algebras of type II_1 (see Takesaki [36, chapter XVI]).

Let R be a separable von Neumann algebra of type II_1 with center $\mathcal{C}$. It follows from the Dixmier approximation theorem that we can obtain a normal faithful center-valued trace $T\colon R \to \mathcal{C}$. Since $\mathcal{C}$ is a separable abelian von Neumann algebra, there exists a normal faithful (tracial) state τ on $\mathcal{C}$. Then $\tau \circ T$ extends to a normal faithful tracial state on R, which is still denoted by τ. As defined in [36], a system $\{e_{i,j} : 1 \leq i, j \leq n\}$ is called a *submatrix unit* of R if

$$e_{i,j} = e^*_{j,i} \text{ and } e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l},$$

where the sum $\sum_{i=1}^n e_{i,i}$ could be a proper projection in R. If $\sum_{i=1}^n e_{i,i} = 1$, we call $\{e_{i,j} : 1 \leq i, j \leq n\}$ a *matrix unit* of R. A submatrix unit is called *admissible* if there exists some $m \in \mathbb{N}$ such that $Sp(T(e_{i,i})) \subseteq 2^{-m}\mathbb{N}$ for all $i = 1, \ldots, n$. If e is a projection in R, we say that e is admissible if $Sp(T(e)) \subseteq 2^{-m}\mathbb{N}$ for some $m \in \mathbb{N}$, *i.e.*, $\{e\}$ is an admissible submatrix unit of $R(V)$.

Theorem 5.4 *Let V be a separable injective W^*-TRO of type $II_{1,1}$. Then V is rectangularly AFD.*

Proof Let $V = eR(V)e^{\perp}$ and let τ be a normal faithful tracial state on $R(V)$ such that $\tau = \tau \circ T$ (see discussion above). If we assume that V is represented on the Hilbert spaces induced by τ, then the strong operator topology coincides with the $\tau(\langle\cdot|\cdot\rangle)$ topology on V.

If e is an admissible projection, then there exist $\{\frac{r_1}{2^m}, \ldots, \frac{r_k}{2^m}\}$ and a central partition $\{z_1, \ldots, z_k\}$ of unity such that

$$T(e) = \sum_{i=1}^k \frac{r_j}{2^m} z_j.$$

We may find a matrix unit $\{u_{i,j} : 1 \leq i, j \leq 2^m\}$ of $R(V)$ such that e (and thus $e^{\perp}$) is contained in the finite dimensional C^*-subalgebra $N_0 = \sum_{j=1}^k {}^{\oplus} M_{2^m} z_j$, where

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we let M_{2^m} denote the subfactor of type I_{2^m} spanned by $\{u_{i,j} : 1 \leq i, j \leq 2^m\}$. It is known from [36] that there exists an increasing sequence of finite dimensional C^*-subalgebras $\{N_n\}$ of $R(V)$ such that $e, e^\perp \in N_0 \subseteq N_n$ and $\bigcup N_n$ is strong operator dense in $R(V)$. Then $\{eN_ne^\perp\}$ is an increasing sequence of finite dimensional TRO's contained in V such that $\bigcup eN_ne^\perp$ is strong operator dense in V. This shows that V is a rectangular AFD W^*-TRO.

If e is not admissible, then $T(e)$ is a positive and contractive element in the center $\mathcal{C}$. If we identify $\mathcal{C}$ with $L_\infty(X,\mu)$ for some measure space (X,μ), then the range of $T(e)$, which can be identified with a function on (X,μ), is contained in the interval $[0,1]$. We may consider a 2^{n_1} equal length partition of the range interval $[0,1]$ and thus obtain a central partition of unit $\{z_1,\ldots,z_{2^{n_1}}\}$ in $\mathcal{C}$ such that

$$\sum_{j=0}^{2^{n_1}-1} \frac{j}{2^{n_1}} z_j \leq T(e) \leq \sum_{j=0}^{2^{n_1}-1} \frac{j+1}{2^{n_1}} z_j.$$

Then there exists a matrix unit $\{g_{i,j} : 1 \leq i, j, \leq 2^{n_1}\}$ in $R(V)$ such that

$$(g_{1,1} + \cdots + g_{j,j})z_j \leq ez_j \leq (g_{1,1} + \cdots + g_{j+1,j+1})z_j.$$

Let u_j be the partial isometries such that

$$u_j^* u_j = (g_{1,1} + \cdots + g_{j,j})z_j \text{ and } u_j u_j^* \leq ez_j.$$

Then $e_1 = u_1 u_1^* + \cdots + u_{2^{n_1}} u_{2^{n_1}}^*$ is an admissible subprojection of e such that

$$\tau(e - e_1) \leq \frac{1}{2^{n_1}}$$

(see more details in [36]). Applying this procedure to the projection $e - e_1$, and so on, we may obtain a sequence of mutually orthogonal admissible subprojections $\{e_k\}$ of e such that the sum $\sum_{k=1}^\infty e_{n_k}$ converges to e in strong operator topology. Similarly, we can construct a sequence of mutually orthogonal admissible subprojections $\{e_k^\perp\}$ of $e^\perp$ such that the sum $\sum_{k=1}^\infty e_{n_k}^\perp$ converges to $e^\perp$ in strong operator topology. From this, we obtain an increasing sequence of TRO's $\{e_n V e_n^\perp\}$ of V such that $\bigcup e_n V e_n^\perp$ is strong operator dense in V. Since $e_n V e_n^\perp$ is rectangularly AFD, we can conclude from Corollary 5.2 that V is also rectangularly AFD. This completes the proof. ■

We note that if V is a separable injective W^*-TRO of type $II_{1,1}$ such that $R(V)$ is a II_1 factor (with trivial center), then we may obtain a much easier proof for Theorem 5.4. For example, if e is admissible with $\tau(e) = \frac{r}{2^m}$ for some r and m, then there exists a projection $e_1 \leq e$ with $\tau(e_1) = \frac{1}{2^m}$ such that V is actually TRO-isomorphic to $M_{r,2^m-r}(e_1 R(V) e_1)$. Since $e_1 R(V) e_1$ is an injective II_1 factor, it is AFD and thus V is rectangularly AFD by Proposition 5.3. One can also give a very clear construction for non-admissible case. The readers are encouraged to work out the details for this special case, which could help to understand the proof given in Theorem 5.4.

We may summarize our results in the following theorem.

Theorem 5.5 *Let V be a separable W^*-TRO. Then V is injective if and only if V is rectangular AFD.*

In von Neumann algebra theory, it is known that all AFD factors of type II_1 are $*$-isomorphic. The following proposition shows that this is not true anymore for W^*-TRO's. For example, if we let R_0 denote the AFD factor of II_1, then R_0 and $M_{1,2}(R_0)$ are both AFD W^*-TRO's of type $II_{1,1}$. Their linking von Neumann algebras $M_2(R_0)$ and $M_3(R_0)$ are $*$-isomorphic by the uniqueness of AFD II_1 factors. However, R_0 and $M_{1,2}(R_0)$ are not TRO-isomorphic.

Proposition 5.6 *Let R be a finite von Neumann algebra. Then $M_{1,2}(R)$ is not TRO-isomorphic to R.*

Proof Let us first assume that $M_{1,2}(R)$ is TRO-isomorphic to R, and let $\varphi\colon M_{1,2}(R) \to R$ be a TRO-isomorphism from $M_{1,2}(R)$ onto R. Since the row vector $v = [0, 1]$ is a partial isometry in $M_{1,2}(R)$ such that $vv^*x = x$ for any $x \in M_{1,2}(R)$, its image $w = \varphi(v)$ must be a partial isometry in R such that $ww^*y = y$ for all $y \in R$. This implies that $ww^* = 1$ in R. Since R is a finite von Neumann algebra, we must have $w^*w = 1$. This shows that $yw^*w = y$ for all $y \in R$.

On the other hand, there exists a non-zero element $x_1 = [1, 0]$ in $M_{1,2}(R)$ such that

$$\varphi(x_1)w^*w = \varphi(x_1v^*v) = 0 \neq \varphi(x_1).$$

This contradiction shows that $M_{1,2}(R)$ cannot be TRO-isomorphic to R. ∎

6 The Operator Preduals of Injective W^*-TRO's

Let us recall that an operator space X is said to be an $\mathcal{OL}_{1,1^+}$ *space* (respectively, a *rectangular* $\mathcal{OL}_{1,1^+}$ *space*) if for every finite dimensional subspace $E \subseteq X$ and $\lambda > 1$ there exists a finite dimensional subspace F with $E \subseteq F \subseteq X$ and there exists a linear isomorphism $T\colon F \to B_*$ from F onto the operator predual B_* of a finite dimensional C^*-algebra (respectively, a finite dimensional TRO) B such that

$$\|T\|_{cb}\|T^{-1}\|_{cb} < \lambda.$$

We call X a *rigid* $\mathcal{OL}_{1,1^+}$ space (respectively, a *rigid* rectangular $\mathcal{OL}_{1,1^+}$ space) if there exists a family of finite dimensional subspaces $\{F_i\}$ such that $\bigcup F_i$ is norm dense in X and each F_i is completely isometric to the operator predual of a finite dimensional C^*-algebra (respectively, a finite dimensional TRO). Using a standard perturbation argument, it is easy to show that if an operator space is a rigid $\mathcal{OL}_{1,1^+}$ (respectively, a rigid rectangular $\mathcal{OL}_{1,1^+}$) space, then it must be an $\mathcal{OL}_{1,1^+}$ space (respectively, a rectangular $\mathcal{OL}_{1,1^+}$ space).

It was shown in [8] that the operator predual of a separable injective von Neumann algebra is an $\mathcal{OL}_{1,1^+}$ space (equivalently, a rigid $\mathcal{OL}_{1,1^+}$ space). On the other hand, Ng and Ozawa proved in [25] that if a separable operator space X is an $\mathcal{OL}_{1,1^+}$ space, then its operator dual $V = X^*$ is completely isometric to an injective von

https://doi.org/10.4153/CJM-2004-038-7 Published online by Cambridge University Press

Neumann algebra. However, Ng and Ozawa's result is no longer true if the separability is removed. For example if I is an (infinite) uncountable index set, the space of all rectangular trace classes $T_{\infty,I}$ is a rigid $\mathcal{OL}_{1,1^+}$ space. But its operator dual $M_{\infty,I} = B(\ell_2(I), \ell_2(\mathbb{N}))$ is not completely isometric to any von Neumann algebra. Indeed, if we assume that there is a complete isometry T from $M_{\infty,I}$ onto a von Neumann algebra R, then T must be a TRO-isomorphism. If we let 1_R denote the unital element in R, then $v = T^{-1}(1_R)$ is a partial isometry in $M_{\infty,I}$ such that

$$vv^*x = T^{-1}(1_R 1_R^* T(x)) = x \text{ and } xv^*v = T^{-1}(T(x)1_R^* 1_R) = x \tag{6.1}$$

for all $x \in M_{\infty,I}$. Then (6.1) implies that $vv^* = I_{\ell_2(\mathbb{N})}$ and $v^*v = I_{\ell_2(I)}$ *i.e.*, $I_{\ell_2(\mathbb{N})}$ and $I_{\ell_2(I)}$ are equivalent projections in $M_{\infty,I}$. This induces a contradiction since I is uncountable.

In the following theorem we show that for general injective dual operator spaces, their operator preduals can be characterized as rectangular $\mathcal{OL}_{1,1^+}$ spaces. To prove this result, we need to recall a notion introduced in [22]. An operator space X is said to have the *local lifting property* if given any operator spaces $Z \subseteq Y$ and any complete contraction $\varphi\colon X \to Y/Z$, for every finite dimensional subspace $E \subseteq X$ and $\varepsilon > 0$, there exists a completely bounded linear map $\tilde{\varphi}\colon E \to Y$ such that $\|\tilde{\varphi}\|_{cb} < 1 + \varepsilon$ and $q \circ \tilde{\varphi} = \varphi_{|E}$, *i.e.*, we have the commutative diagram

$$\begin{tikzcd} & & Y \arrow[d, "q"] \\ E \arrow[r, hook] \arrow[urr, "\tilde{\varphi}"] & V \arrow[r, "\varphi"] & Y/W, \end{tikzcd}$$

where we let $q\colon Y \to Y/W$ denote the complete quotient map from Y onto Y/W. It was shown in [7, Proposition 3.2] that an operator space X has the local lifting property if and only if its operator dual $V = X^*$ is an injective operator space.

Theorem 6.1 *Let X be an operator space. Then the following are equivalent.*

(1) *X is a rigid rectangular $\mathcal{OL}_{1,1^+}$ space,*
(2) *X is a rectangular $\mathcal{OL}_{1,1^+}$ space,*
(3) *$V = X^*$ is an injective operator space* (*and thus is an injective W^*-TRO*).

Proof It is obvious that (1) ⇒ (2). If we have (2), then for every finite dimensional subspace $E \subseteq X$ and $\varepsilon > 0$, there exists a finite dimensional subspace F with $E \subseteq F \subseteq X$ and there exists a completely bounded linear isomorphism $T\colon F \to B_*$ from F onto the operator predual B_* of a finite dimensional TRO B such that $\|T\|_{cb}\|T^{-1}\|_{cb} < 1+\varepsilon$. Suppose that we are given operator spaces $Z \subseteq Y$. Since B is a finite dimensional TRO, we have the (completely) isometric isomorphisms

$$CB(B_*, Y/Z) = B \check{\otimes} (Y/Z) \cong (B \check{\otimes} Y)/(B \check{\otimes} Z). \tag{6.2}$$

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Given any complete contraction $\varphi\colon X \to Y/Z$, it is easy to see that $\varphi \circ T^{-1}\colon B_* \to Y/Z$ is a completely bounded map, and it follows from (6.2) that there exists a completely bounded map $\psi\colon B_* \to Y$ such that $q \circ \psi = \varphi \circ T^{-1}$ and

$$\|\psi\|_{cb} < (1+\varepsilon)\|\varphi \circ T^{-1}\|_{cb} \le (1+\varepsilon)\|T^{-1}\|_{cb}.$$

Then $\tilde{\varphi} = \psi \circ T_{|E}\colon E \to Y$ is a completely bounded map such that

$$\|\tilde{\varphi}\|_{cb} \le (1+\varepsilon)\|T^{-1}\|_{cb}\|T\|_{cb} < (1+\varepsilon)^2$$

and $q \circ \tilde{\varphi} = q \circ \psi \circ T_{|E} = \varphi_{|E}$. This shows that X has the local lifting property, and thus its operator dual X^* is an injective operator space by [7, Proposition 3.2]. In this case, X^* is actually an injective W^*-TRO by [7, Theorem 1.3].

To prove (3) $\Rightarrow$ (1), let us first assume that X is a separable operator space. Then the injectivity of $V = X^*$ implies that V is a separable rectangular AFD W^*-TRO by Theorem 5.5. If V is of type III, then it is TRO-isomorphic to an AFD von Neumann of type III, and thus $X = V_*$ is a rigid $\mathcal{OL}_{1,1+}$ space by [8]. Type I case is clear by Theorem 4.1. So we only need to consider type II case. In this case, its linking von Neumann algebra $R(V)$ is a AFD von Neumann algebra of II.

As we discussed in Theorem 5.1, there exists an increasing sequence of finite dimensional TRO's $\{V_n\}$ contained in V such that $\bigcup V_n$ is strong operator dense in V. Considering the C^*-algebras $A_n = A(V_n)$ generated by V_n, we obtain an increasing sequence of finite dimensional C^*-subalgebras $\{A_n\}$ in $R(V)$ such that $\bigcup A_n$ is strong operator dense in $R(V)$. In general, A_n need not be unital in $R(V)$, but we may consider their unitalization A_n^1 in $R(V)$. Let $\iota_n\colon A_n^1 \hookrightarrow R(V)$ denote the canonical inclusions from A_n^1 into $R(V)$. Since $R(V)$ is of type II, for each n there exists a (tracial invariant) normal conditional expectation $P_n\colon R(V) \to A_n^1$. From this, we may obtain a sequence of complete contractions $r_n\colon (V_n)_* \to X$ and $s_n\colon X \to (V_n)_*$ such that $s_n r_n = id_{(V_n)_*}$ and $r_n s_n \to id_X$ in the point-norm topology on X. Therefore, $\{r_n((V_n)_*)\}$ is an increasing sequence of subspaces of X such that each $r_n((V_n)_*)$ is completely isometric to the operator predual $(V_n)_*$ of the finite dimensional TRO V_n and $\bigcup r_n((V_n)_*)$ is norm dense in X. Therefore, X is a rigid $\mathcal{OL}_{1,1^+}$ space.

In general, we may reduce the problem to separable case. To see this, let us assume that $V = eR(V)e^{\perp}$ and $X = V_* = e^{\perp} \cdot R(V)_* \cdot e$. Given any finite dimensional subspace F in V_*, we let $\{f_1, \dots, f_n\}$ be a basis for F. Then we may find positive normal linear functionals $\{\varphi_1, \dots, \varphi_n\} \in R(V)_*$ such that $f_j = e^{\perp} \cdot \varphi_j \cdot e$. We set $\varphi = \varphi_1 + \cdots + \varphi_n$. It is known from Haagerup's argument (see [12, Appendix]) that we may construct a separable von Neumann subalgebra R_F of R such that $e, e^{\perp} \in R_F$ and there exists a φ invariant normal conditional expectation P_F from $R(V)$ onto R_F. Then $V_F = eR_F e^{\perp}$ is a separable W^*-TRO and $P_{|V}$ is a normal conditional expectation from V onto V_F. Then its pre-adjoint induces a completely isometric inclusion ι_F from $(V_F)_*$ into $X = V_*$, and we may completely isometrically identify F with a finite dimensional subspace $\tilde{F}$ of $(V_F)_*$ spanned by $\{\tilde{f}_j = f_{j|V_F}\}$. Then we may obtain the result by applying separable result to $\tilde{F} \subseteq (V_F)_*$. ∎

Finally we end this paper by the following remark.

Remark 6.2 Since W^*-TRO's are rectangular corners of von Neumann algebras, we can study rectangular L_p-spaces arising from W^*-TRO's (see [29] for the special case of $V = M_{\infty,1}(R)$ or $V = M_{1,\infty}(R)$). For this purpose, we only need to consider semifinite (*i.e.*, type I and type II) W^*-TRO's since every stable W^*-TRO is TRO-isomorphic to a von Neumann algebra. In this case, the linking von Neumann algebra $R(V)$ is semifinite and thus we may fix a normal faithful semifinite trace φ on $R(V)$. We may define $L_p(V, \varphi)$ to be the norm closure of

$$L_p^0(V, \varphi) = \{x \in V : \varphi((x^*x)^{\frac{p}{2}}) < \infty\}$$

with respect to the L_p norm

$$\|x\|_p = \varphi((x^*x)^{\frac{p}{2}})^{\frac{1}{p}}.$$

We may also use the Pisier's complex interpolation method to obtain a canonical operator space structure on $L_p(V, \varphi)$. Then as a consequence of Theorem 5.5 we can prove that if V is a separable injective W^*-TRO, then $L_p(V, \varphi)$ has a very nice rectangular $\mathcal{OL}_p$ structure (*i.e.*, it is a rigid rectangular $\mathcal{OL}_{p,1^+}$ space) for every $1 < p < \infty$. We will discuss these details somewhere else.

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