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A UNIFIED MODEL FOR KAKUTANI'S INTERVAL SPLITTING AND RÉNYI'S RANDOM PACKING

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Abstract

One-dimensional random packing, known as the car-parking problem, was first analyzed by Rényi (1958). A stochastic version of Kakutani's (1975) interval splitting is another typical model on a one-dimensional interval. We consider a generalized car-parking problem which contains the above two models as special cases. In the generalized model, one can park a car of length l, if there is a space not less than 1. We give the limiting packing density and the limiting distribution of the length of randomly selected gaps between cars. Our results bridge the two models of Rényi and Kakutani.

GENERALIZED RANDOM PACKING; PACKING DENSITY; DISTRIBUTION OF GAPS; ASYMPTOTIC BEHAVIOR
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1. Introduction

We consider a generalization of Rényi's random packing problem. Cars of length, l $(0 \leqq l \leqq 1)$ are allowed to park on the interval $[0, x]$ if there are spaces not less than 1 unit in length. Cars are sequentially parked, with the preferred locations of the front of each car being uniformly distributed on $[0, x]$. The car actually parks at this site if it fits within a gap of length >1. If not, it resamples its preferred location. The procedure continues until none of the gaps are >1. Here, we study this model and obtain closed formulas for the limiting packing densities and the limiting distribution functions of gaps. These bridge the two well-known formulas for the Rényi and Kakutani models.

Rényi's random packing is the special case $(l = 1)$ of this problem. The case of $l = 0$ is related to a stochastic version of Kakutani's interval splitting procedure and a stopping rule which was discussed by van Zwet (1978). For details of this relation, see van Zwet (1978).

2. The limiting random packing density

Let $M_l(x)$ be the expected number of cars allowed to park on the interval $[0, x]$. We have for $x \geqq 1 - l$,

$$M_l(x + l) = \frac{1}{x} \int_0^x \{M_l(y) + M_l(x - y) + 1\}\, dy,$$

with $M_l(x) = 0$ for $0 \leqq x < 1$. Put $w_l(s) = e^s \int_0^\infty M_l(x) \exp(-sx)\, dx$. We have

$$\frac{d}{ds} w_1(s) = \left\{(1 - l) - \frac{2\exp(-ls)}{s}\right\} w_l(s) - \frac{1 - l}{s} - \frac{1}{s^2}. \tag{1}$$

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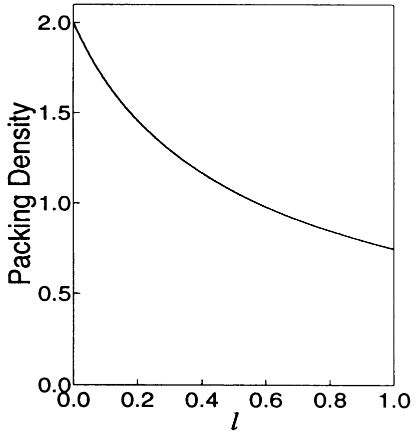

Figure 1. The limiting packing density $\lim_{x\to\infty} M_l(x)/x$ for $0 \leqq l \leqq 1$

The solution of (1), using the boundary condition $\lim_{s\to\infty} w_l(s) = 0$, is

$$w_l(s) = \frac{1}{s^2}\int_s^\infty \{1 + (1-l)t\} \exp\{-(1-l)(t-s)\} \exp\left(-2\int_s^t \frac{1-\exp(-lu)}{u}\,du\right) dt.$$

By the Tauberian theorem used in Rényi (1958),

$$\lim_{x\to\infty} \frac{M_l(x)}{x} = \lim_{s\to 0} s^2 w_l(s). \tag{2}$$

From this equality we obtain the following result.

Theorem 2.1. The limiting packing density of the cars of length l $(0 \leqq l \leqq 1)$ on the street of x in length is given by

$$\lim_{x\to\infty} \frac{M_l(x)}{x} = \int_0^\infty \{1 + (1-l)t\} \exp\{-(1-l)t\} \exp\left(-2\int_0^t \frac{1-\exp(-lu)}{u}\,du\right) dt. \tag{3}$$

When $l = 1$, the formula (3) coincides with the result obtained by Rényi (1958) in which $\lim_{x\to\infty} M_1(x)/x \simeq 0.748$. When $l = 0$, $\lim_{x\to\infty} M_0(x)/x = 2$ obtained from (3) corresponds to $M_0(x) = 2x - 1$ $(1 \leqq x)$ for a stochastic version of Kakutani's interval splitting problem. The limiting packing densities (3) are shown in Figure 1.

3. The limiting distribution of a randomly chosen gap

Let $m_{l,h}(x)$ be the sum of the expected number of cars and that of gaps which are bigger than h. We have for $x \geqq 1 - l$,

$$m_{l,h}(x+l) = \frac{1}{x}\int_0^x \{m_{l,h}(y) + m_{l,h}(x-y) + 1\}\,dy,$$

and the initial condition

$$m_{l,h}(x) = \begin{cases} 0, & 0 \leqq x < h \\ 1, & h \leqq x < 1. \end{cases}$$

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Put $\tilde{w}_{l,h}(s) = \exp(hs)\int_1^\infty m_{l,h}(x)\exp(-sx)\,dx$. We have

$$\frac{d}{ds}\tilde{w}_{l,h} = \left\{(h-l) - \frac{2\exp(-sl)}{s}\right\}\tilde{w}_{l,h} - \frac{2\exp(h-l)s}{s}\int_{1-l}^1 m_{l,h}(x)\exp(-sx)\,dx$$
$$-\frac{1-l}{s}\exp(-(1-h)s)m_{l,h}(1) - \frac{\exp(-(1-h)s)}{s^2},$$

where

$$m_{l,h}(1) = \begin{cases} 1, & h+l \geqq 1 \\ 3 - \dfrac{2h}{1-l}, & h+l < 1. \end{cases}$$

By an argument similar to the derivation of (2), we have

$$(4)\quad \lim_{x\to\infty}\frac{m_{l,h}(x)}{x} = \int_0^\infty \left[2t\int_{1-l}^1 m_{l,h}(x)\exp(-tx)\,dx\right.$$
$$\left. + \{(1-l)m_{l,h}(1)t + 1\}\exp(-(1-l)t)\right]\exp\left(-2\int_0^t \frac{1-\exp(-lu)}{u}\,du\right)dt.$$

By an argument analogous to the proof for Theorem 1 in Bankövi (1962),

$$(5)\qquad \lim_{x\to\infty}\Pr(I_{x,l} \leqq h) = 2 - \frac{\lim_{x\to\infty} m_{l,h}(x)/x}{\lim_{x\to\infty} m_{l,1}(x)/x}.$$

Here, $I_{x,l}$ is the length of the randomly chosen gap.

We can get the following theorem from (4) and (5).

Theorem 3.1. Let $I_{x,l}$ be the length of a randomly chosen gap. The limiting distribution function of $I_{x,l}$ as $x \to \infty$ is given by

$$\lim_{x\to\infty}\Pr(I_{x,l} \leqq h)$$
$$(6)\quad = \begin{cases} 1 - 2\dfrac{\displaystyle\int_0^\infty(-\exp(-t) + \exp(-ht))\exp\left(-2\int_0^t\frac{1-\exp(-lu)}{u}\,du\right)dt}{\displaystyle\int_0^\infty\{(1-l)t+1\}\exp(-(1-l)t)\exp\left(-2\int_0^t\frac{1-\exp(-lu)}{u}\,du\right)dt}, & h+l \geqq 1 \\[3em] 1 - 2\dfrac{\displaystyle\int_0^\infty\{-\exp(-t) + \exp(-(1-l)t) + (1-l-h)t\exp(-(1-l)t)\}\exp\left(-2\int_0^t\frac{1-\exp(-lu)}{u}\,du\right)dt}{\displaystyle\int_0^\infty\{(1-l)t+1\}\exp(-(1-l)t)\exp\left(-2\int_0^t\frac{1-\exp(-lu)}{u}\,du\right)dt}, & h+l < 1. \end{cases}$$

The limiting density functions of $I_{x,l}$ for several values of l are illustrated in Figure 2. When $l = 1$, the formula (6) coincides with the result by Bankövi (1962). When $l = 0$, (6) gives $\lim_{x\to\infty}\Pr(I_{x,l} \leqq h) = h$.

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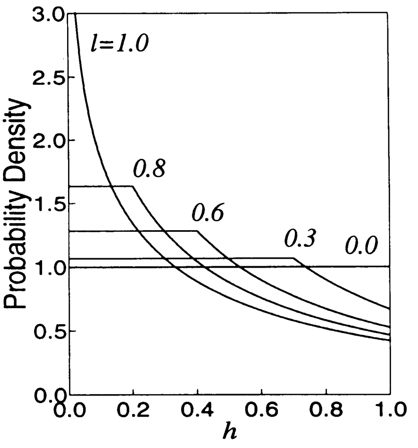

Figure 2. The limiting density functions of a randomly chosen gap between cars for several values of l ($l = 0.0$, 0.3, 0.6, 0.8, 1.0)

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