

SOME COMPLETELY MONOTONIC FUNCTIONS INVOLVING THE GAMMA AND POLYGAMMA FUNCTIONS

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Abstract

The function $[\Gamma(x+1)]^{1/x}(1+1/x)^x/x$ is strictly logarithmically completely monotonic in $(0, \infty)$. The function $\psi''(x+2) + (1+x^2)/x^2(1+x)^2$ is strictly completely monotonic in $(0, \infty)$.

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1. Introduction

It is well known that the classical Euler gamma function $\Gamma(z)$ is defined for $\operatorname{Re} z > 0$ as

$$(1) \quad \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

The psi or digamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed for $x > 0$ and $k \in \mathbb{N}$ as

$$(2) \quad \psi(x) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{1+n} - \frac{1}{x+n} \right),$$

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$$(3) \quad \psi^{(k)}(x) = (-1)^{k+1} k! \sum_{i=0}^{\infty} \frac{1}{(x+i)^{k+1}},$$

$$(4) \quad \psi(x) = -\gamma + \int_0^{\infty} \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt,$$

$$(5) \quad \psi^{(k)}(x) = (-1)^{k+1} \int_0^{\infty} \frac{t^k e^{-xt}}{1 - e^{-t}} dt,$$

where $\gamma = 0.57721566490153286 \dots$ is the Euler-Mascheroni constant.

DEFINITION 1. A function f is said to be *completely monotonic* on an interval I if f has derivatives of all orders on I which alternate successively in sign, that is,

$$(6) \quad (-1)^n f^{(n)}(x) \geq 0$$

for $x \in I$ and $n \geq 0$. If inequality (6) is strict for all $x \in I$ and for all $n \geq 0$, then f is said to be strictly completely monotonic.

DEFINITION 2. A function f is said to be *logarithmically completely monotonic* on an interval I if its logarithm $\ln f$ satisfies

$$(7) \quad (-1)^k [\ln f(x)]^{(k)} \geq 0$$

for $k \in \mathbb{N}$ on I . If inequality (7) is strict for all $x \in I$ and for all $k \in \mathbb{N}$, then f is said to be strictly logarithmically completely monotonic.

The concepts of (logarithmically) completely monotonic function are defined on an arbitrary interval I here, but the main case is when $I = (0, \infty)$, where the completely monotonic functions are characterized by Bernstein's Theorem [8, page 161] as the Laplace transforms of positive measure μ in $(0, \infty)$. Bernstein's Theorem states that a function f is completely monotonic in $(0, \infty)$ if and only if

$$(8) \quad f(x) = \int_0^{\infty} e^{-xs} d\mu(s),$$

where $\mu(s)$ is a nonnegative measure, or say that $\mu(s)$ is nondecreasing, on $(0, \infty)$ such that the integral converges for all $x > 0$. Hence we conclude that a completely monotonic function which is non-identically zero cannot vanish at any point in $(0, \infty)$. It is clear that a completely monotonic function f in $(0, \infty)$ is strictly completely monotonic if and only if $\mu(s)$ has mass in the open interval $(0, \infty)$. Therefore the sharpenings with 'strict' in Definition 1 and Definition 2 are not very interesting.

To the best of our knowledge, the terminology or the notion 'logarithmically completely monotonic function' was explicitly introduced in [5, 6, 7] and it was also

proved in [5, 6] that a logarithmically completely monotonic function is completely monotonic. However, it cannot be said to be new, since in [2] this notion appears implicitly in Lemma 2.4 (ii) which can be rephrased as [5, Theorem 1] or [6, Theorem 4].

Completely monotonic functions have applications in many branches. For example, they play a role in potential theory, probability theory, physics, numerical and asymptotic analysis, and combinatorics. Some related references are listed in [1].

It is easy to prove that the function $(1 + 1/x)^{-x}$ is completely monotonic in $(0, \infty)$ through proving that it is logarithmically completely monotonic in $(0, \infty)$. A stronger result that the function $(1 + 1/x)^{-x}$ is a Stieltjes transform in $(0, \infty)$ follows from [1, Remark 3, page 457]. A function f is called a Stieltjes transform if it is of the form

$$(9) \quad f(x) = a + \int_0^\infty \frac{d\mu(s)}{s + x},$$

where $a \geq 0$ and μ is a nonnegative measure on $[0, \infty)$ satisfying

$$\int_0^\infty \frac{1}{1 + s} d\mu(s) < \infty.$$

From (9) we can see directly that a Stieltjes transform is a completely monotonic function.

Among other things, the following results were obtained in [6]: For $\alpha \leq 0$, the function $x^\alpha / [\Gamma(x + 1)]^{1/x}$ is strictly logarithmically completely monotonic in $(0, \infty)$. For $\alpha \geq 1$, the function $[\Gamma(x + 1)]^{1/x} / x^\alpha$ is strictly logarithmically completely monotonic in $(0, \infty)$. It should be noted that a similar but stronger result is contained in [2, Theorem 3.2]. The statement of [2] is that the function

$$\varphi(x) = \frac{1}{x[\Gamma(1 + 1/x)]^x}$$

is a Stieltjes transform and hence completely monotonic. However, it is well known (see, for example, [3, page 127]) that if $\varphi(x)$ is a Stieltjes transform, then so is $1/\varphi(1/x)$ and this is exactly the function $[\Gamma(x + 1)]^{1/x} / x$, which is then completely monotonic, since it is a Stieltjes transform.

In [4] the following two inequalities are presented: For $x \in (0, 1)$, we have

$$\frac{x}{[\Gamma(x + 1)]^{1/x}} < \left(1 + \frac{1}{x}\right)^x < \frac{x + 1}{[\Gamma(x + 1)]^{1/x}}.$$

For $x \geq 1$,

$$(10) \quad \left(1 + \frac{1}{x}\right)^x \geq \frac{x + 1}{[\Gamma(x + 1)]^{1/x}}.$$

Equality in (10) occurs for $x = 1$.

It is easy to obtain, using the standard argument, that

$$\lim_{x \rightarrow \infty} \frac{[\Gamma(x + 1)]^{1/x}}{x} \left(1 + \frac{1}{x}\right)^x = 1.$$

Out of curiosity, the (logarithmically) completely monotonic property of the quotient between two (logarithmically) completely monotonic functions (Stieltjes transforms) $[\Gamma(x + 1)]^{1/x}/x$ and $(1 + 1/x)^{-x}$ will be considered in this article. The main result of this consideration is

THEOREM 1.1. *The function $x^{-1}(\Gamma(x + 1))^{1/x}(1 + 1/x)^x$ is strictly logarithmically completely monotonic in $(0, \infty)$.*

As a direct consequence of the proof of Theorem 1.1, we have

COROLLARY 1.2. *The function*

$$\psi''(x) + \frac{x^4 + 5x^3 + 7x^2 + 7x + 2}{x^3(x + 1)^3} = \psi''(x + 2) + \frac{1 + x^2}{x^2(1 + x)^2}$$

is strictly completely monotonic in $(0, \infty)$.

2. Proof of Theorem 1.1

Define

$$(11) \quad F(x) = \frac{[\Gamma(x + 1)]^{1/x}}{x^c} \left(1 + \frac{a}{x}\right)^{x+b}$$

for $x > 0$ and some fixed real numbers a, b and c .

Taking the logarithm of $F(x)$ and differentiating yields

$$\ln F(x) = (x + b) \ln \left(1 + \frac{a}{x}\right) + \frac{\ln \Gamma(x + 1)}{x} - c \ln x,$$

$$[\ln F(x)]' = \ln \left(1 + \frac{a}{x}\right) - \frac{a(x + b)}{x(x + a)} + \frac{x\psi(x + 1) - \ln \Gamma(x + 1)}{x^2} - \frac{c}{x}, \quad \text{and}$$

$$\begin{aligned} [\ln F(x)]^{(n)} &= (-1)^{n-1}(n - 1)!(x + b) \left[\frac{1}{(x + a)^n} - \frac{1}{x^n} \right] \\ &\quad + (-1)^n(n - 2)!n \left[\frac{1}{(x + a)^{n-1}} - \frac{1}{x^{n-1}} \right] + \frac{h_n(x)}{x^{n+1}} + (-1)^n(n - 1)! \frac{c}{x^n} \\ &= (-1)^n(n - 2)! \left[\frac{(n - 1)(b + c) - x}{x^n} + \frac{x + na - (n - 1)b}{(x + a)^n} \right] + \frac{h_n(x)}{x^{n+1}}, \end{aligned}$$

where $n \geq 2$, $\psi^{(-1)}(x + 1) = \ln \Gamma(x + 1)$, $\psi^{(0)}(x + 1) = \psi(x + 1)$, and

$$h_n(x) = \sum_{k=0}^n \frac{(-1)^{n-k} n! x^k \psi^{(k-1)}(x + 1)}{k!},$$

$$h'_n(x) = x^n \psi^{(n)}(x + 1) \begin{cases} > 0, & \text{if } n \text{ is odd;} \\ < 0, & \text{if } n \text{ is even.} \end{cases}$$

Therefore, we have

$$\begin{aligned} & (-1)^n x^{n+1} [\ln F(x)]^{(n)} + (-1)^{n+1} h_n(x) \\ &= (n - 2)! \left\{ (n - 1)(b + c) - x + \frac{x^n [x + na - (n - 1)b]}{(x + a)^n} \right\} x \end{aligned}$$

and

$$\begin{aligned} & \frac{d \{ (-1)^n x^{n+1} [\ln F(x)]^{(n)} \}}{dx} \\ &= (-1)^n x^n \psi^{(n)}(x + 1) + (n - 2)! \left\{ (n - 1)(b + c) - 2x \right. \\ & \quad \left. + \frac{x^n [a(b + an + an^2 - bn^2) + (2a + b + 2an - bn)x + 2x^2]}{(x + a)^{n+1}} \right\} \\ &= x^n \left\{ (-1)^n \psi^{(n)}(x + 1) + (n - 2)! \left[\frac{(n - 1)(b + c) - 2x}{x^n} \right. \right. \\ & \quad \left. \left. + \frac{a(b + an + an^2 - bn^2) + (2a + b + 2an - bn)x + 2x^2}{(x + a)^{n+1}} \right] \right\} \\ &= x^n \left\{ (-1)^n \psi^{(n)}(x) + \frac{n!}{x^{n+1}} + (n - 2)! \left[\frac{(n - 1)(b + c) - 2x}{x^n} \right. \right. \\ & \quad \left. \left. + \frac{a(b + an + an^2 - bn^2) + (2a + b + 2an - bn)x + 2x^2}{(x + a)^{n+1}} \right] \right\}. \end{aligned}$$

By letting $a = c = 1$ and $b = 0$, we have

$$\begin{aligned} & \frac{d \{ (-1)^n x^{n+1} [\ln F(x)]^{(n)} \}}{dx} \\ &= x^n \left\{ (-1)^n \psi^{(n)}(x) + \frac{n!}{x^{n+1}} \right. \\ & \quad \left. + (n - 2)! \left[\frac{n - 1 - 2x}{x^n} + \frac{n(n + 1) + 2(n + 1)x + 2x^2}{(x + 1)^{n+1}} \right] \right\} \\ &= x^n \left\{ (-1)^n \psi^{(n)}(x) + (n - 2)! \left[\frac{n(n - 1) + (n - 1)x - 2x^2}{x^{n+1}} \right. \right. \\ & \quad \left. \left. + \frac{n(n + 1) + 2(n + 1)x + 2x^2}{(x + 1)^{n+1}} \right] \right\} \end{aligned}$$

$$\triangleq x^n \{(-1)^n \psi^{(n)}(x) + (n - 2)!g_n(x) + (n - 2)!h_n(x)\}.$$

By induction, it follows that $g'_n(x) = -(n - 1)g_{n+1}(x)$ and $h'_n(x) = -(n - 1)h_{n+1}(x)$. This implies $g_2^{(n-2)}(x) = (-1)^n (n - 2)!g_n(x)$ and $h_2^{(n-2)}(x) = (-1)^n (n - 2)!h_n(x)$. Therefore,

$$\frac{d \{(-1)^n x^{n+1} [\ln F(x)]^{(n)}\}}{dx} = (-1)^n x^n [\psi''(x) + g_2(x) + h_2(x)]^{(n-2)}.$$

It is a well-known fact that, for $x > 0$ and $r > 0$,

$$(12) \quad \frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-xt} dt.$$

From formulae (3), (5) and (12), for $x \in (0, \infty)$ and any nonnegative integer i , we have

$$\begin{aligned} \phi(x) &\triangleq \psi''(x) + g_2(x) + h_2(x) = \psi''(x) + \frac{2 + x - 2x^2}{x^3} + \frac{2(3 + 3x + x^2)}{(x + 1)^3} \\ &= \psi''(x) + \frac{x^4 + 5x^3 + 7x^2 + 7x + 2}{x^3(x + 1)^3} \\ &= \psi''(x) + \frac{2}{x^3} + \frac{1}{x^2} - \frac{2}{x} + \frac{2}{(1 + x)^3} + \frac{2}{(1 + x)^2} + \frac{2}{1 + x} \\ &= \frac{1}{x^2} - \frac{2}{x} + \frac{2}{(1 + x)^2} + \frac{2}{1 + x} - 2 \sum_{i=2}^\infty \frac{1}{(x + i)^3} \\ &= \psi''(x + 2) + \frac{1}{x^2} - \frac{2}{x} + \frac{2}{(1 + x)^2} + \frac{2}{1 + x} = \psi''(x + 2) + \frac{1 + x^2}{x^2(1 + x)^2} \\ &= \int_0^\infty t e^{-xt} dt - 2 \int_0^\infty e^{-xt} dt + 2 \int_0^\infty t e^{-(x+1)t} dt \\ &\quad + 2 \int_0^\infty e^{-(x+1)t} dt - \int_0^\infty \frac{t^2 e^{-(x+2)t}}{1 - e^{-t}} dt \\ &= \int_0^\infty [t - 2 + (t + 4)e^{-t} - (t^2 + 2t + 2)e^{-2t}] \frac{e^{-xt}}{1 - e^{-t}} dt \triangleq \int_0^\infty \frac{q(t)e^{-xt}}{1 - e^{-t}} dt, \\ \phi^{(i)}(x) &= (-1)^i \int_0^\infty q(t) \frac{t^i e^{-xt}}{1 - e^{-t}} dt, \end{aligned}$$

and

$$\begin{aligned} q'(t) &= (2 + 2t + 2t^2 - 3e^t + e^{2t} - te^t)e^{-2t} \triangleq p(t)e^{-2t}, \\ p'(t) &= 2 + 4t - 4e^t + 2e^{2t} - te^t, \quad p''(t) = 4 - 5e^t + 4e^{2t} - te^t, \\ p'''(t) &= (8e^t - t - 6)e^t > 0. \end{aligned}$$

Hence, $p''(t)$ increases in $(0, \infty)$. Since $p''(0) = 3 > 0$, we have $p''(t) > 0$ and $p'(t)$ is increasing. Because $p'(0) = 0$, it follows that $p'(t) > 0$ in $(0, \infty)$, and then $p(t)$ is increasing. From $p(0) = 0$, it is deduced that $p(t) > 0$ and $q'(t) > 0$ in $(0, \infty)$, then $q(t)$ increases. As a result of $q(0) = 0$, we obtain $q(t) > 0$ in $(0, \infty)$. Therefore, we have $\phi(x) > 0$ in $(0, \infty)$, and then for all nonnegative integer i , we have $(-1)^i \phi^{(i)}(x) > 0$ in $(0, \infty)$. This means that the function $\psi''(x) + g_2(x) + h_2(x)$ is strictly completely monotonic in $(0, \infty)$.

Thus the function $(-1)^n x^{n+1} [\ln F(x)]^{(n)}$ is increasing in $x \in (0, \infty)$. Since

$$\lim_{x \rightarrow 0} \{(-1)^n x^{n+1} [\ln F(x)]^{(n)}\} = 0,$$

we have $(-1)^n x^{n+1} [\ln F(x)]^{(n)} > 0$, then $(-1)^n [\ln F(x)]^{(n)} > 0$ for $n \geq 2$ in $(0, \infty)$. Since $[\ln F(x)]'' > 0$, the function $[\ln F(x)]'$ is increasing. It is not difficult to obtain $\lim_{x \rightarrow \infty} [\ln F(x)]' = 0$, so $[\ln F(x)]' < 0$ and $\ln F(x)$ is decreasing in $(0, \infty)$. In conclusion, the function $\ln F(x)$ is strictly completely monotonic in $(0, \infty)$. The proof is complete. \square

3. An open problem

We would like to pose the following open problem:

OPEN PROBLEM. *Under what conditions on a , b and c is the function $F(x)$ defined by (11) completely monotonic, or logarithmically completely monotonic, or a Stieltjes transform on $(0, \infty)$?*

In some subsequent papers, we will discuss the above open problem and publish its solutions.

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