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GENERIC RESULTS FOR COCYCLES WITH VALUES IN A SEMIDIRECT PRODUCT

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ABSTRACT Let $A \propto B$ be the semidirect product of two local compact Hausdorff topological groups We prove that for a nonsingular ergodic automorphism T of a Lebesgue probability space, a generic cocycle taking values in $A \propto B$ is nontrivial and recurrent

0. **Introduction.** Let A and B be two second countable locally compact (necessarily countably generated) Hausdorff topological groups, each with a translation invariant metric. We denote both metrics on A and B by d to be understood from the context which metric is under consideration. The group operation on A is denoted by multiplication, the identity by 1 and the inverse of $a \in A$ by a^{-1}. The group B is assumed to be abelian and noncompact; the group operation is denoted by addition, the identity by 0, and the inverse of $b \in B$ by $-b$. The group A acts on B by group automorphisms; for simplicity we shall denote the action by multiplication: $b \xrightarrow{a} ab$. Furthermore, the map $(a,b) \longrightarrow ab$ is assumed to be uniformly jointly continuous, that is for every $\epsilon > 0$ there exist $\delta_1, \delta_2 > 0$ such that $d(ab, a'b') < \epsilon$ whenever $d(a,a') < \delta_1$ and $d(b,b') < \delta_2$. Let $A \propto B$ be the semidirect product of B by A relative to the given action. That is, the elements have the form $(a,b) \in A \times B$, and group operation $\circ$ defined as follows:

$$(a,b) \circ (a',b') = (aa', b + ab').$$

The identity element is $(1,0)$, and $(a,b)^{-1} = (a^{-1}, -a^{-1}b)$.

Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space, and G a countable group (with identity e) that acts nonsingularly, ergodically, and freely on X. We denote this action by multiplication: $x \xrightarrow{g} gx$. We shall consider cocycles on X taking values in the semidirect product $A \propto B$. That is, we shall consider measurable functions $F: G \times X \longrightarrow A \propto B$ with the property that

$$F(g'g, x) = F(g,x) \circ F(g', gx). \tag{1}$$

The above identity is called the *cocycle identity* and it implies that $F(e,x) = (1,0)$. We let $\psi: G \times X \longrightarrow A$ and $f: G \times X \longrightarrow B$ denote the projections of F onto the first and second coordinates respectively. Then $F(g,x) = \big(\psi(g,x), f(g,x)\big) \equiv (\psi, f)(g,x)$ and together with (1) imply

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(i) $\psi(g'g,x) = \psi(g,x)\psi(g',gx)$ and $\psi(e,x) = 1$ *i.e.*, ψ is a multiplicative A valued cocycle,

(ii) $f(g'g,x) = f(g,x) + \psi(g,x)f(g',gx)$ and $f(e,x) = 0$; *i.e.*, f is a ψ-cocycle (or twisted) B valued cocycle.

Throughout this paper (ψ,f) will always mean that ψ is an A valued cocycle and f a B valued ψ-cocycle. All equalities are understood to hold a.e.

We study generic properties of nontrivial and recurrent cocycles (ψ,f) in terms of their coordinate functions; this work is a generalization of the results in [D]. In Section 1, we define the notion of ψ-cohomology for ψ-cocycles, and investigate its connection with the recurrence properties and cohomology of cocycles taking values in the semidirect product $A \propto B$. In Section 2 we define the essential range, $\bar{E}_\psi(f)$, of a ψ-cocycle f to be a certain closed subgroup of $\bar{B}$, the one point compactification of B. We give sufficient conditions for triviality (being a coboundary) and recurrence of (ψ,f) in terms of $\bar{E}_\psi(f)$. Let $\bar{E}(\psi,f)$ and $E(\psi,f)$ denote the essential range and finite essential range of the cocycle (ψ,f) (see [K], [S1], and [S3]). We show that $E(\psi,f)$ is always an extension of the abelian group $E_\psi(f)$ by $E_f(\psi)$, where $E_\psi(f) = \bar{E}_\psi(f) \cap B$ and $E_f(\psi)$ consists of all elements in the finite essential range of ψ that appear as a first coordinate of some element in $E(\psi,f)$. We give sufficient conditions under which this extension is split, that is, $E(\psi,f)$ is a semidirect product. We topologize the set of ψ-cocycles by extending appropriately the topology of convergence in measure. In Section 3 we prove that orbit equivalence induces a topological group isomorphism between the corresponding sets of twisted cocycles which preserve triviality, the notions of recurrence, full essential range, and infinity in the essential range. In Section 4 we prove that if T is a nonsingular ergodic automorphism, then for a certain class of A valued cocycles ψ which simultaneously recur with the cocycle of the Radon-Nikodym derivative, there is a dense G_δ set of ψ-cocycles f whose essential range contains infinity and for which the cocycle (ψ,f) is recurrent. Using techniques similar to those in [PS] (see also [D]) this is first done for cocycles of a particular countable group action Γ on $\{0,1\}^{\mathbb{N}}$ (see §4 for a definition) which is orbit equivalent to the action of $\mathbb{Z}$ by powers of T ([S1] §8), then orbit equivalence (see §3) allows us to transfer the results back to T.

1. ψ-Cohomology.

DEFINITION 1.1. Two cocycles F and H on X taking values in $A \propto B$ are said to be *cohomologous* if there exists a measurable function $K: X \longrightarrow A \propto B$ such that $F(g,x) = K(x) \circ H(g,x) \circ K(gx)^{-1}$ for $g \in G$ and a.e. $x \in X$. The function K is called a *transfer function*. If $F(g,x) = K(x) \circ K(gx)^{-1}$ (*i.e.*, F is cohomologous to the constant function $(1,0)$), then F is called a *coboundary*. Similar definitions hold for two A valued cocycles ψ and ϕ on X.

DEFINITION 1.2. Two ψ-cocycles f and h on X are said to be ψ-*cohomologous* if there exists a measurable function $\beta: X \longrightarrow B$ such that $f(g,x) = \beta(x) + h(g,x) - \psi(g,x)\beta(gx)$. The function β is called a ψ-*transfer function*. If f is ψ-cohomologous to the constant function 0, then f is called ψ-*coboundary*.

PROPOSITION 1.3. *A cocycle (ψ,f) is a coboundary with transfer function (α,β) if and only if ψ is a coboundary with transfer function α and f is a ψ-coboundary with ψ-transfer function β.*

REMARK 1.4. Let ψ and ϕ be two A valued cohomologous cocycles with transfer function α, *i.e.*, $\psi(g,x) = \alpha(x)\phi(g,x)\alpha(gx)^{-1}$. Let h be a ϕ-cocycle, then the function $\alpha h\colon G\times X \longrightarrow B$ defined by $\alpha h(g,x) = \alpha(x)h(g,x)$ is a ψ-cocycle.

PROPOSITION 1.5. *Two cocycles (ψ,f) and (ϕ,h) are cohomologous with transfer function (α,β) if and only if ψ and ϕ are cohomologous with transfer function α, and f and αh are ψ-cohomologous with ψ-transfer function β.*

PROOF. Let $(\psi,f)(g,x) = \big(\alpha(x),\beta(x)\big)\circ(\phi,h)(g,x)\circ\big(\alpha(gx)^{-1},-\alpha(gx)^{-1}\beta(gx)\big)$. Then

$$\psi(g,x) = \alpha(x)\phi(g,x)\alpha(gx)^{-1},$$

and

$$\begin{aligned} f(g,x) &= \beta(x) + \alpha(x)h(g,x) - \alpha(x)\phi(g,x)\alpha(gx)^{-1}\beta(gx) \\ &= \beta(x) + \alpha(x)h(g,x) - \psi(g,x)\beta(gx). \end{aligned}$$

That is, ψ and ϕ are cohomologous with transfer function α, and f and αh are ψ-cohomologous with ψ-transfer function β. The converse is proved by reversing the above steps. ∎

COROLLARY 1.6. *If the group A is abelian, then (ψ,f) and (ψ,h) are cohomologous with transfer function (α,β) if and only if α equals a constant α_0 and f is ψ-cohomologous to $\alpha_0 h$ with ψ-transfer function β.*

PROOF. Suppose (ψ,f) and (ψ,h) are cohomologous with transfer function (α,β), from the above proposition we only need to show that α is a constant. Since A is abelian it follows that $\alpha(x) = \alpha(gx)$ and hence by ergodicity of the G action, α is equal to some constant α_0. Conversely, suppose $\alpha(x)\equiv\alpha_0$ and $f(g,x) = \beta(x)+\alpha(x)h(g,x) - \psi(g,x)\beta(gx)$. Since A is abelian, it follows that $\psi(g,x) = \alpha(x)\psi(g,x)\alpha(gx)^{-1}$ so that $(\psi,f)(g,x) = \big(\alpha(x),\beta(x)\big)\circ(\psi,h)(g,x)\circ\big(\alpha(x),\beta(x)\big)^{-1}$. ∎

DEFINITION 1.7. A cocycle (ψ,f) is said to be *recurrent* if for every $C\in\mathcal{B}$ of positive measure, and for each neighborhood $U\subseteq A$ of 1 and $V\subseteq B$ of 0, there exists $g\in G$ different from the identity such that

$$\mu\big(C\cap g^{-1}C\cap\{x:\psi(g,x)\in U\}\cap\{x:f(g,x)\in V\}\big) > 0.$$

Similar definitions hold for the coordinate functions ψ and f.

REMARK 1.8. $(\psi,0)$ is recurrent if and only if ψ is recurrent.

PROPOSITION 1.9. *If (ψ,f) and (ϕ,h) are cohomologous cocycles, then (ψ,f) is recurrent if and only if (ϕ,h) is recurrent.*

PROOF. Assume (ϕ,h) is recurrent and let $\psi(g,x) = \alpha(x)\phi(g,x)\alpha(gx)^{-1}$, and $f(g,x)=\beta(x)+\alpha(x)h(g,x)-\psi(g,x)\beta(gx)$. Let $\epsilon>0$ there exist $0<\delta_1,\delta_2<\frac{\epsilon}{2}$ such that $d(ab,a',b')<\frac{\epsilon}{2}$ whenever $d(a,a')<\delta_1$ and $d(b,b')<\delta_2$. Choose sequences $\{a_n\}$ in A and $\{b_n\}$ in B such that the sequences of neighborhoods $U_n=\{a\in A : d(a,a_n)<\frac{\delta_1}{4}\}$ and $V_n=\{b\in B : d(b,b_n)<\frac{\delta_2}{2}\}$ cover A and B respectively. Now, let $C\in\mathcal{B}$ with $\mu(C)>0$. For $n,m\in\mathbb{N}$, let $C_{n,m}=\{x\in C : \alpha(x)\in U_n \text{ and } \beta(x)\in V_m\}$. Since $C=\bigcup_{n,m}C_{n,m}$ there exist $n,m\in\mathbb{N}$ such that $\mu(C_{n,m})>0$. By recurrence of (ϕ,h) there exist $g\in G$, $g\neq e$ such that

$$\mu\Big(C_{n,m}\cap g^{-1}C_{n,m}\cap\Big\{x : d\big(\phi(g,x),1\big)<\frac{\delta_1}{2}\Big\}\cap\Big\{x : d\big(h(g,x),0\big)<\delta_2\Big\}\Big)>0.$$

Since,

$$\begin{aligned}C_{n,m}\cap g^{-1}C_{n,m}\cap\Big\{x : d\big(\phi(g,x),1\big)<\frac{\delta_1}{2}\Big\}\cap\Big\{x : d\big(h(g,x),0\big)<\delta_2\Big\}\subseteq\\ C\cap g^{-1}C\cap\Big\{x : d\big(\psi(g,x),1\big)<\epsilon\Big\}\cap\Big\{x : d\big(f(g,x),0\big)<\epsilon\Big\}\end{aligned}$$

we have that $\mu\Big(C\cap g^{-1}\cap\big\{x : d\big(\psi(g,x),1\big)<\epsilon\big\}\cap\big\{x : d\big(f(g,x),0\big)<\epsilon\big\}\Big)>0$. Therefore, (ψ,f) is recurrent. The converse is proved similarly. ∎

PROPOSITION 1.10. *If ψ is recurrent and f is a ψ-coboundary, then the cocycle (ψ,f) is recurrent.*

PROOF. From Proposition 1.5, we have that (ψ,f) and $(\psi,0)$ are cohomologous with transfer function $(1,\beta)$, where β is the ψ-transfer function of f. Remark 1.8 implies that $(\psi,0)$ is recurrent, and hence by Proposition 1.9 (ψ,f) is recurrent. ∎

2. **Essential range.** Let $(\psi,f)\colon G\times X\longrightarrow A\propto B$ be a cocycle, and consider its essential range $\bar{E}(\psi,f)$ which is a subgroup of $(A\propto B)^{-}$, the one point compactification of $A\propto B$. Let $E(\psi,f)=\bar{E}(\psi,f)\cap(A\propto B)$, the finite essential range which is a subgroup of $A\propto B$. Similarly the essential range and the finite essential range of ψ are denoted by $\bar{E}(\psi)$ and $E(\psi)$ respectively (see [S1] and [S3]). Let $\bar{B}=B\cup\{\infty\}$ be the one point compactification of B. For $\lambda\in B$ let $B_\epsilon(\lambda)=\{b\in B : d(b,\lambda)<\epsilon\}$, and $B_\epsilon(\infty)=\{b\in B : d(b,0)>1/\epsilon\}$. We define the essential range of f to be the set $\bar{E}_\psi(f)$ consisting of all $\lambda\in\bar{B}$ such that for every $\epsilon>0$ and for every subset C of X of positive measure, there exists $g\in G$ such that

$$\mu\Big(C\cap g^{-1}C\cap\Big\{x : d\big(\psi(g,x),1\big)<\epsilon\Big\}\cap\Big\{x : f(g,x)\in B_\epsilon(\lambda)\Big\}\Big)>0.$$

That is, $\lambda\in\bar{E}_\psi(f)$ if and only if $(1,\lambda)$ belongs to the essential range of the cocycle (ψ,f) in the usual sense. Let $E_\psi(f)=\bar{E}_\psi(f)\cap B$. Since 0 is trivially an element of $E_\psi(f)$, it follows that $E_\psi(f)\neq\emptyset$.

PROPOSITION 2.1. *$E_\psi(f)$ is a closed subgroup of B.*

PROOF. Assume that $\lambda, \lambda' \in E_\psi(f)$, we want to show that $\lambda + \lambda' \in E_\psi(f)$. Assume with no loss of generality that $\lambda, \lambda' \neq 0$. Let $\epsilon > 0$ and $C \subseteq X$ with $\mu(C) > 0$. By joint continuity of the action of A on B there exist $\delta_1, \delta_2 < \frac{\epsilon}{2}$ such that $d(ab, \lambda') < \frac{\epsilon}{2}$ whenever $d(a, 1) < \delta_1$ and $d(b, \lambda') < \delta_2$. Since $\lambda' \in E_\psi(f)$ there exists $g' \in G$ such that

$$\mu\Big(C \cap g'^{-1}C \cap \big\{x : d\big(\psi(g', x), 1\big) < \delta_1\big\} \cap \big\{x : d\big(f(g', x), \lambda'\big) < \delta_2\Big) > 0.$$

Let $D = C \cap g'^{-1}C \cap \big\{x : d\big(\psi(g', x), 1\big) < \delta_1\big\} \cap \big\{x : d\big(f(g', x), \lambda'\big) < \delta_2\big\}$. Then $\mu(D) > 0$ and there exists a $g \in G$ such that $\mu\Big(D \cap g^{-1}D \cap \big\{x : d\big(\psi(g, x), 1\big) < \delta_1\big\} \cap \big\{x : d\big(f(g, x), \lambda\big) < \frac{\epsilon}{2}\big\}\Big) > 0$. Since,

$$\begin{aligned} D \cap g^{-1}D \cap \big\{x : d\big(\psi(g, x), 1\big) < \delta_1\big\} \cap \Big\{x : d\big(f(g, x), \lambda\big) < \frac{\epsilon}{2}\Big\} \subseteq \\ C \cap (g'g)^{-1}C \cap \big\{x : d\big(\psi(g'g, x), 1\big) < \epsilon\big\} \cap \big\{x : d\big(f(g'g, x), \lambda + \lambda'\big) < \epsilon\big\}, \end{aligned}$$

it follows that

$$\mu\Big(C \cap (g'g)^{-1}C \cap \big\{x : d\big(\psi(g'g, x), 1\big) < \epsilon\big\} \cap \big\{x : d\big(f(g'g, x), \lambda + \lambda'\big) < \epsilon\big\}\Big) > 0.$$

Therefore, $\lambda + \lambda' \in E_\psi(f)$. Now, let $\lambda \in E_\psi(f)$. Note that $\psi(g^{-1}, gx) = \psi(g, x)^{-1}$, and $f(g^{-1}, gx) = -\psi(g, x)^{-1} f(g, x)$ for all $g \in G$. So that $d\big(\psi(g^{-1}, gx), 1\big) = d\big(\psi(g, x)^{-1}, 1\big) = d\big(1, \psi(g, x)\big)$, and

$$\begin{aligned} d\big(f(g^{-1}, gx), -\lambda\big) &= d\big(-\psi(g, x)^{-1} f(g, x), -\lambda\big) \\ &\leq d\big(-\psi(g, x)^{-1} f(g, x), -\psi(g, x)^{-1}\lambda\big) + d\big(-\psi(g, x)^{-1}\lambda, -\lambda\big) \\ &= d\big(\psi(g, x)^{-1} f(g, x), \psi(g, x)^{-1}\lambda\big) + d\big(\psi(g, x)^{-1}\lambda, \lambda\big). \end{aligned}$$

Choose $\delta_1, \delta_2 < \frac{\epsilon}{2}$ such that $d(ab, \lambda) < \frac{\epsilon}{2}$ whenever $d(a, 1) < \delta_1$ and $d(b, \lambda) < \delta_2$. For any set C in X of positive measure and for all $g \in G$, we have

$$\begin{aligned} g\Big(C \cap g^{-1}C \cap \big\{x : d\big(\psi(g, x), 1\big) < \delta_1\big\} \cap \big\{x : d\big(f(g, x), \lambda\big) < \delta_2\big\}\Big) \subseteq \\ C \cap gC \cap \big\{x : d\big(\psi(g^{-1}, x), 1\big) < \delta_1\big\} \cap \big\{x : d\big(f(g^{-1}, x), -\lambda\big) < \epsilon\big\} \end{aligned}$$

Since $\lambda \in E_\psi(f)$ and the G action is nonsingular, there exists $g \in G$ such that

$$\mu\Big(C \cap gC \cap \big\{x : d\big(\psi(g^{-1}, x), 1\big) < \delta_1\big\} \cap \big\{x : d\big(f(g^{-1}, x), -\lambda\big) < \epsilon\big\}\Big) > 0.$$

Therefore, $-\lambda \in E_\psi(f)$. The fact that $E_\psi(f)$ is closed is clear. ∎

PROPOSITION 2.2. *If f and h are ψ-cohomologous ψ-cocycles, then $\bar{E}_\psi(f) = \bar{E}_\psi(h)$.*

PROOF. Suppose $f(g, x) = \beta(x) + h(g, x) - \psi(g, x)\beta(gx)$, where $\beta : X \longrightarrow B$ is a measurable function. Let $\epsilon > 0$ be given, and choose $0 < \delta < \epsilon$ such that $d(b, ab) < \frac{\epsilon}{3}$

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whenever $d(1,a) < \delta$. Since B is a Lindelöf space, there exist a sequence $\{b_n\}$ in B and a countable cover $\{U_n\}$ of X with $U_n = \{b \in B : d(b, b_n) < \frac{\epsilon}{6}\}$. Let $C \subseteq X$ with $\mu(C) > 0$. For each $n \in \mathbb{N}$, let $C_n = \{x \in X : \beta(x) \in U_n\}$. Since $C = \bigcup_n C_n$, it follows that there exists $n \in \mathbb{N}$ such that $\mu(C_n) > 0$. For any $\lambda \in B$ and $g \in G$ we have

$$(*) \qquad \begin{aligned} d\big(f(g,x),\lambda\big) &= d\big(\beta(x) + h(g,x) - \psi(g,x)\beta(gx), \lambda\big) \\ &\le d\big(h(g,x),\lambda\big) + d\big(\beta(x),\beta(gx)\big) + d\big(\beta(gx), \psi(g,x)\beta(gx)\big) \end{aligned}$$

Now, let $\lambda \in E_\psi(h)$. There exists $g \in G$ such that

$$\mu\Big(C_n \cap g^{-1}C_n \cap \big\{x : d\big(\psi(g,x),1\big) < \delta\big\} \cap \Big\{x : d\big(h(g,x),\lambda\big) < \frac{\epsilon}{3}\Big\}\Big) > 0.$$

It follows from $(*)$ that

$$\mu\Big(C \cap g^{-1}C \cap \big\{x : d\big(\psi(g,x),1\big) < \epsilon\big\} \cap \big\{x : d\big(f(g,x),\lambda\big) < \epsilon\big\}\Big) > 0.$$

Therefore, $\lambda \in E_\psi(f)$, *i.e.*, $E_\psi(g) \subseteq E_\psi(f)$. The reverse containment is proved similarly, so that $E_\psi(g) = E_\psi(f)$. Now, let $\infty \in \bar{E}_\psi(g)$ and $\epsilon_1 > 0$ be so that $\frac{\epsilon_1}{1-\epsilon_1^2} < \epsilon$. Choose $0 < \delta_1, \delta_2 < \epsilon$ so that $d(b, ab') < \epsilon_1$ whenever $d(a,1) < \delta_1$ and $d(b,b') < \delta_2$. Let $C \in \mathcal{B}$ be of positive measure, we can find for some $n \in \mathbb{N}$ an element $b_n \in B$ so that the set $C_n = \big\{x \in C : d\big(\beta(x), b_n\big) < \frac{\delta_2}{2}\big\}$ has positive measure. Let $g \in G$ be such that

$$\mu\Big(C_n \cap g^{-1}C_n \cap \big\{x : d\big(\psi(g,x),1\big) < \delta_1\big\} \cap \Big\{x : d\big(h(g,x),0\big) > \frac{1}{\epsilon_2}\Big\}\Big) > 0.$$

Since

$$d\big(f(g,x),0\big) \ge d\big(h(g,x),0\big) - d\big(\beta(x), \psi(g,x)\beta(gx)\big) > \frac{1}{\epsilon_1} - \epsilon_1 > \frac{1}{\epsilon},$$

it follows that

$$\mu\Big(C \cap g^{-1}C \cap \big\{x : d\big(\psi(g,x),1\big) < \epsilon\big\} \cap \Big\{x : d\big(f(g,x),0\big) > \frac{1}{\epsilon}\Big\}\Big) > 0. \qquad \blacksquare$$

PROPOSITION 2.3. *If $\lambda \in E_\psi(f)$ for some $\lambda \neq 0$, then (ψ, f) is recurrent.*

PROOF. Let $\epsilon > 0$ and $C \in \mathcal{B}$ with $\mu(C) > 0$. Choose $0 < \delta_1, \delta_2 < \frac{\epsilon}{2}$ so that $d(ab, \lambda) < \frac{\epsilon}{2}$ whenever $d(a,1) < \delta_1$ and $d(b,\lambda) < \delta_2$. Since $\lambda \neq 0$ there exists $g' \in G$, $g' \neq e$ such that

$$\mu\Big(C \cap g'^{-1}C \cap \Big\{x : d\big(\psi(g',x),1\big) < \frac{\epsilon}{2}\Big\} \cap \Big\{x : d\big(f(g',x),\lambda\big) < \delta_2\Big\}\Big) > 0.$$

Let $D = C \cap g'^{-1}C \cap \big\{x : d\big(\psi(g',x),1\big) < \frac{\epsilon}{2}\big\} \cap \big\{x : d\big(f(g',x),\lambda\big) < \delta_2\big\}$. By Rohlin lemma we can choose a subset D' of D of positive measure such that $D' \cup g'D' \subseteq D$ and $\mu(D' \cap g'D') = \mu(D' \cap g'^{-1}D') = 0$. Since $-\lambda \in E_\psi(f)$ and $\lambda \neq 0$, there exists $g \notin \{e, g', g'^{-1}\}$ such that

$$\mu\Big(D' \cap g^{-1}D' \cap \big\{x : d\big(\psi(g,x),1\big) < \delta_1\big\} \cap \Big\{x : d\big(f(g,x),-\lambda\big) < \frac{\epsilon}{2}\Big\}\Big) > 0.$$

Now, for $x \in D' \cap g^{-1}D' \cap \{x : d(\psi(g,x),1) < \delta_1\} \cap \{x : d(f(g,x),-\lambda) < \frac{\epsilon}{2}\}$, we have

(i) $x \in C \cap (g'g)^{-1}C$,

(ii) $d(\psi(g'g,x),1) = d(\psi(g,x)\psi(g',gx),1) \le d(\psi(g',gx),1) + d(\psi(g,x),1) < \epsilon$,

(iii) $d(f(g'g,x),0) = d(f(g,x) + \psi(g,x)f(g',gx),0) \le d(f(g,x),-\lambda) + d(\psi(g,x)f(g',gx),\lambda) < \epsilon$.

Thus,

$$\mu\Big(C \cap (g'g)^{-1}C \cap \{x : d(\psi(g'g,x),1) < \epsilon\} \cap \{x : d(f(g'g,x),0) < \epsilon\}\Big) > 0.$$

Therefore, (ψ,f) is recurrent. ∎

PROPOSITION 2.4. *If ψ is cohomologous to ϕ with transfer function α and f is a ψ-cocycle, then $E_\psi(f) = B$ if and only if $E_\phi(\alpha^{-1}f) = B$, and $\infty \in \bar{E}_\psi(f)$ if and only if $\infty \in \bar{E}_\phi(\alpha^{-1}f)$.*

PROOF. Let $\lambda \in B$ be any element, and let $\epsilon > 0$ be given. There exist $0 < \delta_1$, $\delta_2 < \frac{\epsilon}{2}$ such that $d(ab,a'b') < \frac{\epsilon}{2}$ whenever $d(a,a') < \delta_1$ and $d(b,b') < \delta_2$. Choose a sequence $\{a_n\}$ in A such that sequence of neighborhoods $\{V_n\}$, with $V_n = \{a \in A : d(a,a_n) < \frac{\delta_1}{2}\}$, covers A. Let $C \in \mathcal{B}$ with $\mu(C) > 0$; there exists $n \in \mathbb{N}$ such that the set $C_n = \{x \in C : \alpha^{-1}(x) \in V_n\}$ has positive measure. Since $a_n^{-1}\lambda \in E_\psi(f)$ there exists $g \in G$ such that

$$\mu\Big(C_n \cap g^{-1}C_n \cap \Big\{x : d(\psi(g,x),1) < \frac{\delta_1}{2}\Big\} \cap \{x : d(f(g,x),a_n^{-1}\lambda) < \delta_2\}\Big) > 0.$$

For $x \in C_n \cap g^{-1}C_n \cap \{x : d(\psi(g,x),1) < \frac{\delta_1}{2}\} \cap \{x : d(f(g,x),a_n\lambda) < \delta_2\}$ we have

$$d(\alpha^{-1}(x)f(g,x),\lambda) \le d(\alpha^{-1}(x)f(g,x),\alpha^{-1}(x)a_n^{-1}\lambda) + d(\alpha^{-1}(x)a_n^{-1}\lambda,\lambda) < \epsilon$$

and

$$\begin{aligned} d(\phi(g,x),1) &= d(\alpha(x)^{-1}\psi(g,x)\alpha(gx),1) \\ &\le d(\psi(g,x),1) + d(\alpha^{-1}(x),\alpha^{-1}(gx)) < \frac{\delta_1}{2} + \delta_1 < \epsilon. \end{aligned}$$

Therefore $\lambda \in E_\phi(\alpha^{-1}f)$. The converse is proved similarly. Also a similar proof shows that $\infty \in E_\phi(f)$ if and only if $\infty \in E_\phi(\alpha^{-1}f)$. ∎

We now look at the algebraic connection between $E(\psi,f)$, $E_\psi(f)$, and $E(\psi)$. We first consider the split exact sequence $0 \longrightarrow B \xrightarrow{\iota} A \propto B \xrightarrow{\pi} A \longrightarrow 1$, where $\iota(b) = (1,b)$ and $\pi(a,b) = a$. Let $E_f(\psi) = \pi(E(\psi,f)) = \{a \in E(\psi) : (a,b) \in E(\psi,f)$ for some $b \in B\}$, and $E'_f(\psi) = \{a \in E(\psi) : (a,0) \in E(\psi,f)\}$. Both $E_f(\psi)$ and $E'_f(\psi)$ are subgroups of $E(\psi)$. We now give an equivalent definition of $E'_f(\psi)$.

PROPOSITION 2.5. *$a \in E'_f(\psi)$ if and only if $(a,b) \in E(\psi,f)$ for all $b \in E_\psi(f)$.*

PROOF. Let $a \in E'_f(\psi)$, then $(a,0) \in E(\psi,f)$. For any $b \in E_\psi(f)$ we have $(1,b) \in E(\psi,f)$. Since $E(\psi,f)$ is a group, then $(a,b) = (1,b) \circ (a,0) \in E(\psi,f)$. The converse is trivial since $0 \in E_\psi(f)$. ∎

PROPOSITION 2.6. *The group $E(\psi,f)$ is an extension of $E_f(\psi)$ by $E_\psi(f)$.*

PROOF. We need to show that the sequence $0 \longrightarrow E_\psi(f) \xrightarrow{\iota} E(\psi,f) \xrightarrow{\pi} E_f(\psi) \longrightarrow 1$ is short exact. Here ι and π denote the restrictions to the appropriate subgroups. By definition $E_f(\psi) = \pi\big(E(\psi,f)\big)$, so that π is surjective. Clearly ι is injective and $\pi\iota(b) = b$ for all $b \in E_\psi(f)$. ∎

The following lemma shows that the group $E_f(\psi)$ acts on the group $E_\psi(f)$, and the action is inherited from that of A on B.

LEMMA 2.7. *If $a \in E_f(\psi)$ and $b \in E_\psi(f)$, then $ab \in E_\psi(f)$.*

PROOF. We need to show that $(1, ab) \in E(\psi,f)$. Since $a \in E_f(\psi)$, there exists $b' \in B$ such that $(a, b') \in E(\psi,f)$. Also $(1, b) \in E(\psi,f)$, so that $(1, ab) = (a, b')\circ(1, b)\circ(a, b')^{-1}\circ (1, b')^{-1} \in E(\psi,f)$. ∎

NOTATION. We denote by $E_f(\psi) \propto E_\psi(f)$, the semidirect product of $E_\psi(f)$ by $E_f(\psi)$ relative to the above inherited action.

PROPOSITION 2.8. *If $E_f(\psi) = E_f'(\psi)$, then $E(\psi,f) = E_f(\psi) \propto E_\psi(f)$.*

PROOF. For this it suffices to show that the sequence $0 \longrightarrow E_\psi(f) \xrightarrow{\iota} E(\psi,f) \xrightarrow{\pi} E_f(\psi) \longrightarrow 1$ is split exact. From the given, we have that $(a, 0) \in E(\psi,f)$ for every $a \in E_f(\psi)$. Define $\alpha\colon E_f(\psi) \longrightarrow E(\psi,f)$ by $\alpha(a) = (a, 0)$. Then, $\pi\alpha(a) = a$ for all $a \in E_f(\psi)$, and hence the above sequence splits. Therefore, $E(\psi,f) = E_f(\psi) \propto E_\psi(f)$. ∎

COROLLARY 2.9. *If $E_\psi(f) = B$, then $E(\psi,f) = E_f(\psi) \propto E_\psi(f)$.*

PROOF. For any $(a, b) \in E(\psi,f)$, we have $(1, b) \in E(\psi,f)$ so that $(a, 0) = (1, b)^{-1} \circ (a, b) \in E(\psi,f)$. This shows that $E_f(\psi) = E_f'(\psi)$ and hence by Proposition 2.8, $E(\psi,f) = E_f(\psi) \propto E_\psi(f)$. ∎

COROLLARY 2.10. *If $E_f'(\psi) = A$, then $E(\psi,f) = E_f(\psi) \propto E_\psi(f)$.*

PROOF. Follows immediately from Proposition 2.8, since in this case $E_f'(\psi) = E_f(\psi)$. ∎

Let $Z_\psi(X, G, B, \mu)$ be the set of ψ-cocycles which is a group under pointwise addition. Let $B_\psi(X, G, B, \mu)$ be the subgroup of ψ-coboundaries (we identify those that agree μ a.e.). We topologize Z_ψ and B_ψ by defining the following notion of convergence: $f^{(n)} \longrightarrow f$ if and only if for each $g \in G, f^{(n)}(g, .) \longrightarrow f(g, .)$ in measure. It is well known that the topology of convergence in measure is given by the metric: $\bar{d}(\beta, \beta') = \int_X \frac{d(\beta(x),\beta'(x))}{1+d(\beta(x),\beta'(x))}\, d\mu$, where $\beta, \beta'\colon X \longrightarrow B$ are measurable. Let $M(X, G, A, \mu)$ be the set of equivalence classes of multiplicative A valued cocycles on X.

Let $\mathcal{C} = \{C_n : n \in \mathbb{Z}\}$ be a countable dense collection in the measure algebra. The following lemma gives a necessary condition for an element of B to be in the essential range of a ψ-cocycle by reducing the verifications to members of $\mathcal{C}$ only (see [CHP], [D]). This will be used in Section 4. Denote by ω, the Radon-Nikodym derivative μ *i.e.*, $\omega(g, x) = \frac{d\mu\circ g}{d\mu}(x)$, where $\mu \circ g(A) = \mu(gA)$. Let $[G]$ denote the full group of G. That is,

$[G]$ consists of all bimeasurable automorphisms $V\colon X \longrightarrow Y$ such that for each $x \in X$ there exists $g \in G$ such that $Vx = gx$. For $V \in [G]$, set $\omega(V,x) = \omega(g,x)$, $\psi(V,x) = \psi(g,x)$, and $f(V,x) = f(g,x)$ where $Vx = gx$.

LEMMA 2.11. *If there exists a $0 < K < 1$ such that for every $\epsilon > 0$ and for every $C \in \mathcal{C}$*

$$\sup_{V\in[G]} \mu\Big(C \cap V^{-1}C \cap \{x : |\omega(V,x) - 1| < \epsilon\} \cap \Big\{x : d\big(\psi(V,x), 1\big) < \epsilon\Big\} \cap \{x : f(V,x) \in B_\epsilon(\lambda)\}\Big) > K\mu(C),$$

then $\lambda \in \bar{E}_\psi(f)$.

PROOF. Let $\epsilon > 0$ and $E \subseteq X$ with $\mu(E) > 0$. Let $c(\epsilon, K) = \frac{(1-\epsilon)K}{(1-\epsilon)(K+1)+1}$. Choose $C \in \mathcal{C}$ such that $\mu(E\Delta C) < c(\epsilon, K)\mu(E)$. By hypothesis, there exists $V \in [G]$ such that

$$\mu\Big(C \cap V^{-1}C \cap \{x : |\omega(V,x) - 1| < \epsilon\} \cap \Big\{x : d\big(\psi(V,x), 1\big) < \epsilon\Big\} \cap \{x : f(V,x) \in B_\epsilon(\lambda)\}\Big) > K\mu(C).$$

Let $\bar{C} = C \cap V^{-1}C \cap \{x : |\omega(V,x) - 1| < \epsilon\} \cap \Big\{x : d\big(\psi(V,x), 1\big) < \epsilon\Big\} \cap \{x : f(V,x) \in B_\epsilon(\lambda)\}$, then

$$\mu(\bar{C}) > K\mu(C) \geq K\mu(E \cap C) \geq K\big(\mu(E) - \mu(E\Delta C)\big) > \mu(E)\big(K - Kc(\epsilon, K)\big) > 0.$$

Let $\bar{E} = C \cap \bar{C}$. Then $\mu(\bar{E}) = \mu(\bar{C}) - \mu(\bar{C} \setminus E) \geq \mu(\bar{C}) - \mu(C\Delta E) > \mu(E)\big(K - (K+1)c(\epsilon, K)\big) > 0$, and $\mu(V\bar{E}) \geq (1-\epsilon)\mu(\bar{E}) > (1-\epsilon)\mu(E)\big(K - (K+1)c(\epsilon, K)\big)$. Since $V\bar{E} \subseteq C$, we have

$$\begin{aligned} \mu(E \cap V\bar{E}) &= \mu(V\bar{E}) - \mu(V\bar{E} \setminus E) \\ &\geq \mu(V\bar{E}) - \mu(C\Delta E) \\ &> \mu(E)\Big((1-\epsilon)K - c(\epsilon, K)\big((1-\epsilon)(K+1) + 1\big)\Big) \geq 0. \end{aligned}$$

by nonsingularity of μ with respect to V, it follows that $\mu(V^{-1}E \cap \bar{E}) > 0$ and hence

$$\mu\Big(E \cap V^{-1}E \cap \{x : |\omega(V,x) - 1| < \epsilon\} \cap \Big\{x : d\big(\psi(V,x), 1\big) < \epsilon\Big\} \cap \{x : f(V,x) \in B_\epsilon(\lambda)\}\Big) > 0.$$

Therefore, $\lambda \in \bar{E}_\psi(f)$. ∎

LEMMA 2.12. *For each $\lambda \in \bar{E}_\psi(f)$ and for each $k, m, n \in \mathbb{N}$, the map*

$$f\colon\longrightarrow \sup_{V\in[G]} \mu\Big(C_k \cap V^{-1}C_k \cap \Big\{x : |\omega(V,x) - 1| < \frac{1}{m}\Big\} \cap \Big\{x : d\big(\psi(V,x), 1\big) < \frac{1}{n}\Big\} \cap \{x : f(V,x) \in B_{1/n}(\lambda)\}\Big)$$

is lower semicontinuous.

PROOF. The result follows from the fact that for each $V \in [G]$ the map

$$f\colon\to \mu\Big(C_k \cap V^{-1}C_k \cap \Big\{x : |\omega(V,x) - 1| < \frac{1}{m}\Big\} \cap \Big\{x : d\big(\psi(V,x), 1\big) < \frac{1}{n}\Big\} \cap \{x : f(V,x) \in B_{1/n}(\lambda)\}\Big)$$

is continuous (see [D]). ∎

3. **Invariance under orbit equivalence.**

THEOREM 3.1. *Let G_i be a nonsingular free action on $(X_i, \mathcal{B}_i, \mu_i)$, $i = 1,2$. If the actions of G_1 and G_2 are orbit equivalent, then there exists a topological group isomorphism $\Lambda\colon M(X_1, G_1, A, \mu_1) \longrightarrow M(X_2, G_2, A, \mu_2)$ such that for every $\psi \in M(X_1, G_1, A, \mu_1)$ the following hold:*

(a) If ϕ is cohomologous to $\Lambda(\psi)$, then $Z_\psi(X_1, G_1, A, \mu_1) \cong Z_\phi(X_2, G_2, A, \mu_2)$ and $B_\psi(X_1, G_1, A, \mu_1) \cong B_\phi(X_2, G_2, A, \mu_2)$ (as topological groups),

(b) under the isomorphism of (a), recurrence, ∞ in the essential range, and full essential range are preserved.

PROOF. Let $F\colon X_1 \longrightarrow X_2$ denote the isomorphism that gives the orbit equivalence. For g_2, G_2 and $x_2 \in X_2$, set $\Lambda(\psi)(g_2, x_2) = \psi(g_1, x_1)$, where $x_2 = F(x_1)$ and $g_2 F(x_1) = F(g_1 x)$. Let $\psi \in M(X_1, G_1, A, \mu_1)$ and $\phi(g_2, x_2) = \alpha(x_2)\Lambda(\psi)(g_2, x_2)\alpha(g_2 x_2)^{-1}$, where $\alpha\colon X_2 \longrightarrow A$ is measurable. For $f \in Z_\psi(X_1, G_1, A, \mu_1)$, set $\bar{f}(g_2, x_2) = \alpha(x_2) f(g_1, x_1)$. Then $\bar{f}$ is a ϕ-cocycle. Since if $g_2, g_2' \in G_2$ and $x_2 \in X_2$, then there exists an $x_1 \in X_1$ and $g_1, g_1' \in G_1$ such that $F(x_2) = x_1$, $F(g_1 x_1) = g_2 x_2$, and $F(g_1' g_1 x_1) = g_2' g_2 x_2$. Then,

$$\begin{aligned}
\bar{f}(g_2' g_2, x_2) &= \alpha(x_2) f(g_1' g_1, x_1) \\
&= \alpha(x_2)\big(f(g_1, x_1) + \psi(g_1, x_1) f(g_1', g_1 x_1)\big) \\
&= \alpha(x_2) f(g_1, x_1) + \alpha(x_2)\Lambda(\psi)(g_2, x_2) f(g_1', g_1 x_1) \\
&= \alpha(x_2) f(g_1, x_1) + \phi(g_2, x_2)\alpha(g_2 x_2) f(g_1', g_1 x_1) \\
&= \bar{f}(g_2, x_2) + \phi(g_2, x_2)\bar{f}(g_2', g_2 x_2).
\end{aligned}$$

If $f(g_1, x_1) = \beta(x_1) - \psi(g_1, x_1)\beta(g_1 x_1)$, then

$$\begin{aligned}
\bar{f}(g_2, x_2) = \alpha(x_2) f(g_1, x_1) &= \alpha(x_2)\big(\beta(x_1) - \psi(g_1, x_1)\beta(g_1 x_1)\big) \\
&= \alpha(x_2)\beta(F^{-1}x_2) - \phi(g_2, x_2)\alpha(g_2 x_2)\beta(F^{-1} g_2 x_2)
\end{aligned}$$

i.e., $\bar{f}$ is a ϕ-coboundary with ϕ-transfer function βF^{-1}. This proves (a). For part (b), proofs similar to those of Propositions 1.2, 2.2, and 2.4 show that (ψ, f) is recurrent if and only if $(\phi, \bar{f})$ is recurrent, $\infty \in \bar{E}_\psi(f)$ if and only if $\infty \in \bar{E}_\phi(\bar{f})$, and $E_\psi(f) = B$ if and only if $E_\phi(\bar{f}) = B$. ∎

4. **A generic model: the binary odometer.** Let $X = \prod_{i=1}^{\infty}\{0,1\}_i$, which is a group under addition, and let $\mathcal{F}$ be the Borel σ-algebra. Let Γ be the subgroup of X consisting of all those sequences with finitely many nonzero coordinates only. Then Γ acts on X by coordinatewise addition ($x \xrightarrow{\gamma} \gamma + x$). Let μ be any nonsingular measure on X which is ergodic with respect to the Γ action. It is well known that the action of Γ on X is orbit equivalent to the binary odometer with respect to the measure μ, and for any nonsingular ergodic hyperfinite action of a countable group G on a Lebesgue probability space Y, there exists a measure μ on X which is nonsingular and ergodic for the Γ action such that the actions of G on Y and Γ on X are orbit equivalent (see [S1] §8).

Let $S\colon X \longrightarrow X$ be the left shift, and for $n \geq 0$ let Γ_n be the finite subgroup of Γ whose members consist of all $\gamma \in \Gamma$ such that $\gamma_m = 0$ for all $m > n$ ($\Gamma_0 = \{\bar{0} = (0,0,\ldots)\}$). Denote by $\bar{\Gamma}_n$ the subgroup of Γ consisting of all those elements whose first n coordinates are all zeros. For $x \in X$, let $x^{(n)} = (x_1,\ldots,x_n,0,0,\ldots)$ and $x_{(n)} = (0,\ldots,0x_{n+1},x_{n+2},\ldots)$, then $x^{(n)} \in \Gamma_n$, $x_{(n)} \in \bar{\Gamma}_n$ and $x = x^{(n)} + x_{(n)}$. For $a_1,a_2,\ldots,a_n \in A$ we denote the product $a_1a_2\cdots a_n$ by $\prod_{i=1}^{n} a_i$. The following proposition is a generalization of Theorem 3.1 in [SP] for A abelian.

PROPOSITION 4.1. *For any cocycle $\psi\colon \Gamma \times X \longrightarrow A$ there exists a sequence of measurable maps $\alpha_k\colon X \longrightarrow A$ such that for each $n \geq 1$ and every $\gamma \in \Gamma_n$,*

$$(*) \qquad \psi(\gamma,x) = \left(\prod_{k=0}^{n-1} \alpha_k \circ S^k(x)\right)\left(\prod_{k=0}^{n-1} \alpha_k \circ S^k(\gamma + x)\right)^{-1}.$$

Conversely, for any sequence of measurable maps α_k, $(*)$ defines a cocycle.

PROOF. For $n \geq 1$, let $\psi_n(x) = \psi(x^{(n)}, x_{(n)})$. Note that if $\gamma \in \Gamma_n$, then $\gamma^{(n)} = \gamma$ and $\gamma_{(n)} = (0,0,\ldots)$.

CLAIM (i). For any $\gamma \in \Gamma_n$, $\psi(\gamma,x) = \psi_n(x)^{-1}\psi_n(\gamma + x)$.

PROOF OF CLAIM (i). Note that $(\gamma + x)^{(n)} = \gamma + x^{(n)}$ and $(\gamma + x)_{(n)} = x_{(n)}$; hence the cocycle identity gives

$$\begin{aligned}\psi_n^{-1}(x)\psi_n(\gamma + x) &= \psi(x^{(n)},x_{(n)})^{-1}\psi\big((\gamma + x)^{(n)},(\gamma + x)_{(n)}\big)\\ &= \psi(x^{(n)},x_{(n)})^{-1}\psi(x^{(n)},x_{(n)})\psi(\gamma,x^{(n)} + x_{(n)})\\ &= \psi(\gamma,x^{(n)} + x_{(n)}) = \psi(\gamma,x).\end{aligned}$$

CLAIM (ii). For any $\gamma \in \Gamma_n$ we have,

$$\psi_n(\gamma + x)\psi_{n+1}(\gamma + x)^{-1} = \psi_n(x)\psi_{n+1}(x)^{-1}.$$

PROOF OF CLAIM (ii).

$$\begin{aligned}\psi_{n+1}(x)^{-1}\psi_{n+1}(\gamma + x) &= \psi(x^{(n+1)},x_{(n+1)})^{-1}\psi\big((\gamma + x)^{(n+1)},(\gamma + x)_{(n+1)}\big)\\ &= \psi(x^{(n+1)},x_{(n+1)})^{-1}\psi(\gamma + x^{(n+1)},x_{(n+1)})\\ &= \psi(\gamma,x^{(n+1)} + x_{(n+1)}) = \psi(\gamma,x) = \psi_n(x)^{-1}\psi_n(\gamma + x).\end{aligned}$$

This shows that $\psi_n(\gamma+x)\psi_{n+1}(\gamma+x)^{-1} = \psi_n(x)\psi_{n+1}(x)^{-1}$. Thus for each $n \geq 1$, the function $\psi_n\psi_{n+1}^{-1}$ is independent of the first n coordinates, and hence there exists a measurable function $\alpha_n\colon X \longrightarrow A$ such that $\alpha_n \circ S^n(x) = \psi_n(x)\psi_{n+1}(x)^{-1}$. Set $\alpha_0(x) = \psi_1(x)^{-1}$, then for $n \geq 1$ and any $\gamma \in \Gamma_n$ we have

$$\psi(\gamma, x) = \left(\prod_{k=0}^{n-1} \alpha_k \circ S^k(x)\right)\left(\prod_{k=0}^{n-1} \alpha_k \circ S^k(\gamma + x)\right)^{-1}.$$

Conversely, let $\{\alpha_k\}$ be a sequence of measurable maps defined on X with values in A. For $n \geq 1$ and $\gamma \in \Gamma_n$ set $\psi(\gamma, x) = \left(\prod_{k=0}^{n-1} \alpha_k \circ S^k(x)\right)\left(\prod_{k=0}^{n-1} \alpha_k \circ S^k(\gamma + x)\right)^{-1}$, and let $\psi(\bar{0}, x) = 1$. We claim that ψ is a cocycle. Let $\gamma \in \Gamma_n$ and $\gamma' \in \Gamma_m$. Assume with no loss of generality that $m \geq n$, then $\gamma' + \gamma \in \Gamma_m$. Also for each $i > n - 1$ we have $\alpha_i \circ S^i(x) = \alpha_i \circ S^i(\gamma + x)$, so that

$$\begin{aligned}
\psi(\gamma' + \gamma, x) &= \left(\prod_{k=0}^{m-1} \alpha_k \circ S^k(x)\right)\left(\prod_{k=0}^{m-1} \alpha_k \circ S^k(\gamma' + \gamma + x)\right)^{-1} \\
&= \left(\prod_{k=0}^{n-1} \alpha_k \circ S^k(x)\right)\left(\prod_{k=0}^{n-1} \alpha_k \circ S^k(\gamma + x)\right)^{-1}\left(\prod_{k=0}^{n-1} \alpha_k \circ S^k(\gamma + x)\right) \\
&\quad \left(\prod_{k=n}^{m-1} \alpha_k \circ S^k(\gamma + x)\right)\left(\prod_{k=0}^{m-1} \alpha_k \circ S^k(\gamma' + \gamma + x)\right)^{-1} \\
&= \left(\prod_{k=0}^{n\;1} \alpha_k \circ S^k(x)\right)\left(\prod_{k=0}^{n-1} \alpha_k \circ S^k(\gamma + x)\right)^{-1} \\
&\quad \left(\prod_{k=0}^{m-1} \alpha_k \circ S^k(\gamma + x)\right)\left(\prod_{k=0}^{m-1} \alpha_k \circ S^k(\gamma' + \gamma + x)\right)^{-1} \\
&= \psi(\gamma, x)\psi(\gamma', \gamma + x).
\end{aligned}$$

■

REMARK. We refer to the sequence $\{\alpha_k\}$ as the sequence associated with ψ.

LEMMA 4.2. *Let ψ be an A valued cocyle, let $\{\alpha_k\}$ be its associated sequence. If $n \geq 1$ and $\beta\colon X \longrightarrow B$ is a measurable map satisfying $\beta(x) = \psi(\gamma, x)\beta(\gamma + x)$ for all $\gamma \in \Gamma_n$, then there exists a measurable map $\beta'\colon X \longrightarrow B$ such that*

$$\beta(x) = \left(\prod_{k=0}^{n-1} \alpha_k \circ S^k(x)\right)\beta' \circ S^n(x).$$

Conversely, suppose $\beta(x) = \left(\prod_{k=0}^{n-1} \alpha_k \circ S^k(x)\right)\beta' \circ S^n(x)$ for some measurable function β', then $\beta(x) = \psi(\gamma, x)\beta(\gamma + x)$ for all $\gamma \in \Gamma_n$.

PROOF. Suppose that $\beta(x) = \psi(\gamma, x)\beta(\gamma + x)$ for all $\gamma \in \Gamma_n$, Proposition 4.1 gives

$$\left(\prod_{k=0}^{n-1} \alpha_k \circ S^k(x)\right)^{-1}\beta(x) = \left(\prod_{k=0}^{n-1} \alpha_k \circ S^k(\gamma + x)\right)^{-1}\beta(\gamma + x).$$

Then the function $(\prod_{k=0}^{n-1}\beta_k\circ S^k)^{-1}\beta$ is independent of the first n coordinates of $x\in X$, hence there exists a measurable function $\beta'\colon X\longrightarrow B$ such that $\left(\prod_{k=0}^{n-1}\alpha_k\circ S^k(x)\right)^{-1}\beta(x)=\beta'\circ S^n(x)$. This shows that

$$\beta(x)=\left(\prod_{k=0}^{n-1}\alpha_k\circ S^k(x)\right)\beta'\circ S^n(x).$$

Conversely, suppose that $\beta(x)=\left(\prod_{k=0}^{n-1}\alpha_k\circ S^k(x)\right)\beta'\circ S^n(x)$. For $\gamma\in\Gamma_n$, $\beta'\circ S^n(x)=\beta'\circ S^n(\gamma+x)$ and

$$\begin{aligned}\psi(\gamma,x)\beta(\gamma+x)&=\left(\prod_{k=0}^{n-1}\alpha_k\circ S^k(x)\right)\left(\prod_{k=0}^{n-1}\alpha_k\circ S^k(\gamma+x)\right)^{-1}\\&\qquad\left(\prod_{k=0}^{n-1}\alpha_k\circ S^k(\gamma+x)\right)\beta'\circ S^n(\gamma+x)\\&=\left(\prod_{k=0}^{n-1}\alpha_k\circ S^k(x)\right)\beta'\circ S^n(x)=\beta(x).\end{aligned}$$ ∎

PROPOSITION 4.3. *If $f\colon\Gamma\times X\longrightarrow B$ is a ψ-cocycle, then there exists a sequence of measurable maps $\beta_n\colon X\longrightarrow B$ such that for $n\geq 1$,*

$$(**)\qquad \psi(\gamma,x)\beta_n(\gamma+x)=\beta_n(x)\ \textit{for}\ \gamma\in\Gamma_n$$

and for $\gamma\in\Gamma$,

$$(***)\qquad f(\gamma,x)=\sum_{n=0}^{\infty}\psi(\gamma,x)\beta_n(\gamma+x)-\beta_n(x).$$

Conversely, if $\beta_n\colon X\longrightarrow B$ is a sequence of measurable maps satisfying $(**)$, then $(***)$ defines a ψ-cocycle.

PROOF. Let $\{\alpha_k\colon X\longrightarrow A\}$ be the sequence associated with the cocycle ψ. Let f be a ψ-cocycle, for $n\geq 1$ set

$$f_n(x)=\left(\prod_{k=0}^{n-1}\alpha_k\circ S^k(x)\right)\left(\prod_{k=0}^{n-1}\alpha_k\circ S^k(x_{(n)})\right)^{-1}f(x^{(n)},x_{(n)}).$$

If $\gamma\in\Gamma_n$, then $\gamma^{(n)}=\gamma$ and $\gamma_{(n)}=(0,0,\ldots)$, so that

$$\begin{aligned}&\psi(\gamma,x)f_n(\gamma+x)-f_n(x)\\&=\left(\prod_{k=0}^{n-1}\alpha_k\circ S^k(x)\right)\left(\prod_{k=0}^{n-1}\alpha_k\circ S^k(x_{(n)})\right)^{-1}f(\gamma+x^{(n)},x_{(n)})\\&\qquad-\left(\prod_{k=0}^{n-1}\alpha_k\circ S^k(x)\right)\left(\prod_{k=0}^{n-1}\alpha_k\circ S^k(x_{(n)})\right)^{-1}f(x^{(n)},x_{(n)})\\&=\left(\prod_{k=0}^{n-1}\alpha_k\circ S^k(x)\right)\left(\prod_{k=0}^{n-1}\alpha_k\circ S^k(x_{(n)})\right)^{-1}\psi(x^{(n)}x_{(n)})f(\gamma,x)\\&=f(\gamma,x).\end{aligned}$$

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Using similar calculations as the above, one can show that for $\gamma \in \Gamma_n$

$$\psi(\gamma,x)f_{n+1}(\gamma+x)-f_{n+1}(x)=f(\gamma,x)=\psi(\gamma,x)f_n(\gamma+x)-f_n(x).$$

So that for each $n \geq 1$, the function $f_{n+1}-f_n$ satisfies

$$\psi(\gamma,x)\big(f_{n+1}(\gamma+x)-f_n(\gamma+x)\big)=f_{n+1}(x)-f_n(x).$$

By Lemma 4.2 for each $n \geq 1$ there exists a measurable function β_n' such that

$$f_{n+1}(x)-f_n(x)=\left(\prod_{k=0}^{n-1}\alpha_k\circ S^k(x)\right)\beta_n'\circ S^n(x).$$

Let $\beta_n(x)=f_{n+1}(x)-f_n(x)$, then for $\gamma \in \Gamma_n$, $\psi(\gamma,x)\beta_n(\gamma+x)=\beta_n(x)$. Set $\beta_0(x)=\beta_0'(x)=f_1(x)$. Let $\gamma \in \Gamma_n$, then

$$\begin{aligned}
\sum_{k=0}^{\infty}\psi(\gamma,x)\beta_k(\gamma+x)-\beta_k(x) &= \sum_{k=0}^{n-1}\psi(\gamma,x)\beta_k(\gamma+x)-\beta_k(x)\\
&= \psi(\gamma,x)f_1(\gamma+x)-f_1(x)\\
&\quad + \psi(\gamma,x)\sum_{k=1}^{n-1}\big(f_{k+1}(\gamma+x)-f_k(\gamma+x)\big)\\
&\quad - \sum_{k=1}^{n-1}\big(f_{k+1}(x)-f_k(x)\big)\\
&= \psi(\gamma,x)f_1(\gamma+x)-f_1(x)\\
&\quad + \psi(\gamma,x)\big(f_n(\gamma+x)-f_1(\gamma+x)\big)\\
&\quad - \big(f_n(x)-f_1(x)\big)\\
&= \psi(\gamma+x)f_n(\gamma+x)-f_n(x)=f(\gamma,x).
\end{aligned}$$

Conversely, let $\{\beta_k : X \longrightarrow B\}$ be a sequence of measurable maps satisfying $(**)$. Let f be as defined in $(***)$ and let $\gamma_1,\gamma_2 \in \Gamma$. There exists $m \geq 1$ such that $\gamma_1,\gamma_2 \in \Gamma_m$, then $\gamma_1+\gamma_2 \in \Gamma_m$ and

$$\begin{aligned}
f(\gamma_1+\gamma_2,x) &= \sum_{n=0}^{m-1}\psi(\gamma_1+\gamma_2,x)\beta_n(\gamma_1+\gamma_2+x)-\beta_n(x)\\
&= \psi(\gamma_1,x)\sum_{n=0}^{m-1}\Big(\psi(\gamma_2,\gamma_1+x)\beta_n\big((\gamma_2+(\gamma_1+x)\big)-\beta_n(\gamma_1+x)\Big)\\
&\quad + \sum_{n=0}^{m-1}\big(\psi(\gamma_1,x)\beta_n(\gamma_1+x)-\beta_n(x)\big)\\
&= \psi(\gamma_1,x)f(\gamma_2,\gamma_1+x)+f(\gamma_1,x).
\end{aligned}$$

Therefore, $(***)$ defines a ψ-cocycle. ∎

NOTATION. Let f be a ψ-cocycle and $\{\beta_n\}$ the sequence satisfying $(**)$ and $(***)$. Then by Lemma 4.2 for each $n \geq 1$ there exists a measurable map β_n' such that $\beta_n(x)=$

$\big(\prod_{k=0}^{n-1}\alpha_k\circ S^k(x)\big)\beta'_n\circ S^n(x)$, and $\beta_0=\beta'_0$. We refer to $\{\beta_n\}$ and $\{\beta'_n\}$ as the sequence and tail sequence of f respectively. Let $Z'_\psi(X,\Gamma,B,\mu)$ be the subgroup of all ψ-cocycles f such that each member in the tail sequence of f depends on finitely many coordinates only. We denote by $F(X,\mu,B)$ the set of all equivalence classes of measurable maps on X with values in B (two functions are identified if they agree μ a.e.). We let $F'(X,\mu,B)$ be the subset consisting of the measurable maps depending on finitely many coordinates only. We give $F(X,\mu,B)$ the topology of convergence in measure.

PROPOSITION 4.4. *The set $Z'_\psi(X,\Gamma,\mu)$ is dense in $Z_\psi(X,\Gamma,\mu)$.*

PROOF. Let f be any ψ-cocycle and $\{\beta_n\}$, $\{\beta'_n\}$ its associated sequence and tail sequence. Let $\epsilon>0$ be given, by joint continuity of the A action on B there exist sequences of real numbers $\{\delta_n\}$ and $\{\delta'_n\}$ such that

(i) for $n\geq 0$, $0<\delta_n<\frac{\epsilon}{2^{4+n}}$,

(ii) for $n\geq 0$, $d(ab,ab')<\frac{\epsilon}{2^{4+n}}$ whenever $d(b,b')<\delta_n$.

Since $F'(X,\mu,B)$ is dense in $F(X,\mu,B)$, it follows that for any finite set $\{\gamma^{(i)}\in\Gamma:1\leq i\leq m\}$ there exists a sequence $\{\tilde{\beta}_k\}$ of measurable maps each depending on finitely many coordinates only such that

(i) $\bar{d}(\beta_0,\tilde{\beta}_0)<\delta_0$ and $\bar{d}(\beta_0\circ\gamma^{(i)},\tilde{\beta}_0\circ\gamma^{(i)})<\delta_0$ for $1\leq i\leq m$,

(ii) $\bar{d}(\beta'_k\circ S^k,\tilde{\beta}_k\circ S^k)<\delta_k$ and $\bar{d}(\beta'_k\circ S^k\circ\gamma^{(i)},\tilde{\beta}_k\circ S^k\circ\gamma^{(i)})<\delta_k$ for $1\leq i\leq m$ and $k\geq 1$.

Thus, for $1\leq i\leq m$ and $k\geq 1$ we have

(a) $\bar{d}\big(\psi(\gamma^{(i)},.)\beta_0\circ\gamma^{(i)},\psi(\gamma^{(i)},.)\tilde{\beta}_0\circ\gamma^{(i)}\big)<\frac{\epsilon}{2^4}$,

(b) $\bar{d}\big((\prod_{j=0}^{k-1}\alpha_j\circ S^j)\beta'_k\circ S^k,(\prod_{j=0}^{k-1}\alpha_j\circ S^j)\tilde{\beta}_k\circ S^k\big)<\frac{\epsilon}{2^{k+4}}$,

(c) $\bar{d}\big(\psi(\gamma^{(i)},.)(\prod_{j=0}^{k-1}\alpha_j\circ S^j\circ\gamma^{(i)})\beta'_k\circ S^k\circ\gamma^{(i)},\psi(\gamma^{(i)},.)(\prod_{j=0}^{k-1}\alpha_j\circ S^j\circ\gamma^{(i)})\tilde{\beta}_k\circ S^k\circ\gamma^{(i)}\big)<\frac{\epsilon}{2^{k+4}}$.

Then the measurable function $\beta(x)=\big(\beta_0(x)-\tilde{\beta}_0(x)\big)+\sum_{k=1}^{\infty}\big(\prod_{j=0}^{k-1}\alpha_j\circ S^j(x)\big)\big(\beta'_k\circ S^k(x)-\tilde{\beta}_k\circ S^k(x)\big)$ is well defined. Set

$$\begin{aligned}\tilde{f}(\gamma,x)&=\psi(\gamma,x)\tilde{\beta}_0(\gamma+x)-\tilde{\beta}_0(x)\\&\quad+\sum_{k=1}^{\infty}\psi(\gamma,x)\Big(\prod_{j=0}^{k-1}\alpha_j\circ S^j(\gamma+x)\Big)\tilde{\beta}_k\circ S^k(\gamma+x)\\&\quad-\Big(\prod_{j=0}^{k-1}\alpha_j\circ S^j(x)\Big)\tilde{\beta}_k\circ S^k(x),\end{aligned}$$

then $\tilde{f}$ defines a ψ-cocycle. Also $f(\gamma,x)=\tilde{f}(\gamma,x)+\psi(\gamma,x)\beta(\gamma+x)-\beta(x)$, and for $1\leq i\leq m$,

$$\bar{d}\big(f(\gamma^{(i)},.),\tilde{f}(\gamma^{(i)},.)\big)=\bar{d}\big(\psi(\gamma^{(i)},.)\beta\circ\gamma^{(i)},\beta\big)<\epsilon.\qquad\blacksquare$$

REMARK 4.5. Let $C=\{C_n:n\in\mathbb{N}\}$ be a countable dense collection in the measure algebra and fix some $0 < K < 1$. For $k, m, n \in \mathbb{N}$ and $0 \neq \lambda \in \bar{B}$, set

$$N_\lambda(k,m,n;\psi) = \Big\{ f \in Z_\psi(X,\Gamma,B,\mu) : \sup_{V \in [\Gamma]} \mu\Big(C_k \cap V^{-1} C_k \cap \Big\{ x : |\omega(V,x) - 1| < \frac{1}{m} \Big\} \cap \Big\{ x : d\big(\psi(V,x),1\big) < \frac{1}{n} \Big\} \cap \{ x : f(V,x) \in B_{1/n}(\lambda) \} \Big) > K\mu(C_k) \Big\}.$$

By Lemma 2.12, $N_\lambda(k,m,n;\psi)$ is open. Note that Lemma 2.11 implies

$$\bigcap_{k,m,n} N_\lambda(k,m,n;\psi) = \{ f \in Z_\psi(X,\Gamma,B,\mu) : \lambda \in \bar{E}_\psi(f) \}.$$

If $\{\lambda_p : p \in \mathbb{N}\}$ is a dense sequence in B, then

$$\bigcap_{k,m,n,p} N_{\lambda_p}(k,m,n;\psi) = \{ f \in Z_\psi(X,\Gamma,\mu) : E_\psi(f) = B \}.$$

This shows that $\{ f \in Z_\psi(X,\Gamma,B,\mu) : \lambda \in \bar{E}_\psi(f) \}$ and $\{ f \in Z_\psi(X,\Gamma,\mu) : E_\psi(f) = B \}$ are G_δ sets in $Z_\psi(X,\Gamma,B,\mu)$.

NOTATION. We denote by $M'(X,\Gamma,A,\mu)$ the set of cocycles $\psi \in M(X,\Gamma,A,\mu)$ that recurs simultaneously with ω, the Radon-Nikodym derivative, and whose associated sequence $\{\alpha_k\}$ depends on finitely many coordinates only. Then, for every $\epsilon > 0$ and for any $C \in \mathcal{F}$ with $\mu(C) > 0$, there exist a $\gamma \in \Gamma$, $\gamma \neq \bar{0}$ such that $\mu\Big(C \cap \gamma^{-1} C \cap \{ x : |\omega(\gamma,x) - 1| < \epsilon \} \cap \big\{ x : d\big(\psi(\gamma,x),1\big) < \epsilon \big\} \Big) > 0$.

PROPOSITION 4.6. *For each $\psi \in M'(X,\Gamma,A,\mu)$ and for each $k,m,n \in \mathbb{N}$, the set $N_\infty(k,m,n;\psi)$ is dense in $Z_\psi(X,\Gamma,B,\mu)$.*

PROOF. Let $\{\alpha_k\}$ be the sequence associated with ψ, where each α_k depends on finitely many coordinates only. Choose a positive sequence $\{\epsilon_n\}$ such that $\epsilon_n < \frac{1}{n}$, and $d(ab,b) < \frac{1}{n}$ wherenver $d(a,1) < \epsilon_n$. Let U be any nonempty open set in $Z_\psi(X,\Gamma,B,\mu)$, then by Proposition 4.4 there exists $f \in Z'_\psi(X,\Gamma,B,\mu)$ with $f \in U$. Since f is an interior point of U there is an $\epsilon > 0$ and $\gamma^{(1)}, \ldots, \gamma^{(K)} \in \Gamma$ such that

$$W = \Big\{ h \in Z_\psi(X,\Gamma,B,\mu) : \bar{d}\big(h(\gamma^{(i)},.), f(\gamma^{(i)},.) \big) < \epsilon, 1 \leq i \leq K \Big\} \subseteq U.$$

Let $\{\beta'_k\}$ be the tail sequence of f. Since $f \in Z'_\psi(X,\Gamma,B,\mu)$ each β'_k depends only on finitely many coordinates, and for $\gamma \in \Gamma$

$$f(\gamma,x) = \sum_{n=0}^{\infty} \psi(\gamma,x)\beta_n(\gamma + x) - \beta_n(x).$$

where $\beta_k(x)\big(\prod_{i=0}^{k-1} \alpha_i \circ S^i(x)\big)\beta'_k \circ S^k(x)$ depends only on finitely many coordinates. Then we can find integers $M_1 < M_2$ such that for each $0 \leq j < M_1$ and every $1 \leq i < K$ we have $\alpha_j \circ S^j$, β_j depend only on the first M_2 coordinates

$$f(\gamma^{(i)},x) = \sum_{j=0}^{M_1} \psi(\gamma^{(i)},x)\beta_j(\gamma^{(i)} + x) - \beta_j(x).$$

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Using the simultaneous recurrence of ω and ψ and Rohlin lemma, we can find $\delta^{(1)} \in \Gamma$ different from the identity and a subset $B_1 \subseteq C_k$ of positive measure such that: $\delta^{(1)} \in \bar{\Gamma}_{M_2}$, $B_1 \cap \delta^{(1)} B_1 = \emptyset$, $B_1 \cup \delta^{(1)} B_1 \subseteq C_k$, and for $x \in B_1 \cup \delta^{(1)} B_1$ we have $|\omega(\delta^{(1)}, x) - 1| < \frac{1}{m}$, and $d\big(\psi(\delta^{(1)}, x), 1\big) < \epsilon_1 < 1$. Since $\delta^{(1)} \neq \bar{0}$ there exist positive integers k_1, N_1 such that $M_2 < k_1 \leq N_1$, $\delta^{(1)} \in \bar{\Gamma}_{M_2} \cap \Gamma_{N_1}$, and $(\delta^{(1)})_{k_1} = (\delta^{(1)})_{N_1} = 1$. By hypothesis, we can find an integer $\bar{N}_1 > N_1$ such that $\alpha_j \circ S^j$ depends on the first $\bar{N}_1$ coordinates only for $j \leq N_1$. If $\mu(C_k \backslash B_1 \cup \delta^{(1)} B_1) > 0$, then using again the simultaneous recurrence of ω and ψ and Rohlin lemma, we can find $\delta^{(2)} \in \Gamma$ different from the identity, and a subset $B_2 \subseteq C_k \backslash B_1 \cup \delta^{(1)} B_1$ of positive measure such that: $\delta^{(2)} \in \bar{\Gamma}_{\bar{N}_1}$, $B_2 \cap \delta^{(2)} B_2 = \emptyset$, $B_2 \cup \delta^{(2)} B_2 \subseteq C_k \backslash B_1 \cup \delta^{(1)} B_1$, and for $x \in B_2 \cup \delta^{(2)} B_2$ we have $|\omega(\delta^{(2)}, x) - 1| < \frac{1}{m}$, and $d\big(\psi(\delta^{(2)}, x), 1\big) < \epsilon_2 < \frac{1}{2}$. Since $\delta^{(2)} \neq \bar{0}$ there exist positive integers k_2, N_2 such that $\bar{N}_1 < k_2 \leq N_2$, $\delta^{(2)} \in \bar{\Gamma}_{\bar{N}_1} \cap \Gamma_{N_2}$, and $(\delta^{(2)})_{k_2} = (\delta^{(2)})_{N_2} = 1$. Let $\bar{N}_2 > N_2$ be such that $\alpha_j \circ S^j$ depends on the first $\bar{N}_2$ coordinates only $j \leq N_2$. We continue by an exhasutive argument to find a sequence $\{B_r\}$ of subsets of C_k, sequences of positive integers $\{k_r\}$, $\{N_r\}$, $\{\bar{N}_r\}$, and a sequence $\{\delta^{(r)}\}$ in Γ such that:

(i) $\bar{N}_{r-1} < k_r \leq N_r < \bar{N}_r; \bar{N}_0 = M_2$,

(ii) for $0 \leq j \leq N_r$, $\alpha_j \circ S^j$ depends only on the first $\bar{N}_r$ coordinates,

(iii) $\delta^{(r)} \in \bar{\Gamma}_{N_{r-1}} \cap \Gamma_{N_r}$, and $(\delta^{(r)})_{k_r} = (\delta^{(r)})_{N_r} = 1$,

(iv) $B_r \cap \delta^{(r)} B_r = \emptyset$, $B_r \cup \delta^{(r)} B_r \subseteq C_k \setminus \bigcup_{j<r} B_j \cup \delta^{(j)} B_j$, and $\mu\big(C_k \setminus \bigcup_{r=1}^{\infty} B_r \cup \delta^{(r)} B_r\big) = 0$,

(v) For $x \in B_r \cup \delta^{(r)} B_r$, we have $|\omega(\delta^{(r)}, x) - 1| < \frac{1}{m}$, and $d\big(\psi(\delta^{(r)}, x), 1\big) < \epsilon_n < \frac{1}{n}$.

Define $V \in [\Gamma]$ by

$$Vx = \begin{cases} \delta^{(r)} + x & \text{if } x \in B_r \cup \delta^{(r)} B_r \text{ for some } r \geq 1 \\ x & \text{otherwise.} \end{cases}$$

Using condition (ii) above, we can choose for each $j \geq 1$ an element $b_j \in B$ such that for $x \in X$, $d\Big(\big(\prod_{\iota=0}^{j-1} \alpha_\iota \circ S^\iota(x)\big) b_j, 0\Big) > n + \frac{3}{n}$. Then for any $a \in A$ such that $d(a, 1) < \epsilon_n$ we have $d\Big(a\big(\prod_{\iota=0}^{j-1} \alpha_\iota \circ S^\iota(x)\big) b_j, 0\Big) > n + \frac{2}{n}$. For $j \geq 1$, let $\beta' \colon X \longrightarrow B$ be given by

$$\beta_j'(x) = \begin{cases} b_j & \text{if } x_1 = 0 \\ 0 & \text{if } x_1 = 1, \end{cases}$$

and let $\rho_j(x) = \big(\prod_{\iota=0}^{j-1} \alpha_\iota \circ S^\iota(x)\big) \beta_j' \circ S^j(x)$. Define $h \in Z_\psi(X, \Gamma, \mu)$ by

$$h(\gamma, x) = \sum_{r=1}^{\infty} \psi(\gamma, x) \rho_{N_{r-1}}(\gamma + x) - \rho_{N_{r\ 1}}(x).$$

For a.e. $x \in C_k$ we have that $x \in B_r \cup \delta^{(r)} B_r$ for some $r \geq 1$. Now, either $\rho_{N_{r-1}}(\delta^{(r)} + x) = 0$ and $\rho_{N_{r\ 1}}(x) = \big(\prod_{\iota=0}^{N_{r-2}} \alpha_\iota \circ S^\iota(x)\big) b_{N_{r\ 1}}$, or $\rho_{N_{r\ 1}}(\delta^{(r)} + x) = \big(\prod_{\iota=0}^{N_{r-2}} \alpha_\iota \circ S^\iota(x)\big) b_{N_{r\ 1}}$ and $\rho_{N_{r-1}}(x) = 0$. Also, for $1 \leq l \leq r - 1$, $\rho_{N_{l\ 1}}(\delta^{(r)} + x) = \rho_{N_{l-1}}(x)$ so that

$$h(V, x) = h(\delta^{(r)}, x) = \sum_{l=1}^{r} \psi(\delta^{(r)}, x) \rho_{N_{l-1}}(\delta^{(r)} + x) - \rho_{N_{l\ 1}}(x),$$

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and

$$d\big(h(V,x),0\big) \geq d\big(\psi(\delta^{(r)},x)\rho_{N_{r-1}}(\delta^{(r)}+x)-\rho_{N_{r-1}}(x),0\big)$$
$$- d\Big(\sum_{l=1}^{r-1}\psi(\delta^{(r)},x)\rho_{N_{l-1}}(\delta^{(r)}+x)-\rho_{N_{l-1}}(x),0\Big)$$
$$= d\big(\psi(\delta^{(r)},x)\rho_{N_{r-1}}(\delta^{(r)}+x),\rho_{N_{r-1}}(x)\big)$$
$$- d\Big(\psi(\delta^{(r)},x)\sum_{l=1}^{r-1}\rho_{N_{l-1}}(x),\sum_{l=1}^{r-1}\rho_{N_{l-1}}(x)\Big)$$
$$> n+\frac{2}{n}-\frac{1}{n}=n+\frac{1}{n}.$$

Also, for each $1 \leq i \leq K$, we have $h(\gamma^{(i)},x)=0$. Let

$$\bar{f}(\gamma,x)=\sum_{j=0}^{M_1}\psi(\gamma,x)\beta_j(\gamma+x)-\beta_j(x)+h(\gamma,x).$$

Then, for $1 \leq i \leq K$ $\bar{f}(\gamma^{(i)},x) = f(\gamma^{(i)},x)$ so that $\bar{f} \in U$. Let $x \in B_r \cup \delta^{(r)}B_r$, since $M_2 \leq \bar{N}_{r-1}$ we have for $1 \leq j \leq M_1$ $\beta_j(\delta^{(r)}+x)=\beta_j(x)$. Hence,

$$d\Big(\sum_{j=0}^{M_1}\psi(\delta^{(r)},x)\beta_j(\delta^{(r)}+x)-\beta_j(x),0\Big)=d\Big(\psi(\delta^{(r)},x)\sum_{j=0}^{M_1}\beta_j(x),\sum_{j=0}^{M_1}\beta_j(x)\Big)<\frac{1}{n}.$$

Thus,

$$d\big(\bar{f}(V,x),0\big) \geq d\big(h(\delta^{(r)},x),0\big)-d\Big(\sum_{j=0}^{M_1}\psi(\delta^{(r)},x)\beta_j(\delta^{(r)}+x)-\beta_j(x),0\Big)>n.$$

This shows that $\bar{f} \in N_\infty(k,m,n;\psi)\cap U$ and therefore, $N_\infty(k,m,n;\psi)$ is dense. ∎

COROLLARY 4.7. *If $\psi \in M'(X,\Gamma,A,\mu)$, then the set $\{f \in Z_\psi(X,\Gamma,B,\mu) : \infty \in \bar{E}_\psi(f)\}$ is a dense G_δ.*

Corollary 4.7, Theorem 3.1, and the orbit equivalence of the $\mathbb{Z}$ action by powers of T with the Γ action above ([S1] §8), together give the following theorem:

THEOREM 4.8. *Let T be a nonsingular ergodic automorphism of a Lebesgue probability space $(Y,\mathcal{B},\nu)$. Then for each $\psi \in M'(Y,\mathbb{Z},A,\nu)$, the set $\{f \in Z_\psi(Y,\mathbb{Z},B,\nu) : \infty \in \bar{E}_\psi(f)\}$ is a dense G_δ.*

REMARK 4.9. (i) Using similar techniques and notation as in Lemma 2.11 and Lemma 2.12 one can show that:

(a) If for $\epsilon > 0$ and for every C_k (in a countable dense collection in the measure algebra)

$$\sup_{V\in[\Gamma]}\mu\Big(C_k\cap V^{-1}C_k\cap\{x:|\omega(V,x)-1|<\epsilon\}\cap\big\{x:d\big(\psi(V,x),1\big)<\epsilon\big\}$$
$$\cap\big\{x:d\big(f(V,x),0\big)<\epsilon\big\}\cap\{x:Vx\neq x\}\Big)>K\mu(C_k),$$

then (ψ, f) is recurrent.

(b) For each $k, m, n \in \mathbb{N}$ the map

$$f \longrightarrow \sup_{V \in [\Gamma]} \mu\Big(C_k \cap V^{-1}C_k \cap \Big\{x : |\omega(V,x) - 1| < \frac{1}{m}\Big\} \cap \Big\{x : d\big(\psi(V,x), 1\big) < \frac{1}{n}\Big\} \cap \Big\{x : d\big(f(V,x), 0\big) < \frac{1}{n}\Big\} \cap \{x : Vx \neq x\}\Big),$$

is lower semicontinuous,

(ii) Let $R(k,m,n;\psi)$ be the set of $f \in Z_\psi(X, \Gamma, B, \mu)$ such that

$$\sup_{V \in [\Gamma]} \mu\Big(C_k \cap V^{-1}C_k \cap \Big\{x : |\omega(V,x) - 1| < \frac{1}{m}\Big\} \cap \Big\{x : d\big(\psi(V,x), 1\big) < \frac{1}{n}\Big\} \cap \Big\{x : d\big(f(V,x), 0\big) < \frac{1}{n}\Big\} \cap \{x : Vx \neq x\}\Big) > K\mu(C_k),$$

Then (i) part (b) implies that $R(k,m,n;\psi)$ is open, and hence the set

$$\{f \in Z_\psi(X, \Gamma, B, \mu) : (\psi, f) \text{ is recurrent}\} = \bigcap_{k,m,n} R(k,m,n;\psi)$$

is a G_δ.

(iii) If in the proof of Proposition 4.6 we define $h_1(\gamma, x) = \sum_{r=1}^{\infty} \psi(\gamma, x)\rho_{N_r}(\gamma + x) - \rho_{N_r}(x)$, then for $x \in B_r \cup \delta^{(r)}B_r$ we have

$$d\big(h_1(V,x), 0\big) = d\big(h_1(\delta^{(r)}, x), 0\big) = d\Big(\sum_{j=1}^{r-1} \psi(\delta^{(r)}, x)\rho_{N_j}(\delta^{(r)} + x) - \rho_{N_j}(x), 0\Big)$$
$$= d\Big(\psi(\delta^{(r)}, x)\sum_{j=1}^{r-1} \rho_{N_j}(x), \sum_{j=1}^{r-1} \rho_{N_j}(x)\Big) < \frac{1}{n}.$$

Also,

$$d\Big(\sum_{j=0}^{M_1} \psi(\delta^{(r)}, x)\beta_j(\delta^{(r)} + x) - \beta_j(x), 0\Big) = d\Big(\psi(\delta^{(r)}, x)\sum_{j=0}^{M_1} \beta_j(x), \sum_{j=0}^{M_1} \beta_j(x)\Big) < \frac{1}{n}.$$

Set

$$\bar{f}_1(\gamma, x) = \sum_{j=0}^{M_1} \psi(\gamma, x)\beta_j(\gamma + x) - \beta_j(x) + h_1(\gamma, x).$$

For $x \in B_r \cup \delta^{(r)}B_r$, $d\big(\bar{f}_1(V,x), 0\big) = d\big(\bar{f}_1(\delta^{(r)}, x), 0\big) < \frac{1}{n}$; thus $\bar{f}_1 \in R(k,m,n;\psi) \cap U$. Therefore, $R(k,m,n;\psi)$ is dense. Again using orbit equivalence to the Γ action this proves:

THEOREM 4.10. *Let T be a nonsingular ergodıc automorphısm of a Lebesgue probabılity space* $(Y, \mathcal{B}, \nu)$. *Then for each* $\psi \in M'(Y, \mathbb{Z}, A, \nu)$ *the set* $\{f \in Z_\iota(Y, \mathbb{Z}, \nu) : (\psi, f)$ *ıs recurrent and* $\infty \in \bar{E}_\psi(Y, \mathbb{Z}, \nu)\}$ *ıs a dense* G_δ.

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