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GENERALIZED HADAMARD'S INEQUALITIES BASED ON GENERAL EULER 4-POINT FORMULAE

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Abstract

We present a general closed 4-point quadrature rule based on Euler-type identities. We use this rule to prove a generalization of Hadamard's inequalities for $(2r)$-convex functions $(r \geq 1)$.

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1. Introduction

Let f be a convex function on $[a, b] \subset \mathbb{R}$, $a \neq b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known in the literature as Hadamard's inequalities (see for example [10, page 137]) for convex functions.

Hadamard's inequalities can be generalized in the following way.

THEOREM 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then for every $x \in [a, (a+b)/2]$*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\, dt \\ &\qquad \geq \frac{1}{b-a} \int_a^b f(t)\, dt - \frac{f(x)+f(a+b-x)}{2}, \end{aligned} \tag{1.2}$$

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and for every $x \in [(3a+b)/4, (a+b)/2]$

$$\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(x)+f(a+b-x)}{2} \geq 0. \tag{1.3}$$

PROOF. Let $x \in [a, (a+b)/2]$. Since f is convex on $[a, b]$, the right-hand side of (1.1) gives

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(t)\,dt \\
&= \frac{1}{b-a}\left[\int_a^x f(t)\,dt + \int_x^{a+b-x} f(t)\,dt + \int_{a+b-x}^b f(t)\,dt\right] \\
&\leq \frac{1}{b-a}\left[(x-a)\frac{f(a)+f(x)}{2} + (a+b-2x)\frac{f(x)+f(a+b-x)}{2}\right. \\
&\quad \left. +(x-a)\frac{f(a+b-x)+f(b)}{2}\right] \\
&= \frac{1}{2}\left[\frac{x-a}{b-a}(f(a)+f(b)) + \frac{b-x}{b-a}(f(x)+f(a+b-x))\right].
\end{aligned} \tag{1.4}$$

Since f is convex on $[a, b]$, for any $h > 0$ and $x_1, x_2 \in [a, b]$ such that $x_1 \leq x_2$ we have (see, for example, [11, pages 5,6])

$$f(x_1+h) - f(x_1) \leq f(x_2+h) - f(x_2). \tag{1.5}$$

Consider now $x \in [a, (a+b)/2]$. If we apply (1.5) on $h = x - a$, $x_1 = a$ and $x_2 = a + b - x$, we obtain

$$f(x) - f(a) \leq f(b) - f(a+b-x). \tag{1.6}$$

For $x \in [a, (a+b)/2]$ we have $a + b - 2x \geq 0$, so for such x the inequality (1.6) can be rewritten as

$$(a+b-2x)\frac{f(x)-f(a)}{b-a} \leq (a+b-2x)\frac{f(b)-f(a+b-x)}{b-a},$$

that is,

$$(a+b-2x)\frac{f(x)-f(a)}{b-a} + (2x-a-b)\frac{f(b)-f(a+b-x)}{b-a} \leq 0.$$

From this, a simple calculation gives us

$$\begin{aligned}
&\frac{2(x-a)}{b-a}[f(a)+f(b)] + \frac{2(b-x)}{b-a}[f(x)+f(a+b-x)] \\
&\quad \leq f(a)+f(b)+f(x)+f(a+b-x).
\end{aligned} \tag{1.7}$$

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Combining (1.4) and (1.7) we obtain

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)+f(x)+f(a+b-x)}{4},$$

from which we get

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \ge \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(x)+f(a+b-x)}{2},$$

and this completes the proof of (1.2).

Now let $x \in [(3a+b)/4, (a+b)/2]$. Since f is convex on $[a,b]$, the left-hand side of (1.1) gives

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(t)\,dt &= \frac{1}{b-a}\left[\int_a^{(a+b)/2} f(t)\,dt + \int_{(a+b)/2}^b f(t)\,dt\right]\\ &\ge \frac{1}{b-a}\left[\frac{b-a}{2}f\left(\frac{3a+b}{4}\right) + \frac{b-a}{2}f\left(\frac{a+3b}{4}\right)\right]\\ &= \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right]. \qquad (1.8)\end{aligned}$$

If we apply (1.5) again on $h = (4x-3a-b)/4$, $x_1 = (3a+b)/4$ and $x_2 = a+b-x$, we obtain

$$f(x) - f\left(\frac{3a+b}{4}\right) \le f\left(\frac{a+3b}{4}\right) - f(a+b-x),$$

that is,

$$f(x) + f(a+b-x) \le f\left(\frac{a+3b}{4}\right) + f\left(\frac{3a+b}{4}\right). \qquad (1.9)$$

Combining (1.9) with (1.8) we obtain

$$\frac{1}{b-a}\int_a^b f(t)\,dt \ge \frac{f(x)+f(a+b-x)}{2},$$

so the inequality (1.3) is proved. □

REMARK 1. If in (1.2) and (1.3) we let $x = (a+b)/2$, we obtain

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \ge \frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \ge 0,$$

which is one of Bullen's results from [3]. His result was generalized for $(2r)$-convex functions ($r \in \mathbb{N}$) in [6].

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The goal of this paper is to obtain a variant of Inequalities (1.2) and (1.3) for $(2r)$-convex functions ($r \in \mathbb{N}$). To achieve this goal we will construct a general closed 4-point rule based on Euler-type identities established in [4].

We recall that a function $f : [a, b] \to \mathbb{R}$ is said to be n-convex on $[a, b]$ for some $n \geq 0$ if for any choice of $n+1$ points $x_0, \dots, x_n$ from $[a, b]$ we have $[x_0, \dots, x_n]f \geq 0$, where $[x_0, \dots, x_n]f$ is the n-th order divided difference of f. If f is n-convex, then $f^{(n-2)}$ exists and is an convex function in the ordinary sense. Also, if $f^{(n)}$ exists, then f is n-convex if and only if $f^{(n)} \geq 0$. For more details see for example [10].

It should be noted that each continuous n-convex function on $[a, b]$ is the uniform limit of a sequence of the corresponding Bernstein's polynomials (see, for example, [10, page 293]). Bernstein polynomials of any continuous n-convex function are also n-convex functions, so when stating our results for a continuous $(2r)$-convex function f without any loss in generality we may assume that $f^{(2r)}$ exists and is continuous. Actually, our results are valid for any continuous $(2r)$-convex function f.

In Section 2 we present a general closed 4-point quadrature rule based on the extended Euler formulae and we also give two estimations of the remainder. In Section 3 we use the obtained results to prove a generalization of Hadamard's inequalities for $(2r)$-convex functions ($r \in \mathbb{N}$).

2. General closed 4-point quadrature rule

In the paper [4] two identities, named the extended Euler formulae, have been proved. They are given in the following theorem.

THEOREM A. *Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a, b]$ for some $n \in \mathbb{N}$. Then for every $x \in [a, b]$*

$$f(x) = \frac{1}{b-a} \int_a^b f(t)\,dt + T_n(x) + R_n^1(x) \tag{2.1}$$

and

$$f(x) = \frac{1}{b-a} \int_a^b f(t)\,dt + T_{n-1}(x) + R_n^2(x), \tag{2.2}$$

where

$$T_m(x) = \sum_{k=1}^{m} \frac{(b-a)^{k-1}}{k!} B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right], \tag{2.3}$$

$$R_n^1(x) = -\frac{(b-a)^{n-1}}{n!} \int_{[a,b]} B_n^*\left(\frac{x-t}{b-a}\right) df^{(n-1)}(t),$$

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and

$$R_n^2(x) = -\frac{(b-a)^{n-1}}{n!}\int_{[a,b]}\left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right] df^{(n-1)}(t).$$

Here, as in the rest of the paper, the functions $B_k(\cdot)$ $(k \geq 0)$ are the Bernoulli polynomials, B_k are the Bernoulli numbers and $B_k^*(\cdot)$ are periodic functions of period one, related to the Bernoulli polynomials as

$$B_k^*(x) = B_k(x), \quad 0 \leq x < 1,$$
$$B_k^*(x+1) = B_k^*(x), \quad x \in \mathbb{R}.$$

In this paper we write $\int_{[a,b]} g(t)\, d\varphi(t)$ to denote the Riemann-Stieltjes integral of a function $g : [a,b] \to \mathbb{R}$ with respect to a continuous function $\varphi : [a,b] \to \mathbb{R}$ of bounded variation, and we write $\int_a^b g(t)\, dt$ for the Riemann integral.

To make reading easier, let us recall some of the properties of the Bernoulli polynomials (see, for example, [1, 23.1] or [2]). The Bernoulli polynomials are uniquely determined by the following identities:

$$B_0(x) = 1, \quad x \in \mathbb{R},$$
$$B_k'(x) = kB_{k-1}(x), \quad k \geq 1,$$
$$B_k(x+1) - B_k(x) = kx^{k-1}, \quad k \geq 0.$$

From that we have$B_1(x) = x - 1/2$, $B_2(x) = x^2 - x + 1/6$, $B_3(x) = x^3 - 3x^2/2 + x/2$, so that B_0^* and B_1^* are discontinuous functions with jumps of -1 at each integer. Also, it follows that $B_k(1) = B_k(0) = B_k$ for $k \geq 2$, so that B_k^* are continuous functions for $k \geq 2$. From this we get $(B_k^*)'(x) = kB_{k-1}^*(x)$, $k \geq 1$, for every $t \in \mathbb{R}$ if $k \geq 3$ and for every $t \in \mathbb{R} \setminus \mathbb{Z}$ if $k = 1, 2$.

Here we list some of the properties of the Bernoulli polynomials which will be used in this paper (see, for example, [1] or [2]):

$$B_k(1-x) = (-1)^k B_k(x), \quad n \geq 0,\ x \in \mathbb{R},$$
$$B_k(1/2) = -(1 - 2^{1-k})B_k, \quad n \geq 0,$$
$$B_{2k-1}(1/2) = B_{2k-1} = 0, \quad k \geq 1,$$
$$B_k(0) = B_k(1), \quad k \geq 2,$$
$$(-1)^k B_{2k-1}(x) > 0, \quad k \geq 1,\ x \in (0, 1/2),$$
$$(-1)^{k-1} B_{2k} > 0, \quad r \geq 0.$$

For $k \geq 1$ and fixed $x \in [a, (a+b)/2]$ we define functions G_k^x and F_k^x as

$$\begin{aligned} G_k^x(t) &= B_k^*\left(\frac{x-t}{b-a}\right) + B_k^*\left(\frac{a+b-x-t}{b-a}\right) + B_k^*\left(\frac{a-t}{b-a}\right) + B_k^*\left(\frac{b-t}{b-a}\right) \\ &= B_k^*\left(\frac{x-t}{b-a}\right) + B_k^*\left(\frac{a+b-x-t}{b-a}\right) + 2B_k^*\left(\frac{a-t}{b-a}\right) \end{aligned}$$

and $F_k^x(t) = G_k^x(t) - \widetilde{B}_k^x$, for all $t \in \mathbb{R}$, where

$$\begin{aligned}\widetilde{B}_k^x &= B_k\left(\frac{x-a}{b-a}\right) + B_k\left(\frac{b-x}{b-a}\right) + B_k(0) + B_k(1)\\ &= [1+(-1)^k]\left[B_k\left(\frac{x-a}{b-a}\right) + B_k\right].\end{aligned}$$

Of course, if $k \geq 2$ we have $\widetilde{B}_k^x = [1+(-1)^k]B_k((x-a)/(b-a)) + 2B_k$. Using the properties of the Bernoulli polynomials which were mentioned in the introduction, we can easily see that for any $x \in [a, (a+b)/2]$

$$\begin{aligned}&\widetilde{B}_k^x = G_k^x(a), \quad k \geq 2, \qquad \widetilde{B}_{2r-1}^x = 0, \quad r \geq 1,\\ &\widetilde{B}_{2r}^x = 2\left[B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\right], \quad r \geq 1,\\ &F_{2i-1}^x(t) = G_{2i-1}^x(t), \quad i \geq 1,\\ &F_{2r}^x(t) = G_{2r}^x(t) - 2\left[B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\right], \quad r \geq 1,\\ &F_k^x(a) = F_k^x(b) = 0, \quad k \geq 1,\\ &G_k^x(a) = G_k^x(b) = [1+(-1)^k]B_k\left(\frac{x-a}{b-a}\right) + 2B_k, \quad k \geq 1.\end{aligned}$$

We can also easily check that for all $r \geq 1$

$$F_{2r-1}^x\left(\frac{a+b}{2}\right) = G_{2r-1}^x\left(\frac{a+b}{2}\right) = 0$$

and

$$\begin{aligned}G_{2r}^x\left(\frac{a+b}{2}\right) &= 2B_{2r}\left(\frac{1}{2} - \frac{x-a}{b-a}\right) + 2B_{2r}\left(\frac{1}{2}\right),\\ F_{2r}^x\left(\frac{a+b}{2}\right) &= G_{2r}^x\left(\frac{a+b}{2}\right) - \widetilde{B}_{2r}^x\\ &= 2\left[B_{2r}\left(\frac{1}{2} - \frac{x-a}{b-a}\right) - B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\left(\frac{1}{2}\right) - B_{2r}\right]\\ &= 2\left[B_{2r}\left(\frac{1}{2} - \frac{x-a}{b-a}\right) - B_{2r}\left(\frac{x-a}{b-a}\right) + 2(2^{-2r}-1)B_{2r}\right].\end{aligned}$$

Now let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[a, b]$ for some $n \geq 1$. We introduce the following notation for each $x \in [a, (a+b)/2]$:

$$D(x) = [f(x) + f(a+b-x) + f(a) + f(b)]/4.$$

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Furthermore, we define

$$\widetilde{T}_0(x) = 0,$$
$$\widetilde{T}_m(x) = \frac{1}{4}[T_m(x) + T_m(a+b-x) + T_m(a) + T_m(b)], \quad 1 \le m \le n,$$

where T_m is given by (2.3). It can be easily checked that

$$\widetilde{T}_m(x) = \frac{1}{4}\sum_{k=1}^{m} \frac{(b-a)^{k-1}}{k!}\widetilde{B}_k^x \left[f^{(k-1)}(b) - f^{(k-1)}(a)\right].$$

For further use we will denote

$$\widetilde{T}_m^V(x) = \frac{T_m(x) + T_m(a+b-x)}{2} \quad \text{and} \quad \widetilde{T}_m^F = \frac{T_m(a) + T_m(b)}{2}.$$

Obviously, $\widetilde{T}_m(x) = \left(\widetilde{T}_m^V(x) + \widetilde{T}_m^F\right)/2$.

THEOREM 2.1. *Let $f : [a,b] \to \mathbb{R}$, $a < b$, be such that for some $n \in \mathbb{N}$, the derivative $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$. Then for every $x \in [a,b]$*

$$\frac{1}{b-a}\int_a^b f(t)\,dt = D(x) - \widetilde{T}_n(x) + \widetilde{R}_n^1(x) \quad \textit{and} \tag{2.4}$$
$$\frac{1}{b-a}\int_a^b f(t)\,dt = D(x) - \widetilde{T}_{n-1}(x) + \widetilde{R}_n^2(x), \tag{2.5}$$

where

$$\widetilde{R}_n^1(x) = \frac{(b-a)^{n-1}}{4n!}\int_{[a,b]} G_n^x(t)\,df^{(n-1)}(t) \quad \textit{and}$$
$$\widetilde{R}_n^2(x) = \frac{(b-a)^{n-1}}{4n!}\int_{[a,b]} F_n^x(t)df^{(n-1)}(t).$$

PROOF. Put $x \equiv x, a+b-x, a, b$ in the formula (2.1) to get four new formulae. Then multiply these formulae by $1/4$ and add. The result is (2.4), and (2.5) is obtained from (2.2) by the same procedure. □

REMARK 2. If in Theorem 2.1 we choose $x = a$ we obtain the Euler trapezoidal rule [5], and if we choose $x = (a+b)/2$ we obtain the Euler bitrapezoidal rule [6].

Our next goal is to give an estimation of the remainder $\widetilde{R}_n^2(x)$. For the sake of simplicity we will temporarily introduce two new variables:

$$\xi = \frac{x-a}{b-a} \quad \text{and} \quad s = \frac{t-a}{b-a}.$$

It can be easily seen that for $x, t \in [a, b]$ we have $\xi, s \in [0, 1]$. Using direct calculations, for each $\xi \in [0, 1/2]$ we obtain

$$G_1^{\xi}(s) = F_1^{\xi}(s) = \begin{cases} -4s+1, & 0 \le s \le \xi, \\ -4s+2, & \xi < s \le 1-\xi, \\ -4s+3, & 1-\xi < s < 1, \end{cases}$$

$$G_2^{\xi}(s) = \begin{cases} 4s^2 - 2s + 2\xi^2 - 2\xi + 2/3, & 0 \le s \le \xi, \\ 4s^2 - 4s + 2\xi^2 + 2/3, & \xi < s \le 1-\xi, \\ 4s^2 - 6s + 2\xi^2 - 2\xi + 8/3, & 1-\xi < s < 1, \end{cases}$$

$$F_2^{\xi}(s) = \begin{cases} 4s^2 - 2s, & 0 \le s \le \xi, \\ 4s^2 - 4s + 2\xi, & \xi < s \le 1-\xi, \\ 4s^2 - 6s + 2, & 1-\xi < s < 1, \end{cases}$$

$$G_3^{\xi}(s) = \begin{cases} -4s^3 + 3s^2 - 2s(3\xi^2 - 3\xi + 1), & 0 \le s \le \xi, \\ -4s^3 + 6s^2 - 2s(3\xi^2 + 1) + 3\xi^2, & \xi < s \le 1-\xi, \\ -4s^3 + 9s^2 - 2s(3\xi^2 - 3\xi + 4) + 6\xi^2 - 6\xi + 3, & 1-\xi < s < 1, \end{cases}$$
$$= F_3^{\xi}(s).$$

Next we present some properties of the functions G_k^{ξ} and F_k^{ξ}. First we prove that the functions G_k^{ξ} and F_k^{ξ} are symmetric for even k and skew-symmetric for odd k with respect to $1/2$.

LEMMA 2.2. *Let $\xi \in [0, 1/2]$ be fixed. For $k \ge 2$ and $s \in [0, 1]$, we have*

$$G_k^{\xi}(1-s) = (-1)^k G_k^{\xi}(s) \quad \text{and} \quad F_k^{\xi}(1-s) = (-1)^k F_k^{\xi}(s).$$

PROOF. As stated at the beginning of this section, for $k \ge 2$ and $s \in [0, 1]$, we have

$$\begin{aligned} G_k^{\xi}&(1-s) \\ &= B_k^*(\xi - 1 + s) + B_k^*(-\xi + s) + 2B_k^*(s) \\ &= \begin{cases} B_k(\xi + s) + B_k(1 - \xi + s) + 2B_k(s), & 0 \le s \le \xi, \\ B_k(\xi + s) + B_k(-\xi + s) + 2B_k(s), & \xi < s \le 1-\xi, \\ B_k(\xi - 1 + s) + B_k(-\xi + s) + 2B_k(s), & 1-\xi < s \le 1, \end{cases} \\ &= (-1)^k \begin{cases} B_k(1 - \xi - s) + B_k(\xi - s) + 2B_k(1-s), & 0 \le s \le \xi, \\ B_k(1 - \xi - s) + B_k(1 + \xi - s) + 2B_k(1-s), & \xi < s \le 1-\xi, \\ B_k(2 - \xi - s) + B_k(1 + \xi - s) + 2B_k(1-s), & 1-\xi < s \le 1, \end{cases} \\ &= (-1)^k G_k^{\xi}(s), \end{aligned}$$

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which proves the first identity. Further, we know that $F_k^\xi(s) = G_k^\xi(s) - G_k^\xi(0)$. If $k = 2i-1$, $i \geq 2$, then $G_{2i-1}^\xi(0) = G_{2i-1}^\xi(1) = 0$, so we immediately have

$$F_{2i-1}^\xi(1-s) = G_{2i-1}^\xi(1-s) = (-1)^{2i-1}G_{2i-1}^\xi(s) = (-1)^{2i-1}F_{2i-1}^\xi(s).$$

On the other hand, if $k = 2i$, $i \geq 1$, then $(-1)^{2i} = 1$, so we obtain

$$\begin{aligned} F_{2i}^\xi(1-s) &= G_{2i}^\xi(1-s) + G_{2i}^\xi(0) \\ &= (-1)^{2i}G_{2i}^\xi(s) + (-1)^{2i}G_{2i}^\xi(0) = (-1)^{2i}F_{2i}^\xi(s), \end{aligned}$$

and this proves the second identity. □

REMARK 3. It is obvious that analogous assertions hold true for the functions G_k^x and F_k^x, $k \geq 2$. In other words, if $x \in [a, (a+b)/2]$ and $t \in [a, b]$ we have

$$G_k^x(b-t) = (-1)^k G_k^x(t) \quad \text{and} \quad F_k^x(b-t) = (-1)^k F_k^x(t).$$

LEMMA 2.3. *If $\xi \in [0, 1/2 - 1/(4\sqrt{6}))$, then for all $s \in (0, 1/2)$, $G_3^\xi(s) < 0$. Also*

$$\begin{aligned} G_3^{1/2-1/(4\sqrt{6})}(s) &< 0, \quad s \in (0, 1/2) \setminus \{3/8\}, \\ G_3^{1/2}(s) &< 0, \quad s \in (0, 1/4), \\ G_3^{1/2}(s) &> 0, \quad s \in (1/4, 1/2). \end{aligned}$$

PROOF. For the sake of simplicity we will denote

$$\begin{aligned} G_3^\xi(s) &= \begin{cases} -4s^3 + 3s^2 - 2s(3\xi^2 - 3\xi + 1), & 0 \leq s \leq \xi, \\ -4s^3 + 6s^2 - 2s(3\xi^2 + 1) + 3\xi^2, & \xi < s \leq 1-\xi, \\ -4s^3 + 9s^2 - 2s(3\xi^2 - 3\xi + 4) + 6\xi^2 - 6\xi + 3, & 1-\xi < s < 1, \end{cases} \\ &= \begin{cases} H_1^\xi(s), & 0 \leq s \leq \xi, \\ H_2^\xi(s), & \xi < s \leq 1-\xi, \\ H_3^\xi(s), & 1-\xi < s \leq 1. \end{cases} \end{aligned}$$

If we write $H_1^\xi(s)$ as $H_1^\xi(s) = s\big[-4s^2 + 3s - 2(3\xi^2 - 3\xi + 1)\big]$, we can see that $H_1^\xi(0) = 0$ and that $H_1^\xi(\xi) = \xi(-10\xi^2 + 9\xi - 2)$, so if for a given $\xi \in [0, 1/2]$ the number $-10\xi^2 + 9\xi - 2$ is negative it means that the joining point $\big(\xi, H_1^\xi(\xi)\big) = \big(\xi, H_2^\xi(\xi)\big)$ is under the x-axis. This will be true for $\xi \in [0, 2/5)$. The sign of $H_1^\xi(s)$ is determined by the sign of the function $y(s) = -4s^2 + 3s - 2(3\xi^2 - 3\xi + 1)$. This function will have zeros $s_1 = 3/8 - (\sqrt{D})/8$ and $s_2 = 3/8 + (\sqrt{D})/8$ if $D = -96\xi^2 + 96\xi - 23 \geq 0$, that is, if $\xi \in \big[1/2 - 1/(4\sqrt{6}), 1/2\big]$. Furthermore, $y(0) = -2(3\xi^2 - 3\xi + 1) < 0$ which means that (if they exist) both zeros s_1 and

s_2 are positive. Of course, if $\xi = 1/2 - 1/(4\sqrt{6})$ the function y has only one zero $s = 3/8$. We want to know if it is possible for $\xi \in (1/2 - 1/(4\sqrt{6}), 2/5)$ to have $\xi < s_1$ (because this will imply that $H_1^{\xi}(s) < 0$ for all $0 \le s \le \xi$). This in fact is not possible because if $\xi < s_1$ then we have $\xi < 3/8$, and $3/8 < 1/2 - 1/(4\sqrt{6})$. This means that $H_1^{\xi}(s) \le 0$ for all $s \in (0, \xi)$ can be true only if $D \le 0$, and this will be true for $\xi \in [0, 1/2 - 1/(4\sqrt{6})] \subset [0, 2/5)$.

Now we must check H_2^{ξ} for such ξ. If $\xi < s \le 1/2$ we have

$$H_2^{\xi\prime}(s) = -12s^2 + 12s - 2(3\xi^2 + 1),$$
$$H_2^{\xi\prime\prime}(s) = -24s + 12 = 12(1 - 2s) > 0,$$

which means that H_2^{ξ} is convex for any choice of such ξ. Since $H_2^{\xi}(\xi) < 0$ and $H_2^{\xi}(1/2) = 0$, we can deduce that $H_2^{\xi}(s) < 0$ for all $s \in (\xi, 1/2)$. This means that if $\xi \in [0, 1/2 - 1/(4\sqrt{6}))$, then $G_3^{\xi}(s) < 0$, $s \in (0, 1/2)$, and for $\xi = 1/2 - 1/(4\sqrt{6})$ we have $G_3^{\xi}(s) < 0$, $s \in (0, 1/2) \setminus \{3/8\}$.

On the other hand, if $\xi \in (2/5, 1/2]$ the joining point $(\xi, H_1^{\xi}(\xi)) = (\xi, H_2^{\xi}(\xi))$ is above the x-axis, and we want $H_1^{\xi}(s)$ to be positive for all $s \in (0, \xi)$. This, of course, cannot be true because $(2/5, 1/2] \subset (1/2 - 1/(4\sqrt{6}), 1/2]$, which means that H_1^{ξ} surely has a zero $s_1 < 3/8 < 2/5 < \xi$.

And in the end, we must separately investigate $G_3^{1/2}$ because at this special point $\xi = 1/2$ the function G_3^{ξ} has only one branch for $s \in [0, 1/2]$, that is, we have

$$G_3^{1/2}(s) = s(-4s^2 + 3s - 1/2), \quad s \in [0, 1/2].$$

We can easily see that $G_3^{1/2}(s) < 0$, $s \in (0, 1/4)$ and $G_3^{1/2}(s) > 0$, $s \in (1/4, 1/2)$. □

Of course, from the above results we have $G_3^{x}(t) < 0$, $t \in (a, (a+b)/2)$ for any $x \in [a, (a+b)/2 - (b-a)/(4\sqrt{6}))$, and also

$$G_3^{(a+b)/2-(b-a)/(4\sqrt{6})}(s) < 0, \quad s \in (a, (a+b)/2) \setminus \{(5a+3b)/8\},$$
$$G_3^{(a+b)/2}(t) < 0, \quad t \in (a, (a+b)/4),$$
$$G_3^{(a+b)/2}(t) > 0, \quad t \in ((3a+b)/4, (a+b)/2).$$

LEMMA 2.4. *For $r \ge 2$ and $x \in [a, (a+b)/2 - (b-a)/(4\sqrt{6}))$, the function G_{2r-1}^{x} has no zeros in the interval $(a, (a+b)/2)$. The sign of this function is determined by*

$$(-1)^{r-1} G_{2r-1}^{x}(t) > 0, \quad t \in (a, (a+b)/2).$$

Also,

$$(-1)^{r-1} G_{2r-1}^{(a+b)/2-(b-a)/(4\sqrt{6})}(t) > 0, \quad t \in (a, (a+b)/2) \setminus \{(5a+3b)/8\},$$
$$(-1)^{r-1} G_{2r-1}^{(a+b)/2}(t) > 0, \quad t \in (a, (3a+b)/4),$$
$$(-1)^{r-1} G_{2r-1}^{(a+b)/2}(t) < 0, \quad t \in ((3a+b)/4, (a+b)/2).$$

PROOF. Let $x \in \big[a, (a+b)/2 - (b-a)/(4\sqrt{6})\big)$. If $r = 2$, then the assertion follows from Lemma 2.3. Assume now that $r \geq 3$. In that case we have $2r - 1 \geq 5$ and the function G_{2r-1}^x is continuous and at least twice differentiable. We know that

$$(G_{2r-1}^x)'(t) = -\frac{2r-1}{b-a}\, G_{2r-2}^x(t),$$
$$(G_{2r-1}^x)''(t) = \frac{(2r-1)(2r-2)}{(b-a)^2}\, G_{2r-3}^x(t), \tag{2.6}$$

and that $G_{2r-1}^x(a) = G_{2r-1}^x\big((a+b)/2\big) = 0$.

Suppose that G_{2r-1}^x has another zero $\alpha \in \big(a, (a+b)/2\big)$. Then inside each of the intervals (a, α) and $\big(\alpha, (a+b)/2\big)$ the derivative $(G_{2r-1}^x)'$ must have at least one zero, say $\beta_1 \in (a, \alpha)$ and $\beta_2 \in \big(\alpha, (a+b)/2\big)$. Therefore, the second derivative $(G_{2r-1}^x)''$ must have at least one zero inside the interval $(\beta_1, \beta_2) \subset \big(a, (a+b)/2\big)$. Thus, from the assumption that G_{2r-1}^x has a zero inside the interval $\big(a, (a+b)/2\big)$ it follows that G_{2r-3}^x also has a zero inside the interval $\big(a, (a+b)/2\big)$. From this we could deduce that the function G_3^x also has a zero inside the interval $\big(a, (a+b)/2\big)$ which is not true. Thus G_{2r-1}^x cannot have a zero inside the interval $\big(a, (a+b)/2\big)$. Furthermore, if $G_{2r-3}^x(t) > 0$ for $t \in \big(a, (a+b)/2\big)$, then from (2.6) it follows that G_{2r-1}^x is convex on $\big(a, (a+b)/2\big)$, and hence $G_{2r-1}^x(t) < 0$ for $t \in \big(a, (a+b)/2\big)$. Similarly, if $G_{2r-3}^x(t) < 0$ for $t \in \big(a, (a+b)/2\big)$, then from (2.6) it follows that G_{2r-1}^x is concave on $\big(a, (a+b)/2\big)$, and hence $G_{2r-1}^x(t) > 0$ for $t \in \big(a, (a+b)/2\big)$. Since $G_3^x(t) < 0$ for $t \in \big(a, (a+b)/2\big)$, we can conclude that

$$(-1)^{r-1} G_{2r-1}^x(t) > 0, \quad t \in \big(a, (a+b)/2\big).$$

For the special cases $x = (a+b)/2 - (b-a)/(4\sqrt{6})$ and $x = (a+b)/2$, the proof is similar so we skip the details. □

COROLLARY 2.5. *For $r \geq 2$ and $x \in \big[a, (a+b)/2 - (b-a)/(4\sqrt{6})\big]$, the functions $(-1)^r F_{2r}^x(t)$ and $(-1)^r G_{2r}^x(t)$ are strictly increasing on the interval $\big(a, (a+b)/2\big)$ and strictly decreasing on the interval $\big((a+b)/2, b\big)$. Consequently, a and b are the only zeros of F_{2r}^x in the interval $[a, b]$ and*

$$\max_{t \in [a,b]} \left|F_{2r}^x(t)\right| = 2\left|B_{2r}\left(\frac{1}{2} - \frac{x-a}{b-a}\right) - B_{2r}\left(\frac{x-a}{b-a}\right) + 2(2^{-2r} - 1)B_{2r}\right|,$$
$$\max_{t \in [a,b]} \left|G_{2r}^x(t)\right| = \left\{2\left|B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\right|, 2\left|B_{2r}\left(\frac{1}{2} - \frac{x-a}{b-a}\right) + B_{2r}\left(\frac{1}{2}\right)\right|\right\}.$$

PROOF. Let $r \geq 2$ and $x \in \big[a, (a+b)/2 - (b-a)/(4\sqrt{6})\big)$. We know that

$$\big[(-1)^r F_{2r}^x(t)\big]' = \big[(-1)^r G_{2r}^x(t)\big]' = \frac{2r}{b-a}(-1)^{r-1} G_{2r-1}^x(t),$$

and by Lemma 2.4 we also know that $(-1)^{r-1}G_{2r-1}^{x}(t) > 0$ for all $t \in \big(a, (a+b)/2\big)$. Thus the functions $(-1)^r F_{2r}^x(t)$ and $(-1)^r G_{2r}^x(t)$ are strictly increasing on the interval $\big(a, (a+b)/2\big)$. Also, by Lemma 2.2, we have $F_{2r}^x(b-t) = F_{2r}^x(t)$ and $G_{2r}^x(b-t) = G_{2r}^x(t)$ for $t \in [a,b]$, which implies that $(-1)^r F_{2r}^x(t)$ and $(-1)^r G_{2r}^x(t)$ are strictly decreasing on the interval $\big((a+b)/2, b\big)$. Further, $F_{2r}^x(a) = F_{2r}^x(b) = 0$, which implies that $|F_{2r}^x(t)|$ achieves its maximum at $t = (a+b)/2$, that is,

$$\max_{t\in[a,b]} \left|F_{2r}^x(t)\right| = \left|F_{2r}^x\left(\frac{a+b}{2}\right)\right| = 2\left|B_{2r}\left(\frac{1}{2} - \frac{x-a}{b-a}\right) - B_{2r}\left(\frac{x-a}{b-a}\right) + 2(2^{-2r}-1)B_{2r}\right|.$$

Also,

$$\max_{t\in[a,b]} \left|G_{2r}^x(t)\right| = \max\left\{\left|G_{2r}^x(a)\right|, \left|G_{2r}^x\left(\frac{a+b}{2}\right)\right|\right\} = \max\left\{2\left|B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\right|, 2\left|B_{2r}\left(\frac{1}{2} - \frac{x-a}{b-a}\right) + B_{2r}\left(\frac{1}{2}\right)\right|\right\}.$$

The special case $x = (a+b)/2 - (b-a)/(4\sqrt{6})$ can be investigated similarly. □

COROLLARY 2.6. *For $r \geq 2$ the functions $(-1)^r F_{2r}^{(a+b)/2}(t)$ and $(-1)^r G_{2r}^{(a+b)/2}(t)$ are strictly increasing on the intervals $\big(a, (3a+b)/4\big)$ and $\big((a+b)/2, (3a+b)/4\big)$, and strictly decreasing on the intervals $\big((3a+b)/4, (a+b)/2\big)$ and $\big((3a+b)/4, b\big)$. Consequently, a, $(a+b)/2$ and b are the only zeros of $F_{2r}^{(a+b)/2}$ in the interval $[a,b]$ and*

$$\max_{t\in[a,b]} \left|F_{2r}^{(a+b)/2}(t)\right| = \left|F_{2r}^{(a+b)/2}\big((3a+b)/4\big)\right| = 2^{2-2r}(2-2^{1-2r})|B_{2r}|,$$

$$\max_{t\in[a,b]} \left|G_{2r}^{(a+b)/2}(t)\right| = \left|G_{2r}^{(a+b)/2}\big((3a+b)/4\big)\right| = 2^{2-2r}(1-2^{1-2r})|B_{2r}|.$$

PROOF. The proof follows similarly to the proof of Corollary 2.5, using the fact that $F_{2r}^{(a+b)/2}\big((a+b)/2\big) = 2\big[B_{2r} - B_{2r}(1/2) + 2(2^{-2r}-1)B_{2r}\big] = 0$. □

COROLLARY 2.7. *For $r \geq 2$ and $x \in \big[a, (a+b)/2 - (b-a)/(4\sqrt{6})\big]$, we have*

$$\frac{1}{b-a}\int_a^b \left|F_{2r-1}^x(t)\right| dt = \frac{1}{b-a}\int_a^b \left|G_{2r-1}^x(t)\right| dt = \frac{1}{r}\left|F_{2r}^x\left(\frac{a+b}{2}\right)\right| = \frac{2}{r}\left|B_{2r}\left(\frac{1}{2} - \frac{x-a}{b-a}\right) - B_{2r}\left(\frac{x-a}{b-a}\right) + 2(2^{-2r}-1)B_{2r}\right|.$$

Also, we have

$$\frac{1}{b-a}\int_a^b \left|F_{2r}^x(t)\right| dt = 2\left|B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\right|$$

and

$$\frac{1}{b-a}\int_a^b \left|G_{2r}^x(t)\right| dt \le 4\left|B_{2r}\left(\frac{x-a}{b-a}\right)+B_{2r}\right|.$$

PROOF. Let $r \ge 2$ and $x \in \left[a, (a+b)/2 - (b-a)/(4\sqrt{6})\right]$. Using Lemmas 2.2 and 2.4 we get

$$\begin{aligned}\int_a^b \left|G_{2r-1}^x(t)\right| dt &= 2\left|\int_a^{(a+b)/2} G_{2r-1}^x(t)\,dt\right| \\ &= 2\left|-\frac{b-a}{2r}G_{2r}^x(s)\Big|_a^{(a+b)/2}\right| = \frac{b-a}{r}\left|G_{2r}^x\left(\frac{a+b}{2}\right)-G_{2r}^x(a)\right| \\ &= \frac{b-a}{r}F_{2r}^x\left(\frac{a+b}{2}\right),\end{aligned}$$

which proves the first assertion. Using Corollary 2.5 and the fact that $F_{2r}^x(a) = F_{2r}^x(b) = 0$, we can deduce that the function F_{2r}^x does not change its sign on the interval (a, b). Therefore we have

$$\begin{aligned}\int_a^b \left|F_{2r}^x(t)\right| dt &= \left|\int_a^b F_{2r}^x(t)\,dt\right| = \left|\int_a^b \left[G_{2r}^x(t)-\widetilde{B}_{2r}^x\right] dt\right| \\ &= \left|-\frac{b-a}{2r+1}G_{2r+1}^x(t)\Big|_a^b - (b-a)\widetilde{B}_{2r}^x\right| = (b-a)\left|\widetilde{B}_{2r}^x\right| \\ &= 2(b-a)\left|B_{2r}\left(\frac{x-a}{b-a}\right)+B_{2r}\right|,\end{aligned}$$

which proves the second assertion. Finally, we use the triangle inequality to obtain the third formula. □

COROLLARY 2.8. *For $r \ge 2$, we have*

$$\int_a^b \left|F_{2r-1}^{(a+b)/2}(t)\right| dt = \int_a^b \left|G_{2r-1}^{(a+b)/2}(t)\right| dt = \frac{b-a}{r}2^{4-2r}(1-2^{-2r})|B_{2r}|.$$

Also,

$$\frac{1}{b-a}\int_a^b \left|F_{2r}^{(a+b)/2}(t)\right| dt = 2^{2-2r}|B_{2r}| \quad \text{and} \quad \frac{1}{b-a}\int_a^b \left|G_{2r}^{(a+b)/2}(t)\right| dt \le 2^{3-2r}|B_{2r}|.$$

PROOF. The proof is similar to the proof of Corollary 2.7. □

LEMMA 2.9. *Let $x \in \left[a, (a+b)/2 - (b-a)/(4\sqrt{6})\right]$. If $f : [a, b] \to \mathbb{R}$ is such that for some $r \ge 2$ the derivative $f^{(2r)}$ is continuous on $[a, b]$, then there exists a point $\eta \in [a, b]$ such that*

$$\widetilde{R}_{2r}^2(x) = -\frac{(b-a)^{2r}}{2(2r)!}\left[B_{2r}\left(\frac{x-a}{b-a}\right)+B_{2r}\right]f^{(2r)}(\eta). \tag{2.7}$$

PROOF. Let $x \in \left[a, (a+b)/2 - (b-a)/(4\sqrt{6})\right]$. For $n = 2r \geq 4$ and f such that $f^{(2r)}$ is continuous on $[a, b]$ we can rewrite $R_{2r}^2(f)$ as

$$\widetilde{R}_{2r}^2(x) = (-1)^r \frac{(b-a)^{2r-1}}{4(2r)!} \int_a^b (-1)^r F_{2r}^x(t) f^{(2r)}(t)\, dt = (-1)^r \frac{(b-a)^{2r-1}}{4(2r)!} I_r,$$

where

$$I_r = \int_a^b (-1)^r F_{2r}^x(t) f^{(2r)}(t)\, dt.$$

If $m = \min_{[a,b]} f^{(2r)}(t)$ and $M = \max_{[a,b]} f^{(2r)}(t)$, then $m \leq f^{(2r)}(t) \leq M$, $t \in [a, b]$. From Corollary 2.5 we have$(-1)^r F_{2r}^x(t) \geq 0$, $t \in [a, b]$, so

$$m \int_a^b (-1)^r F_{2r}^x(t)\, dt \leq I_r \leq M \int_a^b (-1)^r F_{2r}^x(t)\, dt.$$

Since

$$\int_a^b F_{2r}^x(t)\, dt = -(b-a)\widetilde{B}_{2r}^x = -2(b-a)\left[B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\right],$$

we obtain

$$2m(-1)^{r-1}(b-a)\left[B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\right]$$
$$\leq I_r \leq 2M(-1)^{r-1}(b-a)\left[B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\right].$$

By the continuity of $f^{(2r)}$ on $[a, b]$ it follows that there must exist a point $\eta \in [a, b]$ such that

$$I_r = 2(-1)^{r-1}(b-a)\left[B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\right] f^{(2r)}(\eta).$$

From that we can easily obtain (2.7). □

LEMMA 2.10. *If $f : [a, b] \to \mathbb{R}$ is such that for some $r \geq 2$ the derivative $f^{(2r)}$ is continuous on $[a, b]$, then there exists a point $\eta \in [a, b]$ such that*

$$\widetilde{R}_{2r}^2\left(\frac{a+b}{2}\right) = -\frac{(b-a)^{2r}}{(2r)!} 2^{-2r} B_{2r} f^{(2r)}(\eta).$$

PROOF. The proof follows analogously to the proof of Lemma 2.9. □

THEOREM 2.11. *Let $x \in \left[a, (a+b)/2 - (b-a)/(4\sqrt{6})\right]$. Assume that $f : [a, b] \to \mathbb{R}$ is such that $f^{(2r)}$ is continuous on $[a, b]$ for some $r \geq 2$. If f is a $(2r)$-convex or $(2r)$-concave function, then there exists a point $\vartheta \in [0, 1]$ such that*

$$\widetilde{R}_{2r}^2(x) = \vartheta\left[B_{2r}\left(\frac{1}{2} - \frac{x-a}{b-a}\right) - B_{2r}\left(\frac{x-a}{b-a}\right) + 2(2^{-2r} - 1)B_{2r}\right]$$
$$\times \frac{(b-a)^{2r-1}}{2(2r)!}\left[f^{(2r-1)}(b) - f^{(2r-1)}(a)\right].$$

PROOF. By Corollary 2.5 for $t \in [a, b]$ we have

$$0 \le (-1)^{r-1} F_{2r}^{x}(t) \le (-1)^{r-1} F_{2r}^{x}\big((a+b)/2\big).$$

The rest of the proof is similar to the proof of Lemma 2.9. □

Theorem 2.11 can be improved in a way that the derivative $f^{(2r)}$ need not be continuous on $[a, b]$. To obtain such a result we use the following theorem from [7, Theorem 1].

THEOREM B. *Let $\varphi : I \to \mathbb{R}$, $I \subset \mathbb{R}$, be a monotonic function, and let $\rho : \mathbb{R} \to \mathbb{R}$ be a periodic function with period P such that for some $a \in \mathbb{R}$ and $n \in \mathbb{N}$ $[a, a+nP] \subset I$. Suppose that there exists some $x_0 \in (a, a+P)$ such that $\rho(x_0) = 0$, $\rho(x) \ge 0$ for all $x \in [a, x_0)$ and $\rho(x) \le 0$ for all $x \in (x_0, a+P]$. Suppose also that $\int_a^{a+P} \rho(x)\,dx = 0$. If φ is increasing on $[a, a+nP]$, then*

$$-\int_a^{a+nP} \rho(x)\varphi(x)\,dx \le \frac{1}{2n}\left(\varphi(a+nP) - \varphi(a)\right) \int_a^{a+nP} |\rho(x)|\,dx, \tag{2.8}$$

and this inequality is sharp. If φ is decreasing on $[a, a+nP]$, then the inequality (2.8) *is reversed.*

THEOREM 2.12. *Assume that the function $f : [a, b] \to \mathbb{R}$ is such that for some $r \ge 2$ the derivative $f^{(2r-1)}$ is continuous and increasing on $[a, b]$. Then for every $x \in \big[a, (a+b)/2 - (b-a)/(4\sqrt{6})\big]$ we have*

$$\begin{aligned}
(-1)^r &\left\{\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)+f(x)+f(a+b-x)}{4} + \tilde{T}_{2r-1}(x)\right\} \\
&\le \frac{(b-a)^{2r-1}}{2(2r)!}\left[f^{(2r-1)}(b) - f^{(2r-1)}(a)\right] \\
&\quad \times \left| B_{2r}\left(\frac{1}{2} - \frac{x-a}{b-a}\right) - B_{2r}\left(\frac{x-a}{b-a}\right) + 2(2^{-2r}-1)B_{2r}\right|,
\end{aligned}$$

and this inequality is sharp.

PROOF. We know that the function F_{2r-1}^{x} is periodic with period $P = b - a$. From Theorem 2.4 and Lemma 2.2 for $r \ge 2$ and $x \in \big[a, (a+b)/2 - (b-a)/(4\sqrt{6})\big]$ we have: $F_{2r-1}^{x}\big((a+b)/2\big) = 0$, $\int_a^b F_{2r-1}^{x}(t)\,dt = 0$ and also

$$(-1)^{r-1} F_{2r-1}^{x}(t) \begin{cases} > 0, & t \in \big(a, (a+b)/2\big), \\ < 0, & t \in \big((a+b)/2, b\big). \end{cases}$$

This means that if in Theorem B we choose $\rho(t) = (-1)^{r-1}F^x_{2r-1}(t)$, $\varphi(t) = f^{(2r-1)}(t)$ and $n = 1$, then from (2.8) we obtain

$$-\int_a^b (-1)^{r-1}F^x_{2r-1}(t)f^{(2r-1)}(t)\,dt \le \frac{1}{2}\left[f^{(2r-1)}(b) - f^{(2r-1)}(a)\right]\int_a^b \left|F^x_{2r-1}(t)\right|dt,$$

and combining this with Corollary 2.7 we obtain

$$(-1)^r\int_a^b F^x_{2r-1}(t)f^{(2r-1)}(t)\,dt \le \frac{b-a}{r}\left[f^{(2r-1)}(b) - f^{(2r-1)}(a)\right] \times\left|B_{2r}\left(\frac{1}{2}-\frac{x-a}{b-a}\right) - B_{2r}\left(\frac{x-a}{b-a}\right) + 2(2^{-2r}-1)B_{2r}\right|.$$

From Theorem 2.1 we know that

$$\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)+f(x)+f(a+b-x)}{4} + \widetilde{T}_{2r-1}(x) = \frac{(b-a)^{2r-2}}{4(2r-1)!}\int_{[a,b]} F^x_{2r-1}(t)f^{(2r-1)}(t)\,dt,$$

so

$$\begin{aligned}
&(-1)^r\left\{\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)+f(x)+f(a+b-x)}{4} + \widetilde{T}_{2r-1}(x)\right\} \\
&= \frac{(b-a)^{2r-2}}{4(2r-1)!}(-1)^r\int_a^b F^x_{2r-1}(t)f^{(2r-1)}(t)\,dt \\
&\le \frac{(b-a)^{2r-1}}{2(2r)!}\left[f^{(2r-1)}(b) - f^{(2r-1)}(a)\right] \\
&\quad\times\left|B_{2r}\left(\frac{1}{2}-\frac{x-a}{b-a}\right) - B_{2r}\left(\frac{x-a}{b-a}\right) + 2(2^{-2r}-1)B_{2r}\right|.
\end{aligned}$$

□

THEOREM 2.13. *Assume that the function $f : [a,b] \to \mathbb{R}$ is such that for some $r \ge 2$ the derivative $f^{(2r-1)}$ is continuous and increasing on $[a,b]$. Then we have*

$$\begin{aligned}
&(-1)^r\left\{\frac{1}{b-a}\int_a^b f(t)dt - \frac{f(a)+f(b)+2f\big((a+b)/2\big)}{4} + \widetilde{T}_{2r-1}\left(\frac{a+b}{2}\right)\right\} \\
&\le \frac{(b-a)^{2r-1}}{(2r)!}\left[f^{(2r-1)}(b) - f^{(2r-1)}(a)\right]2^{1-2r}(1-2^{-2r})|B_{2r}|,
\end{aligned}$$

and this inequality is sharp.

PROOF. The proof is similar to the proof of Theorem 2.12. □

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3. Hadamard's inequalities for $(2r)$-convex functions

Now we can give our main result: a generalization of Hadamard's inequalities for $(2r)$-convex functions, $r \geq 2$.

THEOREM 3.1. *Assume that $f : [a, b] \to \mathbb{R}$ is such that for some $r \geq 2$ the derivative $f^{(2r-1)}$ is continuous on $[a, b]$, and assume that f is $(2r)$-convex on $[a, b]$. If r is odd, then for all $x \in \left[a, (a+b)/2 - (b-a)/(4\sqrt{6})\right] \cup \{(a+b)/2\}$*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt - \widetilde{T}_{2r-1}^F, \\ \geq \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(x)+f(a+b-x)}{2} + \widetilde{T}_{2r-1}^V(x), \tag{3.1}$$

and for all $x \in \left[a + (b-a)/(2\sqrt{3}), (a+b)/2\right]$

$$\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(x)+f(a+b-x)}{2} + \widetilde{T}_{2r-1}^V(x) \geq 0. \tag{3.2}$$

If r is even the above inequalities are reversed.

PROOF. Let $x \in \left[a, (a+b)/2 - (b-a)/(4\sqrt{6})\right]$. In the case $n = 2r \geq 4$, from (2.5) we get

$$\frac{2}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)+f(x)+f(a+b-x)}{2} + 2\widetilde{T}_{2r-1}(x) = 2\widetilde{R}_{2r}^2(f),$$

where

$$\widetilde{R}_{2r}^2(x) = \frac{(b-a)^{2r-1}}{4(2r)!}\int_{[a,b]} F_{2r}^x(t)\,df^{(2r-1)}(t).$$

If f is $(2r)$-convex then $df^{(2r-1)}(t) \geq 0$ on $[a, b]$, and since by Corollary 2.5 we know that $(-1)^r F_{2r}^x(t) \geq 0$, $t \in [a, b]$, we obtain $\widetilde{R}_{2r}^2(x) \geq 0$ for r even and $\widetilde{R}_{2r}^2(x) \leq 0$ for r odd. The same is true if $x = (a+b)/2$. This means that for r odd we have

$$\frac{2}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)+f(x)+f(a+b-x)}{2} + 2\widetilde{T}_{2r-1}(x) \leq 0,$$

that is,

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt - \widetilde{T}_{2r-1}^F \\ \geq \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(x)+f(a+b-x)}{2} + \widetilde{T}_{2r-1}^V(x),$$

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and the above inequality is reversed if r is even. This completes the proof of (3.1).

Now let $x \in \left[a + (b-a)/(2\sqrt{3}), (a+b)/2\right]$ and suppose that r is odd. We can use the analogous results from [9, Theorem 2.1 and Corollary 2.4] to obtain

$$\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(x)+f(a+b-x)}{2} + \widetilde{T}^{V}_{2r-1}(x) \geq 0,$$

and the reverse if r is even. This completes the proof. □

The interested reader can find several sharper variants of (3.2) in [8].

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