

CONCISENESS OF COPRIME COMMUTATORS IN FINITE GROUPS

CRISTINA ACCIARRI✉, PAVEL SHUMYATSKY
and ANITHA THILLAISUNDARAM

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Abstract

Let G be a finite group. We show that the order of the subgroup generated by coprime γ_k -commutators (respectively, δ_k -commutators) is bounded in terms of the size of the set of coprime γ_k -commutators (respectively, δ_k -commutators). This is in parallel with the classical theorem due to Turner-Smith that the words γ_k and δ_k are concise.

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1. Introduction

Let F be the free group freely generated by x_1, \dots, x_n . Any nonidentity element of F is called a group-word in the variables x_1, \dots, x_n . Given a group-word, we think of it primarily as a function of n variables defined on any given group G . The verbal subgroup $w(G)$ of G determined by w is the subgroup generated by the set G_w consisting of all values $w(g_1, \dots, g_n)$, where g_1, \dots, g_n are elements of G . A word w is said to be concise if, whenever G_w is finite for a group G , it always follows that $w(G)$ is finite. More generally, a word w is said to be concise in a class of groups \mathcal{X} if, whenever G_w is finite for a group $G \in \mathcal{X}$, it always follows that $w(G)$ is finite. In the 1960s P. Hall asked whether every word is concise but later Ivanov proved that this problem has a negative solution in its general form [6] (see also [9, page 439]). On the other hand, many important words are known to be concise. For instance, Turner-Smith [15] showed that the *lower central words* γ_k and the *derived words* δ_k are concise; here the words γ_k and δ_k are defined by the positions $\gamma_1 = \delta_0 = x_1$, $\gamma_{k+1} = [\gamma_k, x_{k+1}]$ and $\delta_{k+1} = [\delta_k, \delta_k]$. Wilson showed in [16] that the multilinear commutator words (outer commutator words) are concise. It was proved by Merzlyakov [8] that every word is concise in the class of linear groups.

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In [3], a word w was called boundedly concise in a class of groups \mathcal{X} if for every integer m there exists a number $\nu = \nu(\mathcal{X}, w, m)$ such that whenever $|G_w| \leq m$ for a group $G \in \mathcal{X}$ it always follows that $|w(G)| \leq \nu$. Fernández-Alcober and Morigi [4] showed that every word which is concise in the class of all groups is boundedly concise. Moreover, they showed that whenever w is a multilinear commutator word having at most m values in a group G , one has $|w(G)| \leq (m-1)^{(m-1)}$. Questions on conciseness of words in the class of residually finite groups have been tackled in [1]. It was shown that if w is a multilinear commutator word and q a prime-power, then the word w^q is concise in the class of residually finite groups; and if $w = \gamma_k$ is the k th lower central word and q a prime-power, then the word w^q is boundedly concise in the class of residually finite groups.

The concept of (bounded) conciseness can be applied in a much wider context. Suppose \mathcal{X} is a class of groups and $\phi(G)$ is a subset of G for every group $G \in \mathcal{X}$. One can ask whether the subgroup generated by $\phi(G)$ is finite whenever $\phi(G)$ is finite. In the present paper we show bounded conciseness of coprime commutators in finite groups.

The coprime commutators γ_k^* and δ_k^* were introduced in [13] as a tool to study properties of finite groups that can be expressed in terms of commutators of elements of coprime orders. Let G be a finite group. Every element of G is both a γ_1^* -commutator and a δ_0^* -commutator. Now let $k \geq 2$ and let X be the set of all elements of G that are powers of γ_{k-1}^* -commutators. An element g is a γ_k^* -commutator if there exist $a \in X$ and $b \in G$ such that $g = [a, b]$ and $(|a|, |b|) = 1$. For $k \geq 1$ let Y be the set of all elements of G that are powers of δ_{k-1}^* -commutators. The element g is a δ_k^* -commutator if there exist $a, b \in Y$ such that $g = [a, b]$ and $(|a|, |b|) = 1$. The subgroups of G generated by all γ_k^* -commutators and all δ_k^* -commutators will be denoted by $\gamma_k^*(G)$ and $\delta_k^*(G)$, respectively. One can easily see that if N is a normal subgroup of G and x an element whose image in G/N is a γ_k^* -commutator (respectively, a δ_k^* -commutator), then there exists a γ_k^* -commutator $y \in G$ (respectively, a δ_k^* -commutator) such that $x \in yN$. It was shown in [13] that $\gamma_k^*(G) = 1$ if and only if G is nilpotent and $\delta_k^*(G) = 1$ if and only if the Fitting height of G is at most k . It follows that for every $k \geq 2$ the subgroup $\gamma_k^*(G)$ is precisely the last term of the lower central series of G (which is sometimes denoted by $\gamma_\infty(G)$) while for every $k \geq 1$ the subgroup $\delta_k^*(G)$ is precisely the last term of the lower central series of $\delta_{k-1}^*(G)$. In the present paper we prove the following results.

THEOREM 1.1. *Let $k \geq 1$ and let G be a finite group in which the set of γ_k^* -commutators has size m . Then $|\gamma_k^*(G)|$ is m -bounded.*

THEOREM 1.2. *Let $k \geq 0$ and let G be a finite group in which the set of δ_k^* -commutators has size m . Then $|\delta_k^*(G)|$ is m -bounded.*

We remark that the bounds for $|\gamma_k^*(G)|$ and $|\delta_k^*(G)|$ in the above results do not depend on k . Thus, we observe here the phenomenon that in [4] was dubbed ‘uniform conciseness’. We make no attempts to provide explicit bounds for $|\gamma_k^*(G)|$ and $|\delta_k^*(G)|$ in Theorems 1.1 and 1.2. Throughout the paper we use the term m -bounded to mean that the bound is a function of m .

2. Preliminaries

We begin with a well-known result about coprime actions on finite groups. Recall that $[K, H]$ is the subgroup generated by $\{[k, h] : k \in K, h \in H\}$, and $[K, H] = [[K, H], H]$ for $i \geq 2$.

LEMMA 2.1 [5, Lemma 4.29]. *Let A act via automorphisms on G , where A and G are finite groups, and suppose that $(|G|, |A|) = 1$. Then $[G, A, A] = [G, A]$.*

For the following result from [14], recall that a subset B of a group A is *normal* if B is a union of conjugacy classes of A .

LEMMA 2.2. *Let A be a group of automorphisms of a finite group G with $(|A|, |G|) = 1$. Suppose that B is a normal subset of A such that $A = \langle B \rangle$. Let $k \geq 1$ be an integer. Then $[G, A]$ is generated by the subgroups of the form $[G, b_1, \dots, b_k]$, where $b_1, \dots, b_k \in B$.*

The following is an elementary property of δ_k^* -commutators.

LEMMA 2.3. *Let G be a finite group. For a nonnegative integer k ,*

$$\delta_k^*(\delta_1^*(G)) = \delta_{k+1}^*(G).$$

PROOF. We argue by induction. For $k = 0$, the result is obvious by the definition of δ_0^* -commutators.

Suppose the result holds for $k - 1$. So

$$\delta_{k-1}^*(\delta_1^*(G)) = \delta_k^*(G).$$

It was mentioned in the introduction that $\delta_{k+1}^*(G) = \gamma_\infty(\delta_k^*(G))$. By induction,

$$\delta_{k+1}^*(G) = \gamma_\infty(\delta_{k-1}^*(\delta_1^*(G))),$$

and viewing $\delta_1^*(G)$ as the group under consideration,

$$\gamma_\infty(\delta_{k-1}^*(\delta_1^*(G))) = \delta_k^*(\delta_1^*(G)),$$

as required. □

Here is a helpful observation that we will use in both of our main results. Recall that a Hall subgroup of a finite group is a subgroup whose order is coprime to its index. Also, a finite group G is metanilpotent if and only if $\gamma_\infty(G)$ is nilpotent.

LEMMA 2.4. *Let G be a finite metanilpotent group and P a Sylow p -subgroup of $\gamma_\infty(G)$, and let H be a Hall p' -subgroup of G . Then $P = [P, H]$.*

PROOF. For simplicity, we write K for $\gamma_\infty(G)$. By passing to the quotient $G/O_{p'}(G)$, we may assume that $P = K$.

Let P_1 be a Sylow p -subgroup of G . So $G = P_1H$. Now P_1/P is normal in G/P as G/P is nilpotent, but also $P \leq P_1$; hence, P_1 is normal in G . It follows that $K = [P_1, H]$, since in a nilpotent group all coprime elements commute. By Lemma 2.1, $[P_1, H, H] = [P_1, H] = P$, and so $P = [P_1, H] = [P, H]$. □

In the proofs of our main results we often reduce to the following case.

LEMMA 2.5. *Let i and m be positive integers. Let P be an abelian p -group acted on by a p' -group A such that*

$$|\{[x, a_1, \dots, a_i] : x \in P, a_1, \dots, a_i \in A\}| = m.$$

Then $|[P, A]| = 2^m$, so is m -bounded.

PROOF. We enumerate the set $\{[x, a_1, \dots, a_i] : x \in P, a_1, \dots, a_i \in A\}$ as $\{c_1, \dots, c_m\}$. As P is abelian,

$$[x, a_1, \dots, a_i]^l = [x^l, a_1, \dots, a_i] \quad (\dagger)$$

for all $x \in P, a_1, \dots, a_i \in A$, and a positive integer l .

Consider $g \in [P, A]$, which can be expressed as some product $c_1^{l_1} \cdots c_m^{l_m}$ for nonnegative integers l_1, \dots, l_m . We claim that $l_1, \dots, l_m \in \{0, 1\}$. For, if $l_j > 1$ with $j \in \{1, \dots, m\}$, we know from (\dagger) that $c_j^{l_j} \in \{c_1, \dots, c_m\}$. We replace all such $c_j^{l_j}$ accordingly, so that g is now expressed as $c_1^{k_1} \cdots c_m^{k_m}$ with $k_1, \dots, k_m \in \{0, 1\}$. Hence $|[P, A]| = 2^m$. \square

The well-known focal subgroup theorem [12, Corollary 10.34, page 255] states that if G is a finite group and P a Sylow p -subgroup of G , then $P \cap G'$ is generated by the set of commutators $\{[g, z] \mid g \in G, z \in P, [g, z] \in P\}$. In particular, it follows that $P \cap G'$ can be generated by commutators lying in P . This observation led to the question on generation of Sylow subgroups of verbal subgroups of finite groups. More specifically, the following problem was addressed in [2].

Given a multilinear commutator word w and a Sylow p -subgroup P of a finite group G , is it true that $P \cap w(G)$ can be generated by w -values lying in P ?

The answer to this is still unknown. The main result of [2] is that if G has order $p^a n$, where n is not divisible by p , then $P \cap w(G)$ is generated by n th powers of w -values. In the present paper we will require a result on generation of Sylow subgroups of $\delta_k^*(G)$.

LEMMA 2.6. *Let $k \geq 0$ and let G be a finite soluble group of order $p^a n$, where p is a prime and n is not divisible by p , and let P be a Sylow p -subgroup of G . Then $P \cap \delta_k^*(G)$ is generated by n th powers of δ_k^* -commutators lying in P .*

It seems likely that Lemma 2.6 actually holds for all finite groups. In particular, the result in [2] was proved without the assumption that G is soluble. It seems though that proving Lemma 2.6 for arbitrary groups is a complicated task. Indeed, one of the tools used in [2] is the proof of the Ore conjecture by Liebeck *et al.* [7] that every element of any finite simple group is a commutator. Recently, it was conjectured in [13] that every element of a finite simple group is a commutator of elements of coprime orders. If this is confirmed, then extending Lemma 2.6 to arbitrary groups would be easy. However, the conjecture that every element of a finite simple group is a commutator of elements of coprime orders is proved only for the alternating groups [13] and the

groups $\text{PSL}(2, q)$ [10]. Thus, we prove Lemma 2.6 only for soluble groups, which is adequate for the purposes of the present paper.

Before we embark on the proof of Lemma 2.6, we note a key result from [2] that we will need.

LEMMA 2.7. *Let G be a finite group, and let P be a Sylow p -subgroup of G . Assume that $N \leq L$ are two normal subgroups of G , and use bar notation in the quotient group G/N . Let X be a normal subset of G consisting of p -elements such that $\overline{P} \cap \overline{L} = \langle \overline{P} \cap \overline{X} \rangle$. Then $P \cap L = \langle P \cap X, P \cap N \rangle$.*

We are now ready to prove Lemma 2.6.

PROOF. Let G be a counter-example of minimal order. Then $k \geq 1$.

By induction on the order of G , the lemma holds for every proper subgroup and every proper quotient of G . We observe that $\delta_1^*(G) < G$ since G is not perfect, and by Lemma 2.3, $\delta_{k+1}^*(G) = \delta_k^*(\delta_1^*(G))$. Since the result holds for $\delta_1^*(G)$, it follows that $P \cap \delta_{k+1}^*(G)$ is generated by n th powers of δ_k^* -commutators in G . Note that we made use of [2, Remark 3.2].

If $\delta_{k+1}^*(G) \neq 1$, by induction the result holds for $G/\delta_{k+1}^*(G)$. Combining this with the fact that $P \cap \delta_{k+1}^*(G)$ can be generated by n th powers of δ_k^* -commutators, we get a contradiction by Lemma 2.7. Hence $\delta_{k+1}^*(G) = 1$. Further $O_{p'}(G) = 1$ since G is a minimal counter-example. Therefore, $\delta_k^*(G) \subseteq P$, so $P \cap \delta_k^*(G)$ is generated by n th powers of δ_k^* -commutators lying in P . We have our required contradiction. \square

3. Proofs of the main results

We mention here a result of Schur and Wiegold. The much celebrated Schur theorem states that if G is a group with $|G/Z(G)|$ finite, then $|G'|$ is finite. It is implicit in the work of Schur that if $|G/Z(G)| = m$, then $|G'|$ is m -bounded. However, Wiegold produced a shorter proof of this second statement, which also gives the best possible bound. See Robinson [11, pages 102–103] for details.

For the proof of Theorem 1.2, we require the following result from [13].

LEMMA 3.1. *Let G be a finite group and let y_1, \dots, y_k be δ_k^* -commutators in G . Suppose y_1, \dots, y_k normalise a subgroup N such that $(|y_i|, |N|) = 1$ for every $i = 1, \dots, k$. Then for every $x \in N$ the element $[x, y_1, \dots, y_k]$ is a δ_{k+1}^* -commutator.*

Now we are ready to begin.

PROOF OF THEOREM 1.1. Let X be the set of all γ_k^* -commutators. We wish to show that if $|X| = m$, then $|\gamma_k^*(G)|$ is m -bounded. For convenience we write K for $\langle X \rangle$. Of course, $K = \gamma_\infty(G)$.

The subgroup $C_G(X)$ has index at most $m!$, so $|K/Z(K)| \leq m!$ too. By Schur, K' has m -bounded order. Therefore, by passing to the quotient, we may assume $K' = 1$, and so K is abelian with G metanilpotent.

It is enough to bound the order of each Sylow subgroup of K . We choose a Sylow p -subgroup P . By passing to the quotient $G/O_{p'}(G)$, we may assume $K = P$.

By Lemma 2.4, a Hall p' -subgroup H of G satisfies $P = [P,_{k-1} H]$. We know that P is abelian and P is normal in PH .

We denote the set $\{[x, h_1, \dots, h_{k-1}] : x \in P, h_1, \dots, h_{k-1} \in H\}$ by \hat{X} .

For $x \in P$ and $h_1, \dots, h_{i-1} \in H$, where $i \geq 2$, we note that $[x, h_1, \dots, h_{i-1}]$ is a γ_i^* -commutator. Therefore, $\hat{X} \subseteq X$, and $|\hat{X}| \leq m$.

By Lemma 2.5, it follows that $[[P,_{k-1} H]]$ is m -bounded. Appealing to Lemma 2.4, we conclude that $|P|$ is m -bounded. \square

PROOF OF THEOREM 1.2. Let X be the set of δ_k^* -commutators in G . We wish to show that if $|X| = m$, then $|\delta_k^*(G)|$ is m -bounded. We recall that $\delta_k^*(G) = \gamma_\infty(\delta_{k-1}^*(G))$. For ease of notation we define $Q := \delta_{k-1}^*(G)$, and we write K for $\delta_k^*(G)$.

The subgroup $C_G(X)$ has index at most $m!$ in G , so $|K/Z(K)| \leq m!$ and as in the proof of Theorem 1.1, we may assume $K' = 1$. Hence K is assumed to be abelian with Q metanilpotent. In what follows, we now restrict to the group Q .

It is sufficient to show that the order of each Sylow subgroup of K is m -bounded. We choose P a Sylow p -subgroup of K . By passing to the quotient $G/O_{p'}(G)$, we may assume $K = P$.

By Lemma 2.4, a Hall p' -subgroup H of Q satisfies $P = [P, H]$. By Lemma 2.6, since H is generated by its Sylow subgroups, we have that H is generated by a normal subset B of powers of δ_{k-1}^* -commutators that are of p' order.

Lemma 2.2 now implies that $[P, H]$ is generated by subgroups $[P, b_1, \dots, b_k]$ for $b_1, \dots, b_k \in B$. By Lemma 3.1, if $x \in P$, then $[x, b_1, \dots, b_k]$ is a δ_k^* -commutator, and we deduce that $[[P, b_1, \dots, b_k]]$ is m -bounded.

It follows that the number of generators of $[P, H]$ is at most m , and, furthermore, the exponent of $[P, H]$ is m -bounded. Hence, the finite abelian p -group $P = [P, H]$ has m -bounded order. \square

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CRISTINA ACCIARRI, Department of Mathematics, University of Brasilia,
Brasilia-DF, 70910-900, Brazil
e-mail: acciarri cristina@yahoo.it

PAVEL SHUMYATSKY, Department of Mathematics, University of Brasilia,
Brasilia-DF, 70910-900, Brazil
e-mail: pavel@unb.br

ANITHA THILLAISUNDARAM, Institut für Algebra und Geometrie,
Mathematische Fakultät, Otto-von-Guericke-Universität Magdeburg,
39016 Magdeburg, Germany
e-mail: anitha.t@cantab.net