BULL. AUSTRAL. MATH. SOC. 53C40, 53A07,
VOL. 35 (1987) 161-186. 58C25

SUBMANIFOLDS WITH FINITE TYPE GAUSS MAP

BANG-YEN CHEN AND PAOLO PICCINNI

In this paper we study the following problem: To what extent does the type of the Gauss map of a submanifold of E^m determine the submanifold? Several results in this respect are obtained. In particular, submanifolds with 1-type Gauss map are characterized. Surfaces with 1-type Gauss map and minimal surfaces of S^{m-1} with 2-type Gauss map are completely classified. Some applications are also given.

1. Introduction.

A compact submanifold M of a Euclidean m-space E^m is said to be of finite type if the immersion x of M in E^m can be expressed as a finite sum of E^m-valued eigenfuctions of the Laplacian Δ of M, acting on E^m-valued functions. Minimal submanifolds of a hypersphere

---

Received 12 March 1986. This work was done while the first author was a C.N.R. visiting professor at the University of Rome. He would like to take this opportunity to express his thanks to the Consiglio Nazionale delle Ricerche of Italy for the invitation and financial support, and also to express his many thanks to his colleagues at Rome for their hospitality.

---

and equivariant immersions of a compact homogeneous space are the simplest and best known examples of finite type submanifolds (see [5,7]).

Similarly, a smooth map ϕ of a compact Riemannian manifold M into E^m is said to be of finite type if ϕ is a finite sum of E^m-valued eigenfunctions of Δ. Some fundamental results on finite type maps are given in [9].

For an isometric immersion $x : M \to E^m$ of a compact oriented n-dimensional Riemannian manifold M into E^m, the Gauss map $\nu : M \to G(n,m)$ of x is a smooth map which carries a point p in M into the oriented n-plane in E^m which is obtained from the parallel translation of the tangent space of M at p in E^m (where $G(n,m)$ is the Grassmannian consisting of all oriented n-planes through the origin of E^m). Since $G(n,m)$ is canonically imbedded in $\Lambda^n E^m = E^N$, $N = \binom{m}{n}$, the notion of finite-type Gauss map is naturally defined.

The main purpose of this paper is to study the following problem: *To what extent does the type of the Gauss map of a submanifold of* E^m *determine the submanifold?*

For closed curves in E^m, the type of a curve in E^m coincides with that of its Gauss map (Proposition 3.1). In contrast, for submanifolds of dimension ≥ 2, the two notions are different.

A well-known result of Takahashi says that a compact submanifold of E^m is of 1-type if and only if it is a minimal submanifold of a hypersphere. In Section 4 we study the following problem: Which submanifolds of E^m have 1-type Gauss map? In this respect, we obtain a chacterization theorem for submanifolds with 1-type Gauss map. This result is then applied to obtain some classification theorems of such submanifolds. In Section 5, we show that a standard isometric immersion of an ordinary 2-sphere has 2-type Gauss map if and only if it is not the first standard imbedding. The complete classification of flat minimal tori in S^{m-1} with 2-type Gauss map is given in Section 6. In the last section, we give the complete classification of minimal surfaces of S^{m-1}

with 2-type Gauss map (Theorem 7.1).

2. Preliminaries.

Let M be a compact Riemannian manifold and Δ the Laplacian of M acting on the space $C^\infty(M)$ of smooth functions. Then Δ has an infinite discrete sequence of eigenvalues:

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots \uparrow \infty .$$

For each $k = 0,1,2,\ldots,$ The eigenspace $V_k = \{f \in C^\infty(M) \mid \Delta f = \lambda_k f\}$ is finite-dimensional. With respect to the inner product $(f,g) = \int fg\, dV$ on $C^\infty(M)$, the decomposition $\Sigma_k V_k$ is orthogonal and dense in $C^\infty(M)$. Therefore, for any $f \in C^\infty(M)$, we have $f = f_0 + \Sigma_{t\geq 1} f_t$, where f_0 is a constant and f_t is the projection of f into V_t.

For any smooth map $\phi : M \to E^m$ of a Riemannian manifold M into the Euclidean m-space E^m, we can apply the above decomposition to the E^m-valued function ϕ:

$$\phi = \phi_0 + \sum_{t=1}^{\infty} \phi_t , \tag{2.1}$$

where ϕ_0 is a constant vector which is called the centre of gravity of ϕ. The map ϕ is said to be of finite type if there exist only finitely many nonzero terms in the decomposition (2.1). More precisely, ϕ is said to be of k-type if there exist exactly k nonzero ϕ_t's $(t \geq 1)$ in the decomposition.

If the map ϕ is an isometric immersion, then M is called a submanifold of finite type (or of k-type) if ϕ does.

The following result is known (see [5,7]).

THEOREM 2.1. *Let $x : M \to E^m$ be an isometric immersion of a compact Riemannian manifold M into E^m and let H be the mean curvature vector of M in E^m. Then we have*

(i) M is of finite type if and only if there is a nontrivial polynomial $Q(t)$ such that $Q(\Delta)H = 0$.

https://doi.org/10.1017/S0004972700013162 Published online by Cambridge University Press

(ii) If M is of finite type, there is a unique monic polynomial $P(t)$ of least degree with $P(\Delta)H = 0$.

(iii) If M is of finite type, then M is of k-type if and only if deg $P = k$.

The same results hold if H is replaced by $x - x_0$.

For smooth maps, we have the following result analogous to Theorem 2.1, whose proof is the same as that of Theorem 2.1.

THEOREM 2.2. *Let $\phi : M \to E^m$ be a smooth map from a compact Riemannian manifold M into E^n and let $\tau = div(d\phi)$ be the tension field of ϕ . Then we have*

(i) ϕ is of finite type if and only if there is a nontrivial polynomial $Q(t)$ such that $Q(\Delta)\tau = 0$.

(ii) If ϕ is of finite type, there is a unique monic polynomial $P(t)$ of least degree with $P(\Delta)\tau = 0$.

(iii) If ϕ is of finite type, then ϕ is of k-type if and only if deg $P = k$.

The same results hold if τ is replaced by $\phi - \phi_0$.

The unique monic polynomial P mentioned in Theorem 2.1 (respectively, in Theorem 2.2) is called the minimal polynomial of the finite type submanifold M (respectively, of the finite type map ϕ).

3. Gauss Map.

Let V be an oriented n-plane in E^m . Denote by $e_1,\ldots,e_n$ an oriented orthonormal basis of V . Then $e_1 \wedge \ldots \wedge e_n$ is a decomposable n-vector of norm 1 and $e_1 \wedge \ldots \wedge e_n$ gives the orientation on V . Conversely, for any decomposable n-vector of norm 1 , it determines a unique oriented n-plane in E^m . Consequently, if we denote by $G(n,m)$ the Grassmannian of the oriented n-planes in E^m , then $G(n,m)$ can be identified naturally with the decomposable n-vectors of norm 1 in the $\binom{m}{n}$-dimensional Euclidean space $\Lambda^n E^m = E^N$. Let S^{N-1} , $N = \binom{m}{n}$, be the unit hypersphere in $\Lambda^n E^m = E^N$ centred at 0 . Then $G(n,m)$ is an

https://doi.org/10.1017/S0004972700013162 Published online by Cambridge University Press

$n(m-n)$-dimensional submanifold of S^{N-1}. Thus, we have $G(n,m) \subset S^{N-1} \subset E^N = \Lambda^n E^m$.

Let $x : M \to E^m$ be an isometric immersion of a compact, oriented, n-dimensional Riemannian manifold M into E^m. For each vector X tangent to M, we identify X with its image under dx. Let $e_1, \ldots, e_n$ be an oriented orthonormal frame on M. Then the Gauss map

$$\nu : M \to G(n,m) \subset S^{N-1} \subset E^N = \Lambda^n E^m$$

is given by $\nu(p) = (e_1 \wedge \ldots \wedge e_n)(p)$.

LEMMA 3.1. *For a compact oriented submanifold M in E^m, the Gauss map $\nu : M \to E^N$ is mass-symmetric, that is, the centre of gravity ν_0 coincides with the centre of the hypersphere S^{N-1} (that is, the origin) in E^N.*

Proof. Let $x : M \to E^m$ be the isometric immersion and $e_1, \ldots, e_n$ an oriented orthonormal local frame on M. Denote by $\omega^1, \ldots, \omega^n$, the dual frame of $e_1, \ldots, e_n$. Then we have $dx = e_1\omega^1 + e_2\omega^2 + \ldots + e_n\omega^n$. By direct computation, we have

$$\overbrace{dx \wedge \ldots \wedge dx}^{n \text{ copies}} = n!(e_1 \wedge \ldots \wedge e_n)\omega^1 \wedge \ldots \wedge \omega^n = n!\, \nu\, dV .$$

Thus, we obtain

$$n! \int_M \nu\, dV = \int_M dx \wedge \ldots \wedge dx = \int_M d(x \wedge \overbrace{dx \wedge \ldots \wedge dx}^{n-1 \text{ copies}})$$

$$= 0 .$$

This shows that the centre of gravity $\nu_0 = \int \nu\, dV / \int dV = 0$. □

If M is a closed curve in E^m, we have

PROPOSITION 3.1. *The Gauss map ν of a closed curve C in E^m is of k-type if and only if C is of k-type in E^m.*

Proof. Let $x : C \to E^m$ be the isometric immersion, s the arc length and $e_1 = dx/ds$ the unit tangent vector. Then the Gauss map ν is given by $\nu = e_1 \in S^{m-1} = G(1,m) \subset \Lambda^1 E^m = E^m$. Assume C is of k-type and P is the minimal polynomial of C. Then we have $P(\Delta)(x - x_0) = 0$. Thus

$$P(\Delta)\nu = P(\Delta)\,\frac{d}{ds}\,(x - x_0) = \frac{d}{ds}\,P(\Delta)(x - x_0) = 0 .$$

Thus, by Theorem 2.2 and Lemma 3.1 we see that ν is of h-type with $h \le k$.

Now, if ν is of h-type with minimal polynomial $\bar{P}$, then we have $\bar{P}(\Delta)\nu = 0$. Since d/ds commutes with $\bar{P}(\Delta)$ and $\Delta = -d^2/ds^2$, we get $\bar{P}(\Delta)H = 0$, where $H = de_1/ds$. Therefore, by Theorem 2.1, C is of l-type with $l \le h$. Combining these results, we obtain $l = h = k$. □

In the remaining part of this section, we compute the first Laplacian $\Delta\nu$ of ν for later use.

Let $x : M \to E^m$ be an isometric immersion of an oriented, n-dimensional Riemannian manifold into E^m. We choose an oriented orthonormal local frame $e_1,\dots,e_n,e_{n+1},\dots,e_m$ on M such that $e_1,\dots,e_n$ are tangent to M and hence $e_{n+1},\dots,e_m$ are normal to M. We shall make use of the following convention on the ranges of indices:

$$1 \le i,j,k,\ \dots \le n\ ; \quad n + 1 \le r,s,t,\ \dots \le m .$$

Let ∇ and ∇' be the Levi-Civita connections on M and E^m respectively. Denote by $\omega^A{}_B$, $A, B = 1,\dots,m,$ the connection forms. Then we have

$$\nabla'_{e_i} e_j = \omega^k{}_j(e_i)e_k + h^r{}_{ij}\, e_r , \tag{3.1}$$

$$\nabla'_{e_i} e_r = -h^r{}_{ij} e_j + \omega^s{}_r(e_i)e_s , \quad D_{e_i} e_r = \omega^s{}_r(e_i)e_s , \tag{3.2}$$

where D is the normal connection and $h^r{}_{ij}$ the coefficients of the second fundamental form h. The Einstein convention is used for repeated indices.

By regarding ν as an E^N-valued function on M, we have

$$(3.3)\qquad e_i\nu = e_i(e_1 \wedge \dots \wedge e_n) = h^r_{ij}\, e_1 \wedge \dots \wedge \overset{j\text{th}}{e_r} \wedge \dots \wedge e_n .$$

Since

$$(3.4)\qquad \Delta\nu = -e_ie_i\nu + (\nabla_{e_i}e_i)\nu ,$$

by a direct computation we obtain

$$(3.5)\qquad \Delta\nu = -h^r_{ij,i}\, e_1 \wedge \dots \wedge \overset{j\text{th}}{e_r} \wedge \dots \wedge e_n$$

$$- h^r_{ij}h^s_{ik}\, e_1 \wedge \dots \wedge \overset{k\text{th}}{e_s} \wedge \dots \wedge \overset{j\text{th}}{e_r} \wedge \dots \wedge e_n$$

$$+ \|h\|^2 \nu ,$$

where $\|h\|^2 = h^r{}_{ij}\, h^r{}_{ij}$ and

$$(3.6)\qquad h^r_{jk,i} = e_ih^r_{jk} + h^t_{jk}\omega^r_t(e_i) - \omega^l_j(e_i)h^r_{lk} - \omega^l_k(e_i)h^r_{jl} .$$

By the Codazzi equation $h^r{}_{jk,i} = h^r{}_{ij,k}$, (3.5) yields

$$(3.7)\quad \Delta\nu = -n \sum_i e_1 \wedge \dots \wedge D_{e_i} H \wedge \dots \wedge e_n$$

$$- h^r_{ij}\, h^s_{ik}\, e_1 \wedge \dots \wedge \overset{k\text{th}}{e_s} \wedge \dots \wedge \overset{j\text{th}}{e_r} \wedge \dots \wedge e_n + \|h\|^2 \nu ,$$

where $H = (1/n)h^r_{ii}e_i$ is the mean curvature vector. We recall the following Ricci equation of M in E^m:

$$(3.8)\qquad R^D(e_j,e_k;e_r,e_s) = \langle [A_r,A_s]e_j,e_k\rangle = h^r_{ik}h^s_{ij} - h^r_{ij}h^s_{ik} ,$$

where R^D is the normal curvature tensor and A_r the Weingarten map at e_r. From (3.7) and (3.8) we obtain the following.

LEMMA 3.2. *Let $x : M \to E^m$ be an isometric immersion of an oriented n-dimensional Riemannian manifold M into E^m. Then the Laplacian of the Gauss map $\nu : M \to G(n,m) \subset \Lambda^n E^m$ is given by*

$$(3.9)\qquad \Delta\nu = -n\sum_i e_1 \wedge \dots \wedge D_{e_i}H \wedge \dots \wedge e_n$$

$$+ R^D(e_j,e_k;e_r,e_s)e_1 \wedge \dots \wedge \overset{k\text{th}}{e_s} \wedge \dots \wedge \overset{j\text{th}}{e_r} \wedge \dots \wedge e_n + \|h\|^2\nu$$

Since the first term of the right-hand side of (3.9) is the only term tangent to $G(n,m)$ and other two terms are normal to $G(n,m)$, Lemma 3.2 implies the following result of [13].

COROLLARY 3.1. (Ruh and Vilms [13]). *Let M be a submanifold of E^m. Then the map $\nu : M \to G(n,m)$ is harmonic if and only if M has parallel mean curvature vector in E^m.*

If we consider the map $\bar{\nu} = i \cdot \nu : M \to G(n,m) \longrightarrow S^{N-1}$ (i = the inclusion), then Lemma 3.2 gives

COROLLARY 3.2. *Let M be a submanifold of E^m. Then the map $\bar{\nu} : M \to S^{N-1}$ is harmonic if and only if M has flat normal connection and parallel mean curvature vector.*

COROLLARY 3.3. *Let M be a compact submanifold of E^m. If the map $\bar{\nu} : M \to S^{N-1}$ is harmonic, then all of the Pontrjagin classes and the Euler class of the normal bundle $T^\perp M$ vanish.*

4. Submanifolds with 1-type Gauss Map.

From Theorem 2.2 and Lemma 3.2 we have the following.

THEOREM 4.1. *Let $x : M \to E^m$ be an isometric immersion of a compact, oriented Riemannian manifold M into E^m. Then the Gauss map $\nu : M \to \Lambda^n E^m$ is of 1-type if and only if M has constant scalar curvature, flat normal connection and parallel mean curvature vector in E^m.*

Proof. From Theorem 2.2 and Lemma 3.2 we see that ν is of 1-type if and only if $DH = 0$, $R^D = 0$ and $\|h\|$ is a constant. From Gauss' equation, the scalar curvature τ of M satisfies $n(n-1)\tau = n^2|H|^2 - \|h\|^2$. Since $DH = 0$ implies the constancy of the mean curvature $|H|$, Theorem 4.1 follows. □

If M is a hypersurface of E^m , we have

THEOREM 4.2. *A compact hypersurface* M *of* E^{n+1} *has 1-type Gauss map* $\nu : M \to \Lambda^n E^{n+1}$ *if and only if* M *is a hypersphere in* E^{n+1} .

Proof. Let M be a hypersurface of E^{n+1} . Then M has flat normal connection. Thus, by Theorem 4.1, the Gauss map is of *1*-type if and only if M has constant mean curvature and constant scalar curvature. Since a compact hypersurface of E^{n+1} has constant mean curvature and constant scalar curvature if and only if M is a hypersphere (Corollary 6.1 of [7] which follows easily from Proposition 4.1 of [5, p. 271]), we conclude that ν is of *1*-type if and only if M is a hypersphere of E^{n+1} . □

If M is a compact hypersurface of a hypersphere S^{n+1} of E^{n+2} , then the normal connection of M in E^{n+2} is also flat. Thus, Theorem 4.1 implies that M has *1*-type Gauss map if and only if M has constant scalar curvature and constant mean curvature. Thus, by applying Theorem 2 of [6], we obtain the following.

THEOREM 4.3. *Let* M *be a compact hypersurface of a hypersphere* S^{n+1} *of* E^{n+2} . *Then* M *has 1-type Gauss map if and only if* M *is one of the following submanifolds:*

(a) A mass-symmetric 2-type submanifold of E^{n+2} ;

(b) A small hypersphere of S^{n+1} ;

(c) A minimal hypersurface of S^{n+1} *with constant scalar curvature.*

The following theorem classifies surfaces with *1*-type Gauss map completely.

THEOREM 4.4. *Let* M *be a compact surface in* E^m . *Then* M *has 1-type Gauss map if and only if* M *is one of the following surfaces:*

(a) A sphere $S^2(r) \subset E^3 \subset E^m$; *or*

(b) The product of two plane circles $S^1(a) \times S^1(b) \subset E^4 \subset E^m$.

Proof. By Theorem 4.1 we see that both $S^2(r)$ and $S^1(a) \times S^1(b)$ have *1*-type Gauss map.

Conversely, if M is a compact surface in E^m with *1*-type Gauss map, then we have (i) $DH = 0$, (ii) τ is constant and (iii) $R^D = 0$. Since M is compact, $H \neq 0$. Thus, Theorem 2.1 of [*4*, p. 106] shows that M is either a minimal surface of a hypersphere S^{m-1} of E^m, or it lies in $E^3 \subset E^m$ or in $S^3 \subset E^m$. If M is a minimal surface of S^{m-1}, then by $R^D = 0$, M lies in a $S^3 \subset E^4 \subset E^m$ (Remark 2.1 of [*4*, p. 115]). Consequently, M lies either in E^3 or in S^3. If M lies in E^3, Theorem 4.2 shows that M is a sphere $S^2(r) \subset E^3$. If M lies in $S^3 \subset E^4$, Theorem 4.3 shows that M is a sphere in E^3 or a minimal surface with constant Gauss curvature in S^3 or a *2*-type surface in $S^3 \subset E^4$. If M is a minimal surface of S^3 with constant Gauss curvature, then a result of [*12*] shows that M is the product of two plane circles of the same radius. If M is a *2*-type surface in $S^3 \subset E^4$, Theorem 2 of [*6*] shows that M is mass-symmetric in S^3. Thus a classification theorem of [*5*, p. 279] yields that M is the product of two plane circles of different radius. □

From Theorem 4.1, we obtain immediately the following.

COROLLARY 4.1. *Let $x : M \to E^m$ be an isometric immersion of a compact oriented Riemannian manifold M into E^m. If the Gauss map of x is of 1-type, then all of the Pontrjagin classes and the Euler class of the normal bundle vanish.*

Remark. In [*1*], Bleecker and Weiner had studied compact oriented submanifolds of E^m whose Gauss map satisfies $\Delta\nu = \lambda\nu$ for some constant λ. They obtained results such as Theorems 4.1, 4.2 and 4.4.

5. Surfaces with 2-type Gauss Map.

The main purpose of this and the next two sections is to classify minimal surfaces of S^{m-1} with *2*-type Gauss map. In order to do so, we

https://doi.org/10.1017/S0004972700013162 Published online by Cambridge University Press

need to compute $\Delta^2\nu$.

Let $x : M \to E^m$ be an isometric immersion of a compact oriented surface into E^m . Assume that M lies in the unit hypersphere S^{m-1} of E^m centred at the origin. Then the position vector x is a unit normal vector. In the following, we choose an oriented orthonormal local frame $e_1, e_2, e_3, \ldots, e_m$ in such a way that $e_m = x$. Then we have

$$h^m_{ij} = -\delta_{ij} \quad \text{and} \quad \omega^m_r = 0 \ . \tag{5.1}$$

In the following, we assume that M is a minimal surface of S^{m-1}. Then the first normal space $\operatorname{Im} h$ is of dimension ≤ 2 . Thus, we may also assume that e_3, e_4 lies in $\operatorname{Im} h$. Then, with respect to the local frame chosen above, we have

$$A_5 = \ldots = A_{m-1} = 0 \ , \quad A_m = -I \ . \tag{5.2}$$

Consequently, by Ricci's equation, we obtain

$$R^D(e_i, e_j; e_r, e_s) = 0 \quad \text{for} \quad r, s \neq 3, 4 \ . \tag{5.3}$$

Because $DH = 0$, Lemma 3.2 gives

$$\Delta\nu = 2K^D e_3 \wedge e_4 + \|h\|^2 \ e_1 \wedge e_2 \ , \tag{5.4}$$

where $K^D = R^D(e_1, e_2; e_3, e_4) = h^3{}_{2i}h^4{}_{1i} - h^3{}_{1i}h^4{}_{2i}$.

In the following, we shall make use of the following convention on the ranges of indices:

$$1 \leq i, j, k \leq 2 \ ; \quad 5 \leq \alpha, \beta, \gamma \leq m \ ; \quad 3 \leq r, s, t, \leq m \ .$$

By a straight-forward but lengthy computation, we may obtain

LEMMA 5.1. *Under the hypothesis, we have*

$$\Delta^2\nu = 2\{\Delta K^D + 2K^D(\|h\|^2 - 1) + \sum_\alpha (\|\omega^\alpha{}_3\|^2 + \|\omega^\alpha{}_4\|^2)\}e_3 \wedge e_4$$
$$+ \{\Delta\|h\|^2 + \|h\|^4 + 4(K^D)\}e_1 \wedge e_2$$

https://doi.org/10.1017/S0004972700013162 Published online by Cambridge University Press

$$- 2(e_i\|h\|^2)(h^r_{1i}e_r \wedge e_2 + h^r_{2i}e_1 \wedge e_r)$$

$$+ 4(e_iK^D)\{h^3_{ij}e_j \wedge e_4 + h^4_{ij}e_3 \wedge e_j - \omega^\alpha{}_3(e_i)e_\alpha \wedge e_4 - \omega^\alpha{}_4(e_i)e_3 \wedge e_\alpha\}$$

$$- 4K^D\{h^4_{ij}\omega^\alpha{}_3(e_i) - h^3_{ij}\omega^\alpha{}_4(e_i)\}e_j \wedge e_\alpha$$

$$-2K^D\{(\nabla_{e_i}\omega^\alpha{}_3)e_i + \omega^\beta{}_3(e_i)\omega^\alpha{}_\beta(e_i) - \omega^4{}_3(e_i)\omega^\alpha{}_4(e_i)\}e_\alpha \wedge e_4$$

$$- 2K^D\{(\nabla_{e_i}\omega^\alpha{}_4)e_i + \omega^\beta{}_4(e_i)\omega^\alpha{}_\beta(e_i) - \omega^3{}_4(e_i)\omega^\alpha{}_3(e_i)\}e_3 \wedge e_\alpha .$$

Now, we give some examples of compact minimal surfaces in $S^{m-1} \subset E^m$ with 2-type Gauss map. More examples will be given in Section 6.

The first example is given by Veronese surface in S^4. We recall the Veronese surface as follows (see [5, 10]).

Let (x,y,z) be the natural coordinate system in E^3 and (u^1,u^2,u^3,u^4,u^5) the natural coordinate system in E^5. We consider the mapping defined by

$$u^1 = \frac{1}{\sqrt{3}}yz, \quad u^2 = \frac{1}{\sqrt{3}}zx, \quad u^3 = \frac{1}{\sqrt{3}}xy, \quad u^4 = \frac{1}{2\sqrt{3}}(x^2 - y^2)$$
$$u^5 = \frac{1}{6}(x^2 + y^2 - 2z^2) . \tag{5.5}$$

This defines an isometric minimal immersion of $S^2(\sqrt{3})$ into $S^4 = S^4(1)$. Two points (x,y,z) and $(-x,-y,-z)$ of $S^2(\sqrt{3})$ are mapped into the same point of S^4 and this mapping defines an embedded real projective plane in S^4 which is called the Veronese surface. For the Veronese surface, we have

$$\|h\|^2 = \frac{10}{3}, \quad K^D = \frac{2}{3}. \tag{5.6}$$

Thus, Lemma 5.1 yields

$$\Delta^2\nu = -\frac{56}{9}e_3 \wedge e_4 + \frac{116}{9}e_1 \wedge e_2 . \tag{5.7}$$

From (5.4) we have

(5.8) $$\Delta\nu = -\frac{4}{3} e_3 \wedge e_4 + \frac{10}{3} e_1 \wedge e_2 .$$

Consequently, (5.7) and (5.8) give

(5.9) $$\Delta^2\nu - \frac{14}{3}\Delta\nu + \frac{8}{3}\nu = 0 .$$

Therefore, from (5.4) and (5.9) and Theorem 2.2, we may conclude that the second standard immersion $\psi_2 : S^2(\sqrt{3}) \to S^4 \subset E^5$ defined by (5.5) has 2-type Gauss map. Moreover, the order of the Gauss map is [1,3] (with $\lambda_1 = 2/3$ and $\lambda_3 = 4$).

In general, the k-th standard immersion ψ_k of a 2-sphere S^2 in S^{2k} can be defined as follows.

Let (θ,ϕ) denote the spherical coordinates of $S^2(r_k)$ of radius $r_k = (k(k+1)/2)^{1/2}$. Then the coordinates of $S^2(r_k)$ in E^3 are given by

(5.10) $$x = r_k \cos\phi, \quad y = r_k \sin\phi\cos\theta, \quad z = r_k \sin\phi\sin\theta .$$

In terms of (θ,ϕ) , the k-th standard immersion ψ_k of $S^2(r_k)$ into S^{2k} is given by

(5.11) $$\begin{cases} u^0 = (r_k/\sqrt{2}) \cdot B_k^0 \cdot P_k^0(\cos\phi) , \\ u^i = r_k \cdot B_k^i \cdot P_k^i(\cos\phi) \cdot \cos(i\theta) , \quad i = 1,\ldots,k , \\ u^{k+i} = r_k \cdot B_k^i \cdot P_k^i(\cos\phi) \cdot \sin(i\theta) , \end{cases}$$

where $(u^0,u^1,\ldots,u^{2k})$ is the Euclidean coordinate system of E^{2k+1}. Moreover,

(5.12) $$P_k^j(t) = (1-t^2)^{j/2}\frac{d^{k+j}}{dt^{k+j}}[(1-t^2)^k], \quad j = 0,1,\ldots,k,$$

are the Legendre functions and ${B^j}_k$ are defined by

https://doi.org/10.1017/S0004972700013162 Published online by Cambridge University Press

(5.13) $$B_k^j = \frac{1}{k!2^k}\left[\frac{(k-j)!(2k+1)}{(k+j)!2\pi}\right]^{1/2}, \quad j = 0,1,\dots,k .$$

It is well-known that the k-th standard immersion is an isometric minimal immersion of $S^2(r_k)$ into S^{2k}. If k is odd, it is an imbedding and if k is even, it is a two-to-one map.

THEOREM 5.1. *Let* $x : S^2(r) \to S^{m-1} \subset E^m$ *be a minimal isometric immersion of a 2-sphere* $S^2(r)$ *into* $S^{m-1} \subset E^m$. *If* x *is not totally geodesic, then it has 2-type Gauss map.*

Proof. Let $x : S^2(r) \to S^{m-1} \subset E^m$ be a minimal isometric immersion of $S^2(r)$ into S^{m-1}. Then, by a well-known result of Calabi [3], $r = r_k$ for some natural number k and the immersion x is the k-th standard immersion ψ_k of $S^2(r_k)$ into $S^{2k} \subset S^{m-1}$ (up to rigid motions of S^{m-1}). If $k = 1$, x is a totally geodesic immersion. Thus, we obtain $k \geq 2$ from hypothesis.

Since the k-th standard immersion $\psi_k : S^2(r_k) \to S^{2k} \subset S^{m-1} \subset E^m$ is isotropic (see Theorem 1 and Remark 1 of [8]), Lemma 3 of [8] implies that, with respect to a suitable orthonormal frame $e_1, e_2, e_3, \dots, e_m$ so that $e_m = x$, we have

(5.14) $$A_3 = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}, \quad A_5 = \dots = A_{m-1} = 0, \quad A_m = -I .$$

Since $e_m = x$, we have

(5.15) $$\omega^r{}_{2k+1} = 0, \quad r = 3,\dots,m .$$

Moreover, from (5.14) and equation of Gauss, we find

(5.16) $$c^2 = (k-1)(k+2)/2k(k+1) .$$

Let D denote the normal connection of $S^2(r_k)$ in E^m. Then, by (5.14), (5.15), (5.16), and Codazzi equation, we obtain

(5.17) $$D_{e_1} e_3 + 2\omega^2{}_1(e_1)e_4 = D_{e_2} e_4 - 2\omega^2{}_1(e_2)e_3 ,$$

(5.18) $$D_{e_2}e_3 + 2\omega^2{}_1(e_2)e_4 = -D_{e_1}e_4 + 2\omega^2{}_1(e_1)e_3 \, .$$

From (5.17) and (5.18) we get

(5.19) $$\omega^4{}_3 = -2\omega^2{}_1 \, ,$$

(5.20) $$\omega^\alpha{}_3(e_1) = \omega^\alpha{}_4(e_2), \quad \omega^\alpha{}_3(e_2) = -\omega^\alpha{}_4(e_1), \quad \alpha \geq 5 \, .$$

Moreover, from (5.14) and (5.16), we obtain

(5.21) $$\|h\|^2 = 4(k^2 + k - 1)/k(k + 1) \, ,$$

(5.22) $$K^D = (k - 1)(k + 2)/k(k + 1) \, .$$

Furthermore, (5.21) yields

(5.23) $$\sum_\alpha \|\omega^\alpha{}_3\|^2 = -2(\omega^3{}_\alpha \wedge \omega^\alpha{}_4)(e_1,e_2) \, .$$

On the other hand, by (5.14), (5.19) and structure equation, we have

(5.24) $$-\omega^3{}_\alpha \wedge \omega^\alpha{}_4 = (2K + K^D)\omega^1 \wedge \omega^2 \, ,$$

where $K = 2/k(k + 1)$. Combining (5.20), (5.23) and (5.24), we find

(5.25) $$\sum_\alpha \|\omega^\alpha{}_3\|^2 = \sum_\alpha \|\omega^\alpha{}_4\|^2 = 2K + K^D = (k^2 + k + 2)/k(k + 1) \, .$$

From (5.14) and (5.20), we also get

(5.26) $$h^4{}_{ij}\omega^\alpha{}_3(e_i) = h^3{}_{ij}\omega^\alpha{}_4(e_i) \quad \text{for} \quad j = 1,2 \, .$$

From the structure equations, we obtain

(5.27) $$(d\omega^\alpha{}_4)(e_1,e_2) = -(\omega^\alpha{}_3 \wedge \omega^3{}_4)(e_1,e_2) - (\omega^\alpha{}_\beta \wedge \omega^\beta{}_4)(e_1,e_2) \, ,$$

(5.28) $$(d\omega^\alpha{}_3)(e_1,e_2) = -(\omega^\alpha{}_4 \wedge \omega^4{}_3)(e_1,e_2) - (\omega^\alpha{}_\beta \wedge \omega^\beta{}_3)(e_1,e_2) \, .$$

Thus, by using (5.20), (5.27) and (5.28) we give

(5.29) $$(\nabla_{e_i}\omega^\alpha{}_3)e_i = \omega^4{}_3(e_i)\omega^\alpha{}_4(e_i) - \omega^\beta{}_3(e_i)\omega^\alpha{}_\beta(e_i) \, ,$$

(5.30) $$(\nabla_{e_i}\omega^\alpha{}_4)e_i = \omega^3{}_4(e_i)\omega^\alpha{}_3(e_i) - \omega^\beta{}_4(e_i)\omega^\alpha{}_\beta(e_i) \, .$$

https://doi.org/10.1017/S0004972700013162 Published online by Cambridge University Press

Consequently, (5.21), (5.22), (5.25), (5.26), (5.29), (5.30) and Lemma 5.1 yield

$$(5.31)\qquad \Delta^2\nu = 4\{K^D \|h\|^2 + 2K\}e_3 \wedge e_4 + \{\|h\|^4 + 4(K^D)^2\}e_1 \wedge e_2 .$$

Since $\|h\|$, K and K^D are constant, (5.4), (5.31) and Theorems 2.2 and 4.4 imply that the Gauss map ν is of 2-type. □

From the proof of Theorem 5.1 we have the following.

COROLLARY 5.1. *Let $x : M \to S^{m-1} \subset E^m$ be a minimal isometric immersion of a compact oriented surface M into S^{m-1}. If M is constant isotropic in S^{m-1} (or in E^m), then the Gauss map of x is of either 1- or 2-type.*

6. Classification of Minimal Tori with 2-type Gauss Map.

Let (n,k,m) be a triple of integers with $n,k > 0$. Let Λ be the lattice generated by

$$(6.1)\qquad \{(0,\ 2\sqrt{2/3}\, n\pi),\ (\sqrt{2}\, k\pi,\ \sqrt{2/3}\,(2m - k)\pi)\} .$$

Consider the map $\bar{y}_{(n,k,m)} : \mathbb{R}^2 \to E^6$ defined by

$$(6.2)\qquad \bar{y}_{(n,k,m)}(s,t) = \frac{1}{\sqrt{3}}\Big(\cos \frac{1}{\sqrt{2}}(s + \sqrt{3}\, t),\ \sin \frac{1}{\sqrt{2}}(s + \sqrt{3}\, t),\ \cos \frac{1}{\sqrt{2}}(-s + \sqrt{3}\, t),\ \sin \frac{1}{\sqrt{2}}(-s + \sqrt{3}\, t),\ \cos\sqrt{2}\, s,\ \sin\sqrt{2}\, s\Big) .$$

Then $\bar{y}_{(n,k,m)}$ is an isometric immersion and it induces a minimal isometric immersion of the flat torus $T_{(n,k,m)} = \mathbb{R}^2/\Lambda$ into $S^5 \subset E^6$ which is denoted by $y_{(n,k,m)}$ so we have

$$(6.3)\qquad y_{(n,k,m)} : T_{(n,k,m)} \to S^5 \subset E^6 .$$

The following result completely classifies minimal flat tori in S^{m-1} with 2-type Gauss map.

https://doi.org/10.1017/S0004972700013162 Published online by Cambridge University Press

THEOREM 6.1. *(a) For any triple* (n,k,m) *of integers with* $n,k > 0$, *the minimal isometric immersion (6.3) has 2-type Gauss map.*

(b) Let $y : T^2 \to S^{m-1} \subset E^m$ *be an isometric minimal immersion of a flat torus* T^2 *into* S^{m-1} . *If the Gauss map of* y *is of 2-type, then*

(b.1) T^2 *is isometric to the flat torus* $T_{(n,k,m)}$ *for some natural numbers* k *and* n *and integer* m ;

(b.2) T^2 *is immersed fully in a totally geodesic 5-sphere* S^5 *of* S^{m-1} ; *and*

(b.3) up to rigid motions, y *is given by the composition* i . $y_{(n,k,m)} : T^2 \to S^5 \to S^{m-1} \subset E^m$, *where* i *is the inclusion.*

Proof. (a) Let $y_{(n,k,m)}$ be the isometric immersion of $T_{(n,k,m)}$ given by (6.3), induced from (6.2). Then, by a direct computation, we have $\Delta y_{(n,k,m)} = 2y_{(n,k,m)}$. Thus, by a result of Takahashi, $y_{(n,k,m)}$ is a minimal immersion. Since the Gauss map is given by $\nu = \partial/\partial s \wedge \partial/\partial t$, a straight-forward computation yields

$$\Delta^2\nu - 8\Delta\nu + 12\nu = 0 \ . \tag{6.4}$$

From Theorem 4.4, we know that ν is not of 1-type. Thus, Theorem 2.2 implies that the Gauss map is of 2-type.

(b) Let $y : T^2 \to S^{m-1} \subset E^m$ be an isometric minimal immersion of a flat torus T^2 into S^{m-1} such that the Gauss map of y is of 2-type. Assume that $T^2 = \mathbf{R}^2/\Lambda$, where Λ is a lattice in $\mathbf{R}^2$ which defines the flat torus T^2 . Without loss of generality, we may assume that Λ is given by

$$\Lambda = \{(2h\pi u,\ 2m\pi v + 2h\pi w) \mid h,m \in \mathbf{Z}\} \ , \tag{6.5}$$

where u, v, w are real numbers with $u, v > 0$. The dual lattice of Λ is given by

$$\Lambda^* = \{(\frac{k}{2\pi u} - \frac{nw}{2\pi uv} \ , \ \frac{n}{2\pi v}) \mid k,n \in \mathbf{Z}\} \ . \tag{6.6}$$

It is known that the spectrum of $T^2 = \mathbf{R}^2/\Lambda$ is given by

$$(6.7)\qquad \{(\frac{k}{u} - \frac{mw}{uv})^2 + (\frac{n}{v})^2 \mid k,n \in \mathbf{Z}\}\ .$$

The eigenspace $V(\lambda)$ of Δ with eigenvalue λ is given by

$$(6.8)\qquad \mathrm{Span}\{\cos(\frac{\varepsilon s}{u} + \frac{nt}{v})^2,\ \sin(\frac{\varepsilon s}{u} + \frac{nt}{v})^2 \mid (\frac{\varepsilon}{u})^2 + (\frac{n}{v})^2 = \lambda\}\ ,$$

where $\varepsilon = k - \frac{mw}{v}$.

Since $y : T^2 \to S^{m-1} \subset E^m$ is minimal, $\Delta y = 2y$. Thus, every coordinate function of y is an eigenfunction of Δ with eigenvalue 2. We put

$$(6.9)\qquad P = \left\{(\varepsilon_i, n_i) \;\middle|\; \left(\frac{\varepsilon_i}{u}\right)^2 + \left(\frac{n_i}{v}\right)^2 = 2\right\}\ ,$$

where $\varepsilon_i = k_i - n_i w/v$ and $k_i,\ n_i \in \mathbf{Z}$. Let $\#P = l$ ($\#P$ denotes the cardinal number of P). For simplicity, we may assume $P = \{(\varepsilon_i, n_i) \mid i \in I_l\}$, when $I_l = \{1,2,\dots,l\}$. Then the isometric immersion y may assume to be of the following form:

$$(6.10)\qquad y(s,t) = (\mu_i \cos(\bar{\varepsilon}_i s + \bar{n}_i t),\ \mu_i \sin(\bar{\varepsilon}_i s + \bar{n}_i t))_{i \in I}\ ,$$

where I is a subset of I_l, μ_i are positive constants and

$$(6.11)\qquad \bar{\varepsilon}_i = \varepsilon_i/u,\quad \bar{n}_i = n_i/v,\quad \bar{\varepsilon}_i^2 + \bar{n}_i^2 = 2\ .$$

If $\#I = 2$, then T^2 is a minimal flat torus in S^3. Thus, by a result of [12], T^2 is immersed as a Clifford torus. Thus, by Theorem 4.4, y has 1-type Gauss map which is a contradiction. Thus, we obtain $\#I \geq 3$. Since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$, without loss of generality we may put

$$(6.12)\qquad \bar{n}_i \geq 0 \quad \text{for} \quad i \in I\ .$$

Since y is an isometric immersion of T^2 into S^{m-1}, we have

$$(6.13)\qquad \sum \mu_i^2 = 1\ ,$$

(6.14) $$\sum \mu_i^2 \bar{\varepsilon}_i^2 = \sum \mu_i^2 \bar{n}_i^2 = 1 ,$$

(6.15) $$\sum \mu_i^2 \bar{k}_i \bar{n}_i = 0 .$$

By applying (6.10) we see that the nonzero coordinates of the Gauss map $\nu : T^2 \to \Lambda^2 E^m = E^{m(m-1)/2}$ are given by

(6.16) $$\begin{aligned} \nu(s,t) = (\mu_{ij}(&-\cos((\bar{\varepsilon}_i + \bar{\varepsilon}_j)s + (\bar{n}_i + \bar{n}_j)t) + \\ &+ \cos((\bar{\varepsilon}_i - \bar{\varepsilon}_j)s + (\bar{n}_i - \bar{n}_j)t)) , \\ -\mu_{ij}(&\sin((\bar{\varepsilon}_i + \bar{\varepsilon}_j)s + (\bar{n}_i + \bar{n}_j)t) + \\ &+ \sin((\bar{\varepsilon}_i - \bar{\varepsilon}_j)s + (\bar{n}_i - \bar{n}_j)t)) , \\ -\mu_{ij}(&\sin((\bar{\varepsilon}_i + \bar{\varepsilon}_j)s + (\bar{n}_i + \bar{n}_j)t) - \\ &- \sin((\bar{\varepsilon}_i - \bar{\varepsilon}_j)s + (\bar{n}_i - \bar{n}_j)t)) , \\ \mu_{ij}(&\cos((\bar{\varepsilon}_i + \bar{\varepsilon}_j)s + (\bar{n}_i + \bar{n}_j)t) + \\ &+ \cos((\bar{\varepsilon}_i - \bar{\varepsilon}_j)s + (\bar{n}_i - \bar{n}_j)t)))_{i<j} , \end{aligned}$$

where

(6.17) $$\mu_{ij} = \frac{1}{2} \mu_i \mu_j (\bar{\varepsilon}_i \bar{n}_j - \bar{\varepsilon}_j \bar{n}_i) .$$

By direct computation, we find

(6.18) $$\begin{aligned} \Delta\nu = (\mu_{ij}(&-b_{ij} \cos((\bar{\varepsilon}_i + \bar{\varepsilon}_j)s + (\bar{n}_i + \bar{n}_j)t) + \\ &+ c_{ij} \cos((\bar{\varepsilon}_i - \bar{\varepsilon}_j)s + (\bar{n}_i - \bar{n}_j)t)) , \\ -\mu_{ij}(&b_{ij} \sin((\bar{\varepsilon}_i + \bar{\varepsilon}_j)s + (\bar{n}_i + \bar{n}_j)t) + \\ &+ c_{ij} \sin((\bar{\varepsilon}_i - \bar{\varepsilon}_j)s + (\bar{n}_i - \bar{n}_j)t)) , \\ -\mu_{ij}(&b_{ij} \sin((\bar{\varepsilon}_i + \bar{\varepsilon}_j)s + (\bar{n}_i + \bar{n}_j)t) - \\ &- c_{ij} \sin((\bar{\varepsilon}_i - \bar{\varepsilon}_j)s + (\bar{n}_i - \bar{n}_j)t)) , \\ \mu_{ij}(&b_{ij} \cos((\bar{\varepsilon}_i + \bar{\varepsilon}_j)s + (\bar{n}_i + \bar{n}_j)t) + \\ &+ c_{ij} \cos((\bar{\varepsilon}_i - \bar{\varepsilon}_j)s + (\bar{n}_i - \bar{n}_j)t)))_{i<j} , \end{aligned}$$

(6.19) $$\Delta^2\nu = (\mu_{ij}(-b_{ij}^2 \cos((\bar\varepsilon_i + \bar\varepsilon_j)s + (\bar n_i + \bar n_j)t) + c_{ij}^2 \cos((\bar\varepsilon_i - \bar\varepsilon_j)s + (\bar n_i - \bar n_j)t)),$$
$$- \mu_{ij}(b_{ij}^2 \sin((\bar\varepsilon_i + \bar\varepsilon_j)s + (\bar n_i + \bar n_j)t) + c_{ij}^2 \sin((\bar\varepsilon_i - \bar\varepsilon_j)s + (\bar n_i - \bar n_j)t)),$$
$$- \mu_{ij}(b_{ij}^2 \sin((\bar\varepsilon_i + \bar\varepsilon_j)s + (\bar n_i + \bar n_j)t) - c_{ij}^2 \sin((\bar\varepsilon_i - \bar\varepsilon_j)s + (\bar n_i - \bar n_j)t)),$$
$$\mu_{ij}(b_{ij}^2 \cos((\bar\varepsilon_i + \bar\varepsilon_j)s + (\bar n_i + \bar n_j)t)) + c_{ij}^2 \cos((\bar\varepsilon_i - \bar\varepsilon_j)s + (\bar n_i - \bar n_j)t)))_{i<j},$$

where

(6.20) $$b_{ij} = 4 + 2\bar\varepsilon_i\bar\varepsilon_j + 2\bar n_i \bar n_j ,$$

(6.21) $$c_{ij} = 4 - 2\bar\varepsilon_i\bar\varepsilon_j - 2\bar n_i \bar n_j .$$

If $b_{ij} = c_{ij}$ for all $i < j$, then

(6.22) $$\bar\varepsilon_i\bar\varepsilon_j = -\bar n_i \bar n_j , \quad i < j .$$

This implies that either $\bar\varepsilon_j = c\bar n_j$ for all $j \in I$ or $\bar n_j = c\bar\varepsilon_j$ for all $j \in I$. Thus, by (6.9), we obtain

(6.23) $$\bar n_j^2 = 2/(1 + c^2) \quad \text{or} \quad \bar\varepsilon_j^2 = 2/(1 + c^2) \quad \text{for} \quad j \in I .$$

This gives $\#I \le 2$ which contradicts the assumption. Consequently, there is a pair (i,j) $(i < j)$ such that $b_{ij} \ne c_{ij}$. Without loss of generality, we may assume that $b_{12} \ne c_{12}$. This is equivalent to

(6.24) $$\bar\varepsilon_1\bar\varepsilon_2 \ne -\bar n_1\bar n_2.$$

Thus, from (6.16), (6.18), (6.19) and Theorem 2.2, we find

(6.25) $$\{b_{ij}, c_{ij} \mid i,j \in I,\ i < j\} = \{b_{12}, c_{12}\} .$$

We put

https://doi.org/10.1017/S0004972700013162 Published online by Cambridge University Press

(6.26) $$\beta_{ij} = \frac{1}{2} b_{ij} - 2 , \quad \gamma_{ij} = -\beta_{ij} .$$

From (6.9) and (6.12) we have

(6.27) $$\bar{n}_j = (2 - \bar{\varepsilon}_j^2)^{\frac{1}{2}} .$$

If $b_{1j} = b_{12}$, then (6.20), (6.26) and (6.27) give

(6.28) $$\bar{\varepsilon}_j = \frac{1}{2} \left(\beta_{12}\bar{\varepsilon}_1 \pm \bar{n}_1 \sqrt{4 - \beta_{12}^2} \right) .$$

If $b_{1j} = c_{12}$, we have

(6.29) $$\bar{\varepsilon}_j = - \frac{1}{2} \left(\beta_{12}\bar{\varepsilon}_1 \pm \bar{n}_1 \sqrt{4 - \beta_{12}^2} \right) .$$

From (6.27), (6.28) and (6.29) we get $\#I \leq 5$.

If $\beta_{12}^2 = 4$, then $\#I = 2$, which is impossible. Therefore, we have $\beta_{12}^2 < 4$. This condition is equivalent to the condition $\bar{\varepsilon}_1\bar{n}_2 \neq \bar{\varepsilon}_2\bar{n}_1$. Without loss of generality, we may assume

(6.30) $$\bar{\varepsilon}_1\bar{n}_2 < \bar{\varepsilon}_2\bar{n}_1 .$$

From (6.20), (6.26) and (6.30) we find

(6.31) $$\bar{\varepsilon}_2 = \frac{1}{2} \left(\beta_{12}\bar{\varepsilon}_1 + \bar{n}_1 \sqrt{4 - \beta_{12}^2} \right) .$$

If we put

(6.32) $$\bar{\varepsilon}_3 = \frac{1}{2} \left(\beta_{12}\bar{\varepsilon}_1 - \bar{n}_1 \sqrt{4 - \beta_{12}^2} \right) ,$$

then we have

(6.33) $$\{\bar{\varepsilon}_i\}_{i\in I} \subset \{\bar{\varepsilon}_1, \bar{\varepsilon}_2, -\bar{\varepsilon}_2, \bar{\varepsilon}_3, -\bar{\varepsilon}_3\} .$$

It is clear that $1, 2 \in I$. Moreover, we have $\#I \geq 3$.

If $\bar{\varepsilon}_3, -\bar{\varepsilon}_3 \notin \{\bar{\varepsilon}_i\}_{i\in I}$, then $\#I = 3$ and $\{\bar{\varepsilon}_i\}_{i\in I} = \{\bar{\varepsilon}_1, \bar{\varepsilon}_2, -\bar{\varepsilon}_2\}$ If $\bar{\varepsilon}_3$ or $-\bar{\varepsilon}_3$ belongs to $\{\bar{\varepsilon}_i\}_{i\in I}$, then by (6.9) and (6.32) we may find $|\bar{\varepsilon}_3| = |\bar{\varepsilon}_2|$. Consequently, we always have

https://doi.org/10.1017/S0004972700013162 Published online by Cambridge University Press

$$(6.34)\qquad \{\bar\varepsilon_i\}_{i\in I} = \{\bar\varepsilon_1, \bar\varepsilon_2, -\bar\varepsilon_2\}\ .$$

Without loss of generality, we may assume that $\bar\varepsilon_2 > 0$. Let us simply denote $\bar\varepsilon_2$ by $\bar\varepsilon$ and denote $\bar n_2$ by $\bar n$. Then from (6.34) we have

$$(6.35)\qquad \{(\bar\varepsilon_i, \bar n_i)\}_{i\in I} = \{(-\bar\varepsilon, \bar n), (\bar\varepsilon, \bar n), (\bar\varepsilon_1, \bar n_1)\},\quad \bar\varepsilon > 0,\quad \bar n > 0\ .$$

If we apply our argument of deriving (6.31) to (6.35), we find

$$(6.36)\qquad \bar\varepsilon_1 = \pm\, \bar\varepsilon(3 - 2\bar\varepsilon^2)\ ,\qquad \bar\varepsilon_1 = \pm\, \bar\varepsilon\ .$$

Therefore, by using (6.27), we get

$$(6.37)\qquad \bar n_1 = \{(2 - \bar\varepsilon^2)(1 - 2\bar\varepsilon^2)^2\}^{\frac{1}{2}}\ ,\qquad \bar n_1 \neq \bar n\ .$$

Therefore, (6.20), (6.21), (6.35), (6.36) and (6.37) yield

$$(6.38)\qquad \{b_{ij}, c_{ij}\}_{i<j} = \{4\bar\varepsilon^2,\ 4\bar n^2,\ (\bar\varepsilon + \bar\varepsilon_1)^2 + (\bar n + \bar n_1)^2\ ,$$
$$(\bar\varepsilon - \bar\varepsilon_1)^2 + (\bar n - \bar n_1)^2,\ (\bar\varepsilon - \bar\varepsilon_1)^2 + (\bar n + \bar n_1)^2,\ (\bar\varepsilon + \bar\varepsilon_1)^2 + (\bar n - \bar n_1)^2\}.$$

Since the Gauss map is of 2-type, $\#\{b_{ij}, c_{ij} \mid i < j\} = 2$. Thus, by (6.36), (6.37) and $\bar\varepsilon, \bar n > 0$, we obtain $\bar n_1 = 0$. Therefore, by (6.37), we obtain $\bar\varepsilon^2 = 2$ or $1/2$. If $\bar\varepsilon^2 = 2$, we obtain from (6.27) that $\bar n = 0$ which yields $\#I = 2$ by virtue of (6.35). Hence, we find

$$(6.39)\qquad \bar\varepsilon = \frac{\sqrt{2}}{2}\ ,\quad \bar\varepsilon_1 = \pm\ 2\bar\varepsilon = \pm\sqrt{2}\ ,\quad \bar n = \frac{\sqrt{6}}{2}\ .$$

Since $\bar n_1 = 0$, we may choose $\bar\varepsilon_1 = \sqrt{2}$. Consequently, we obtain

$$(6.40)\qquad \{(\bar\varepsilon_i, \bar n_i)\}_{i\in I} = \left\{\left(\frac{\sqrt{2}}{2}\ ,\ \frac{\sqrt{6}}{2}\right),\ \left(-\ \frac{\sqrt{2}}{2}\ ,\ \frac{\sqrt{6}}{2}\right),\ \left(\sqrt{2}, 0\right)\right\}\ .$$

Substituting (6.40) into (6.13), (6.14) and (6.15) we get $\mu_1^2 = \mu_2^2 = \mu_3^2 = 1/3$. Therefore, we find that the nonzero coordinates of $y : T^2 \to S^{m-1} \subset E^m$ are given by the following functions:

https://doi.org/10.1017/S0004972700013162 Published online by Cambridge University Press

$$y_1 = \frac{1}{\sqrt{3}} \cos \frac{1}{\sqrt{2}} (s + \sqrt{3}\, t) , \quad y_2 = \frac{1}{\sqrt{3}} \sin \frac{1}{\sqrt{2}} (s + \sqrt{3}\, t)$$

(6.41) $$y_3 = \frac{1}{\sqrt{3}} \cos \frac{1}{\sqrt{2}} (-s + \sqrt{3}\, t) , \quad y_4 = \frac{1}{\sqrt{3}} \sin \frac{1}{\sqrt{2}} (-s + \sqrt{3}\, t)$$

$$y_5 = \frac{1}{\sqrt{3}} \cos \sqrt{2}\, s , \qquad y_6 = \frac{1}{\sqrt{3}} \sin \sqrt{2}\, s .$$

Because $y_1, \dots, y_6$ are functions on $T^2 = \mathbf{R}^2/\Lambda$, they are invariant under the action of Λ . From this we see that $(hu + \sqrt{3}(mv + h\omega))/\sqrt{2}$, $(-hu + \sqrt{3}(mv + h\omega))/\sqrt{2}$ and $\sqrt{2}\, hu$ are integers for any integers h , m. In particular, we have

(6.42) $$u = k/\sqrt{2} , \quad v = \sqrt{2/3}\, n , \quad \omega = (2m - k)/\sqrt{6}$$

for some integer m and natural numbers k and n . Therefore, we find that the lattice Λ is generated by

(6.43) $$\{(0,\ 2\sqrt{2/3}\, n\pi) , \ (\sqrt{2}\, k\pi,\ \sqrt{2}(2m - k)\pi/\sqrt{3})\} .$$

It is easy to verify that the functions y_α are invariant under the action of Λ . Thus, we complete the proof of (b). □

7. Classification of Surfaces with 2-type Gauss Map.

We give the following.

THEOREM 7.1 (Classification). *Let* $x : M \to S^{m-1} \subset E^m$ *be a minimal isometric immersion of a compact oriented surface* M *into* S^{m-1} *. Then* x *has 2-type Gauss map if and only if either (1)* M *is a 2-sphere* $S^2(r_k)$ *with radius* $r_k = \sqrt{k(k+1)/2}$ *for some integer* $k \geq 2$ *and* x *is given by the k-th standard immersion* ψ_k *of* $S^2(r_k)$ *or (2)* M *is the flat torus* $T_{(n,k,h)} = \mathbf{R}^2/\Lambda$ *for some integers* n,k,h *with* $n,k > 0$*, where* Λ *is the lattice generated by*

(7.1) $$\{(0,\ 2\sqrt{2/3}\, n\pi),\ (\sqrt{2}\, k\pi,\ \sqrt{2/3}(2h - k)\pi)\} ,$$

and the immersion x *is induced from the isometric immersion* $\bar{x} : \mathbf{R}^2 \to S^5 \subset E^6 \subset E^m$ *defined by*

https://doi.org/10.1017/S0004972700013162 Published online by Cambridge University Press

$$(7.2)\quad \begin{aligned}\bar{x}(s,t) = \frac{1}{\sqrt{3}}\Big(&\cos\frac{1}{\sqrt{2}}(s+\sqrt{3}\,t),\ \sin\frac{1}{\sqrt{2}}(s+\sqrt{3}\,t),\\ &\cos\frac{1}{\sqrt{2}}(-s+\sqrt{3}\,t),\ \sin\frac{1}{\sqrt{2}}(-s+\sqrt{3}\,t),\ \cos\sqrt{2}s,\ \sin\sqrt{2}s,\ 0,\dots,0\Big),\end{aligned}$$

up to rigid motions of S^{m-1}.

Proof. Let $x : M \to S^{m-1} \subset E^m$ be a minimal isometric immersion of a compact oriented surface into S^{m-1}. If the Gauss map is of 2-type, then, by Theorem 2.2, there exist two constants b and c such that the Gauss map ν of x satisfies

$$(7.3)\qquad \Delta^2\nu + b\Delta\nu + c\nu = 0 .$$

By looking at $\nu = e_1 \wedge e_2$, at equation (5.4) and at Lemma 5.1, we find

$$(7.4)\qquad (e_i\,||h||^2)(h^m_{1i}e_m \wedge e_2 + h^m_{2i}e_1 \wedge e_m) = 0 .$$

Since $A_m = -I$, (7.4) implies that $||h||$ is constant. Similarly, by looking at the coefficients of $e_1 \wedge e_2$ of (5.4) and using Lemma 5.1 and (7.3) we obtain

$$(7.5)\qquad ||h||^4 + 4(K^D)^2 + b\,||h||^2 + c = 0 .$$

Because $||h||$, b and c are constant, (7.5) shows that K^D is also constant. If $K^D = 0$, then, by the constancy of $||h||$ and minimality of M in S^{m-1}, we conclude from Theorem 4.1 that the Gauss map is of 1-type which is a contradiction. Thus, K^D is a nonzero constant. Since M is minimal in S^{m-1} and $||h||$ is constant, M has constant Gauss curvature. Therefore, by applying a result of [2], we may conclude that M is either an ordinary 2-sphere $S^2(r)$ of radius r or a flat torus. If M is $S^2(r)$, we conclude from Theorem 4.4 and a result of [3] that $r = r_k = \sqrt{k(k+1)/2}$ for $k \geq 2$ and x is the k-th standard immersion ψ_k. If M is a flat torus, then we conclude from Theorem 6.1 that M is given by $\mathbb{R}^2/\Lambda$ for some lattice generated by (7.1) where n,k,h are integers with $n,k > 0$. Moreover, by Theorem 6.1, we also see that x is induced by the isometric immersion $\bar{x}$ of $\mathbf{R}^2$ into E^m defined by (7.2) up to rigid motions.

The converse of this was given in Theorems 5.1 and 6.1. □

References

[1] D. D. Bleecker and J. L. Weiner, "Extrinsic bounds on λ_1 of Δ on a compact manifold", *Comment Math. Helv.* 51 (1976), 601-609.

[2] R. L. Bryant, "Minimal surfaces of constant curvature in S^n", *Trans. Amer. Math. Soc.* 290 (1985), 259-271.

[3] E. Calabi, "Minimal immersions of surfaces in Euclidean spheres", *J. Differential Geometry* 1 (1967), 111-125.

[4] B. Y. Chen, *Geometry of Submanifolds*, Marcel Dekker, 1973, New York.

[5] B. Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, World Scientific, 1984.

[6] B. Y. Chen, "2-type submanifolds and their applications", *Chinese J. Math.* 14 (1986), no. 1, 1-14.

[7] B. Y. Chen, *Finite Type Submanifolds and Generalizations*, Quaderni del Seminario di Topologia Algebrica e Differenziale, Univ. di Roma, 1985, Rome.

[8] B. Y. Chen and P. Verheyen, "Submanifolds with geodesic normal sections", *Math. Ann.* 269 (1984), 417-429.

[9] B. Y. Chen, J.-M. Morvan and T. Nore, "Energie, tension et ordre des applications à valeurs dan un espace euclidien", *C.R. Acad. Sci. Paris* 301 (1985), 123-126.

[10] S. S. Chern, M. do Carmo and S. Kobayashi, "Minimal submanifolds of a sphere with second fundamental form of constant length", *Functional Analysis and Related Fields*, Springer-Verlag, 1970, 59-75.

[11] K. Kenmotsu, "On minimal immersions of $\mathbf{R}^2$ into S^N ", *J. Math. Soc. Japan* 28 (1976), 182-191.

[12] H. B. Lawson Jr., "Local rigidity theorems for minimal hypersurfaces", *Ann. of Math.*, (2) 89 (1969), 187-197.

[13] E. A. Ruh and J. Vilms, "The tension of the Gauss map", *Trans. Amer. Math. Soc.*, **149** (1970), 569-573.

Department of Mathematics
Michigan State University
East Lansing, Michigan 48824
U.S.A.

Dipartimento di Matematica
Università di Roma "La Sapienza"
00185 Rome, Italy