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ON PRESERVING THE KOBAYASHI PSEUDODISTANCE

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§ 0.

If X is a complex space, the Kobayashi pseudo-distance d_X is an intrinsic pseudometric on X defined as follows. If p and q are points of X, a chain α from p to q consists of intermediate points $p_0, \cdots, p_r$ with $p_0 = p$ and $p_r = q$ together with maps f_i of the unit disc $D = \{z \in C^1 \,|\, |z| < 1\}$ into X and points a_i and b_i in D such that $f_i(a_i) = p_{i-1}$ and $f_i(b_i) = p_i$ for $i = 1, \cdots, r$. If $d_D(a, b)$ denotes the hyperbolic distance between the points a and b in the unit disc, then the length of the chain α is defined as $|\alpha| = d_D(a_1, b_1) + d_D(a_2, b_2) + \cdots + d_D(a_r, b_r)$. The pseudo-distance between p and q is then defined as the infimum of the lengths of all chaini from p to q: $d_X = \inf \{|\alpha| \,|\, \alpha$ a chain from p to $q\}$. It is easy to establish that $d_X(p, q)$ is jointly continuous in p and q and that holomorphic maps are distance decreasing—i.e. if $f: X' \to X$ is holomorphic and $f(p') = p$, $f(q') = q$ then $d_X(p, q) \leq d_{X'}(p', q')$. If d_X is an actual distance—i.e. if $d_X(p, q) \neq 0$ for $p \neq q$—then X is said to be hyperbolic and in that case the metric topology induced by d_X coincides with the original topology of X ([1]). A general reference for this subject is Kobayashi's book [4].

If A is a closed subset of X, then the inclusion map $X - A \to X$ is holomorphic, so that $d_X(p, q) \leq d_{X-A}(p, q)$ for p and q not in A. Removing an analytic set of codimension 1 often changes the pseudo-distance radically. For instance, the pseudo-distance on $C^* = \{z \in C \,|\, z \neq 0\}$ is identically zero, but, if we remove a single point from C^*, what is left is a hyperbolic space. The same sort of phenomenon generally does not occur if A is an analytic set of codimension at least 2. For instance, Kobayashi proves ([4]) that if A is closed and nowhere dense in some hyperplane section of D^n (the unit polydisc in n-space), then removing

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A does not affect the distance between points not in A—i.e. $d_{D^n-A}(p, q) = d_{D^n}(p, q)$ for p and q not in A. The principal results of this paper (Proposition 2 and Theorems 1 and 2) are generalizations of that proposition. That is, theorems to the effect that the pseudo-distance is preserved if a "small" set (generally one of codimension 2) is removed. Such a result does not hold without some restriction on the space, as is shown by the following

EXAMPLE: Let Y be a hyperbolic projective algebraic manifold (for instance, a nonsingular curve of genus greater than 1) embedded in $\boldsymbol{P}^n$ (complex projective n-space). Let $\pi: \boldsymbol{C}^{n+1} - \{0\} \to \boldsymbol{P}^n$ be the map which takes a point to the line containing it, and let X be the cone over Y—i.e. $X = \pi^{-1}(Y) \cup \{0\}$. Then the pseudo-distance on X is identically zero, since X is a union of lines intersecting at the origin. Let $A = \{0\}$. The space $X - A$ has non-trivial pseudo-distance since $d_{X-A}(p, q) \geq d_Y(\pi(p), \pi(q)) > 0$ if p and q do not belong to the same line through the origin. Note that X is singular with singular locus A. By choosing Y to be of large dimension, we can make the codimension of A in X as large as we wish.

The methods we use to attack the problem are different from those used by Kobayashi, and essentially consist of showing that $\mathrm{Hol}(D, X - A)$ is dense in $\mathrm{Hol}(D, X)$ by using the flows of vector fields to push maps $D \to X$ away from A.

§1

For a complex space X we will denote by $\mathrm{Hol}(D, X)$ the set of holomorphic maps of D into X. Note that $\mathrm{Hol}(D, X)$ depends only on the reduction of X. We equip $\mathrm{Hol}(D, X)$ with the compact-open topology. We will use the notation $U \subset \subset X$ to indicate that $\overline{U}$ is a compact subset of X. We wish to consider the following three properties which a closed subset A of X may have:

I. The Kobayashi pseudo-distance on X restricts to that on $X - A$—i.e., $d_{X-A}(p, q) = d_X(p, q)$ for $p, q \in X - A$.

II. $\mathrm{Hol}(D, X - A)$ is dense in $\mathrm{Hol}(D, X)$.

III. Every $f \in \mathrm{Hol}(D, X)$ with $f(D) \subset \subset X$ can be connected to a $g \in \mathrm{Hol}(D, X - A)$ by a curve in $\mathrm{Hol}(D, X)$ which lies entirely in $\mathrm{Hol}(D, X - A)$ except for its initial point—i.e., there is a homotopy

$H(d, t): D \times I \to X$ such that H is holomorphic in d for each fixed t, $H(d, 0) = f(d)$ and $H(d, t) \notin A$ except for $t = 0$.

Remark. It is easy to verify that the two slightly different definitions of III are equivalent.

PROPOSITION 1. III $\Rightarrow$ II $\Rightarrow$ I.

Proof. III $\Rightarrow$ II. Let $M = \{f \in \mathrm{Hol}(D, X) \mid f(D) \subset\subset X\}$. We need only show that M is dense in $\mathrm{Hol}(D, X)$. But if $f \in \mathrm{Hol}(D, X)$ then f_t (defined by $f_t(x) = f(tx)$) belongs to M for every $0 \leq t < 1$ and $f_t \to f$ in the topology of $\mathrm{Hol}(D, X)$ as $t \to 1$.

II $\Rightarrow$ I. Let p and q be two points of X not in A. Let $r = d_X(p, q)$. Choose $\varepsilon > 0$, and let $f_i: D \to X, i = 1, \cdots, m$ be holomorphic maps such that $f_1(0) = p, f_i(a_i) = f_{i+1}(0), f_m(a_m) = q$ for points $a_1, \cdots, a_m \in D$ satisfying $\sum_{i=1}^m d_D(0, a_i) \leq r + \varepsilon$. (We are using a reformulation of the definition given in the introduction. The reformulation is obtained by using hyperbolic translations to map half the points involved to the origin.) Suppose that g_i is a map of D into $X - A, i = 1, \cdots, m$ and we put $y_i = g_i(0)$, $z_i = g_i(a_i)$, $y_0 = p$, $y_{m+1} = q$, $x_i = f_i(a_i) = f_{i+1}(0)$, $x_0 = p$. Then $d_{X-A}(p, q) \leq \sum_{i=0}^m d_{X-A}(y_i, y_{i+1}) \leq \sum_{i=0}^m [d_{X-A}(y_i, z_i) + d_{X-A}(z_i, y_{i+1})] \leq \sum_{i=0}^m d_D (0, a_i) + \sum_{i=0}^m d_{X-A}(z_i, y_{i+1}) \leq r + \varepsilon + \sum_{i=0}^m d_{X-A}(z_i, y_{i+1})$. Now, if the g_i are chosen close to the f_i in $\mathrm{Hol}(D, X)$, then the points $z_i = g_i(a_i)$ and $y_{i+1} = g_{i+1}(0)$ will both be close to $x_i = f_i(a_i) = f_{i+1}(0)$ and hence close to each other. Since d_{X-A} is continuous, $d_{X-A}(z_i, y_{i+1})$ will be small. Choose the g_i so close to the f_i in $\mathrm{Hol}(D, X)$ that $\sum_{i=0}^m d_{X-A}(z_i, y_{i+1}) < \varepsilon$. We obtain $d_{X-A}(p, q) \leq r + 2\varepsilon = d_X(p, q) + 2\varepsilon$. Finally, letting $\varepsilon \to 0$ we obtain $d_{X-A}(p, q) \leq d_X(p, q)$. Since the other inequality $d_{X-A}(p, q) \geq d_X(p, q)$ is always satisfied, we obtain $d_{X-A}(p, q) = d_X(p, q)$. End of proof.

§ 2

In this section we obtain results for open subsets of C^n. Some of these could have been deduced as corollaries of later results but the proofs are easier to follow here and the results somewhat more detailed.

We recall that a subset A of a topological space B is said to be of first category in B if it is contained in a countable union of closed, nowhere dense subsets. We omit the (easy) proof of

LEMMA 1: *Let $f: X \to Y$ be a holomorphic map between complex* *spaces, and let* A *be of the first category in* X. *If* X *is a countable union of compact sets and* $\dim_{f(x)} Y \geq \dim_x X$ (*respectively*, $\dim_{f(x)} Y > \dim_x X$) *for every* $x \in X$, *then* $f(A)$ (*respectively*, $f(X)$) *is of the first category in* Y.

PROPOSITION 2. *Let* U *be an open subset of* $\boldsymbol{C}^n$. *Let* A *be a closed subset of* U *which is of the first category in a nowhere dense closed analytic subset of* U. *Then* $\mathrm{Hol}(D, U - A)$ *is dense in* $\mathrm{Hol}(D, U)$. *Furthermore, if* A *is contained in a closed analytic subset of* U *of codimension* ≥ 2, *then* A *has property* III *as a closed subset of* U.

Proof. Suppose A is of the first category in B, where B is a nowhere dense closed analytic subset of U. Let $M = \{f \in \mathrm{Hol}(D, U) \mid f(D) \subset\subset U\}$. M is dense in $\mathrm{Hol}(D, U)$ (see proof of Proposition 1). Let $g \in M$. Consider the map $G: D \times B \to \boldsymbol{C}^n$ defined by $G(d, b) = g(d) - b$. Since $\dim D \times B \leq n$, Lemma 1 shows that $G(D \times A)$ is of the first category in $\boldsymbol{C}^n$. In particular $G(D \times A)$ contains no neighborhood of the origin. Choose a sequence $c_1, c_2, \cdots$, of points of $\boldsymbol{C}^n$ such that $c_i \to 0$ as $i \to \infty$ and $c_i \notin G(D \times A)$. Define $g_i: D \to \boldsymbol{C}^n$ by $g_i(d) = g(d) - c_i$. Since $g(D) \subset\subset U$ there is a $N > 0$ such that for $i > N$, $g_i(D) \subset U$. The sequence $g_i, i > N$, has g as limit in $\mathrm{Hol}(D, U)$ and $g_i(D) \subset U - A$ by construction. This completes the proof of the first assertion. Now suppose that A is contained in a closed analytic subset of U of codimension ≥ 2. It suffices to consider the case where A is itself a closed analytic subset of U of codimension ≥ 2. Let $f \in M$. Consider the map $F: D \times A \times \boldsymbol{C} \to \boldsymbol{C}^n$ defined by $F(d, a, t) = t(f(d) - a)$. Since $\dim D \times A \times \boldsymbol{C} \leq n$ and since $D \times A \times \boldsymbol{R}$ is of the first category in $D \times A \times \boldsymbol{C}$, Lemma 1 shows that $F(D \times A \times \boldsymbol{R})$ is of the first category in $\boldsymbol{C}^n$ and, in particular, that it is a proper subset of $\boldsymbol{C}^n$. Let $c \in \boldsymbol{C}^n$, $c \notin F(D \times A \times \boldsymbol{R})$. Define $H(d, s)$: $D \times I \to \boldsymbol{C}^n$ by $H(d, s) = f(d) + sc$. Since $f(D)$ is relatively compact in U, $H(d, s) \in U$ for sufficiently small s (independently of d) and, by construction, $H(d, s) \notin A$ for any s and d except when $s = 0$. Obviously H provides the required homotopy. End of proof.

Property III has a certain "staying power". Thus if X has property III for closed analytic subsets of codimension ≥ 2, then so does any smooth holomorphic retract of X (the requirement that the retraction $r: X \to A$, where $A \subset X$, be smooth is imposed so that the inverse image under r of any analytic subset of codimension ≥ 2 is again of codimension ≥ 2). Also, if every open subset of X has property III for its analytic subsets of codimension ≥ 2, then so does any space which is spread over X (that is, admits a locally biholomorphic map onto X). Using the fact that any Stein manifold is a smooth holomorphic retract of a tubular neighborhood of any embedding of it into $\boldsymbol{C}^m$, one easily proves the following proposition (which is also a consequence of Theorem 2 in the next section).

PROPOSITION 3. *Let X be spread over an open subset of a Stein manifold. Then any closed analytic subset of X of codimension $2\geq$ has property* III *as a closed subset of X.*

Note that the analogous proposition for Stein *spaces* would be false, as is shown by the example in § 0.

§ 3

In this section we show how to use the flow along vector fields to accomplish the job done by translations in § 2.

For a complex space X and a point $x \in X$ we denote by $T_x(X)$ (respectively, $TC_x(X)$) the tangent space (respectively, cone) to X at x. If $\varphi: X \to Y$ is a holomorphic mapping of complex spaces, we denote by $d\varphi_x$ the differential of φ at x. If $\varphi(x) = y$, the differential is a linear mapping $T_x(X) \to T_y(Y)$ which sends $TC_x(X)$ to $TC_y(Y)$.

By a vector field on X we will mean an $\mathcal{O}_X$-module homomorphism of Ω_X—the sheaf of germs of holomorphic differential forms of degree one—to $\mathcal{O}_X$. The collection of all vector fields on X is thus the vector space $\mathrm{Hom}_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$, which we will denote by $\Theta(X)$. (See [3] for background). Given a vector field T on X there is, locally, an associated local one-parameter group of automorphisms of X, called the flow along T. The parameter in question is complex and the flow depends holomorphically on all the variables occuring (even T). More precisely, suppose we are given a finite dimensional vector subspace V of $\Theta(X)$. Then for any relatively compact open subsets X_0 of X and V_0 of V there exists an $\varepsilon > 0$ for which a holomorphic map $\varphi: X_0 \times V_0 \times D_\varepsilon \to X$ exists, with $D_\varepsilon = \{z \in D \mid |z| < \varepsilon\}$ and $\varphi(x, T, t) = \varphi_T(x, t)$ i.e. the flow along T, for the "time" t, starting at x. One proves this by showing local existence and uniqueness (where defined) for the flow. On sufficiently small coordinate neighborhoods $X_\alpha \subset U_\alpha \subset \boldsymbol{C}^m$ the result is obtained by lifting a basis for the finite dimensional space of vector fields involved to the ambient $U_\alpha \subset \boldsymbol{C}^m$ and by observing that the flow on U_α thus obtained preserves the ideal sheaf of X_α. Finally one uses the uniqueness to piece together these flows, obtaining a holomorphic map $\varphi: \Omega \to X$ where Ω is the largest connected open neighborhood of $X \times V \times \{0\}$ in $X \times V \times \boldsymbol{C}$ on which the flow can be defined. We denote by $T(x)$ the value of a vector field at a point $x \in X$ and by $V(x) = \{T(x) \mid T \in V\}$ the vector subspace of $T_x(X)$ obtained by evaluating vector fields in V at x.

In the following we will show that if A is a closed analytic subset of a reduced complex space X and there are "enough" global vector fields on X then $\mathrm{Hol}\,(D, X - A)$ is dense in $\mathrm{Hol}\,(D, X)$. The intuitive idea here is, given a map $f: D \to X$ and a point $d \in D$ such that $f(d) \in A$, to find a global vector field which is "parallel" neither to A nor to the image of D and to use the flow of that vectorfield to push the image of D away from A at d. If $f(D) \subset\subset X$ then it suffices to find a vectorfield that does the job over all of $f(D)$. To make what we have said more precise and to avoid lengthy repetitions we make the following temporary definition:

DEF: A vector subspace W of $\Theta(X)$ is said to be *sufficiently disjoint from the tangent cone to A* if given any $a \in A$ there exist T_1 and $T_2 \in W$ such that $T_1(a)$ and $T_2(a)$ are linearly independent and no nontrivial linear combination of $T_1(a)$ and $T_2(a)$ lies in $TC_a(A)$.

With this definition the precise meaning of having "enough" global vector fields on X will be that $\Theta(X)$ is sufficiently disjoint from the tangent cone to A. First we prove a

TRANSVERSALITY LEMMA. *Let $\varphi: X \to Y$ be a holomorphic map from a nonsingular complex space X to a complex space Y, and let A be a closed analytic subspace of Y. If for every $x \in \varphi^{-1}(A)$ the image $d\varphi_x(T_x(X))$ of the tangent space contains two linearly independent vectors whose nontrivial linear combinations are never in $TC_{\varphi(x)}(A)$, then $\varphi^{-1}(A)$ is of codimension at least two.*

Proof. Let B be $\varphi^{-1}(A)$ with its reduced complex structure and let $i: B \to X$ be the inclusion. Let b be a nonsingular point of B. We have the following commutative diagram

$$\begin{array}{ccc} T_b(X) & \longrightarrow & T_{\varphi(b)}(Y) \\ {\scriptstyle di_b}\uparrow & & \uparrow \\ T_b(B) & \longrightarrow & TC_{\varphi(b)}(A) \end{array}$$

since φ induces a map $B \to A$ and since $TC_b(B) = T_b(B)$. Since b is a nonsingular point of both B and X, the vector space codimension of $T_b(B)$ in $T_b(X)$ is the same as the codimension of B in X at b. If that codimension is ≤ 1, then some nontrivial linear combination of *any* two vectors in $T_b(X)$ would get mapped to $TC_{\varphi(x)}(A)$, contradicting our assumption. Since the codimension of B is the infimum of its codimensions at regular points, we have the desired result. End of proof.

Given a finite dimensional subspace V of $\Theta(X)$ and the associated flow $\varphi\colon \Omega \to X$ where Ω is an open subset of $X \times V \times \boldsymbol{C}$, the middle partial derivative of φ at a point $\zeta = (x, T, t)$ of Ω is a linear map $D_2\varphi|_\zeta\colon V \to T_{\varphi(\zeta)}(X)$. We need to know that $D_2\varphi|_\zeta$ is, to first order, just $t\cdot$ (evaluation at $\varphi(\zeta)$).

LEMMA: *For any $x \in X$ and $T \in V$, letting $\zeta = (x, T, t)$, we have*

$$D_2\varphi|_\zeta(S) = tS(\varphi(\zeta)) + 0(t^2) \qquad \textit{as } t \textit{ tends to } 0.$$

The bounds implied by "$0(t^2)$" are locally uniform in x and T.

Proof. The desired result is local and φ can be calculated locally, so we may suppose that X is a closed analytic subspace of an open subset U of $\boldsymbol{C}^m$ and even that there is a basis $T_1, \cdots, T_r$ of V that gives rise to vectorfields on X that can be lifted to vectorfields on U. Specifically, suppose that $\sum_j f_{ij}(\partial/\partial z_j)$, $i = 1, \cdots, r$ are vectorfields on U that restrict to T_i on X. Let $F(x, a_1, \cdots, a_r, t) = (F_j(x, a, t))$ denote the flow starting at $x \in U$ for time t along the vectorfield $\sum_{i,j} a_i f_{ij}(\partial/\partial z_j)$. By definition F satisfies

i) $F(x, a, 0) = x$

and

ii) $\dfrac{\partial F_j}{\partial t}(x, a, t) = \sum_i a_i f_{ij}(F(x, a, t))$.

Expanding F in powers of t we obtain

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$F_j(x, a, t) = x + t\sum_i a_i f_{ij}(x)$ + terms involving t^2 and higher powers of t.

Differentiating with respect to a_i we obtain

$$\frac{\partial F_j}{\partial a_i}(x, a, t) = tf_{ij}(x) + 0(t^2)$$

and hence also

$$\frac{\partial F_j}{\partial a_i}(x, a, t) = tf_{ij}(F(x, a, t)) + 0(t^2)$$

But this is the desired result, for, if $S = b_1T_1 + \cdots + b_rT_r$, then $D_2\varphi|_\zeta(S) = \sum_i b_i\left(\frac{\partial F_j}{\partial a_i}\Big|_{(x,a,t)}\right) = t\sum_i b_i f_{ij}(F(x, a, t)) + 0(t^2) = tS(\varphi(\zeta)) + 0(t^2)$. End of proof.

We are now in a position to prove

THEOREM 1. *Let X be a complex space and let A be a closed analytic subspace of X. If $\Theta(X)$ is sufficiently disjoint from the tangent cone to A, then A has property* III *as a closed subset of X (see* §1 *for the definition of property* III).

Proof. Let $f: D \to X$ be a map of the disc into X such that $f(D) \subset\subset X$. As the first step in the proof we show that there is a finite dimensional subspace of Θ which is "sufficiently disjoint from the relevant portion of the tangent cone to A". Let $a \in A$. By hypothesis we can choose vectorfields T_1, T_2 in $\Theta(X)$ such that $\alpha T_1(a) + \beta T_2(a) \notin TC_a(A)$ for $|\alpha| + |\beta| = 1$. Since $\{(\alpha, \beta) | |\alpha| + |\beta| = 1\}$ is compact and $TC(A) = \bigcup_{a\in A} TC_a(A)$ is a closed analytic subset of $T(X) = \bigcup_{x\in X} T_x(X)$ (with its natural analytic structure) and since the map $x \rightsquigarrow \alpha T_1(x) + \beta T_2(x)$ is continuous, there is a neighborhood N_a of a in X such that $\alpha T_1(x) + \beta T_2(x) \notin TC(A)$ for $|\alpha| + |\beta| = 1$ and $x \in N_a$. Since $A \cap \overline{f(D)}$ is compact we can choose a finite number of points $a_1, \cdots, a_\ell$ of A such that the corresponding neighborhoods N_{a_i} cover $A \cap \overline{f(D)}$. Let V be the finite dimensional vector subspace of $\Theta(X)$ spanned by $T_1^{(1)}, T_2^{(1)}, T_1^{(2)}, T_2^{(2)}, \cdots, T_2^{(\ell)}$ where $T_1^{(j)}$ and $T_2^{(j)}$ are the vectorfields that were chosen at the point a_j to define the neighborhood N_{a_j}. V is, in an obvious sense, sufficiently disjoint from the tangent cone to A over $A \cap \overline{f(D)}$.

Let $\varphi: \Omega \to X$ be the flow associated to V, where Ω is an open neighborhood of $X \times V \times \{0\}$ in $X \times V \times \boldsymbol{C}$. Let $\Omega' = (f \times 1 \times 1)^{-1}(\Omega) =$

$\{(d, T, t) \in D \times V \times \boldsymbol{C} \mid (f(d), T, t) \in \Omega\}$, and define $\varphi' : \Omega' \to X$ by $\varphi'(d, T, t) = \varphi(f(d), T, t)$. Choose some norm $|\ |$ on V. For $\varepsilon > 0$ let $V_\varepsilon = \{T \in V \mid |T| < \varepsilon\}$. We wish to show that there is an $\varepsilon > 0$ such that $D \times V_\varepsilon \times D_\varepsilon \subset \Omega'$ and such that the restriction of φ' to $D \times V_\varepsilon \times (D_\varepsilon - \{0\})$ satisfies the hypotheses of the transversality lemma (with $X = D \times V_\varepsilon \times (D_\varepsilon - \{0\}), Y = X, A = A$). Observe first that (by the lemma preceeding this proof) the map $\psi : (\Omega' - (D \times V \times \{0\})) \times V \to T(X)$ given by $\psi((d, T, t), S) = (1/t) D_2\varphi'|_{(d,T,t)}(S)$ can be extended to a map $\tilde{\psi} : \Omega' \times V \to T(X)$ by putting $\tilde{\psi}((d, T, 0), S)$ equal to $S(f(d))$. Next let us construct for each $x \in X$ a neighborhood N_x of x and an $\varepsilon_x > 0$ as follows:

Case I. $x \notin A$. Then $\varphi(x, 0, 0) \notin A$. By continuity there exist a neighborhood N_x of x and an $\varepsilon_x > 0$ such that $(y, T, t) \in \Omega$ and $\varphi(y, T, t) \notin A$ for $y \in N_x$ and $|T|, |t| < \varepsilon$.

Case II. $x \in A \cap \overline{f(D)}$. By the construction of V, there exist T_1, $T_2 \in V$ such that no nontrivial linear combination of $T_1(x)$ and $T_2(x)$ belongs to $TC_x(A)$. We can restate this as $\tilde{\psi}((x, 0, 0), \alpha T_1 + \beta T_2) \notin TC(A)$ for $|\alpha| + |\beta| = 1$ (since ψ is linear in its last variable). By continuity there exist a neighborhood N_x of x and an $\varepsilon_x > 0$ such that $(y, T, t) \in \Omega$ and $\tilde{\psi}((y, T, t), \alpha T_1 + \beta T_2) \notin TC(A)$ for $|\alpha| + |\beta| = 1$, $y \in N_x$, $|T| < \varepsilon_x$ and $|t| < \varepsilon_x$. Note that, by the definition of $\tilde{\psi}$, this says that if $t \neq 0$ and $\zeta = (y, T, t)$ then $D_2\varphi'|_\zeta(T_1)$ and $D_2\varphi'|_\zeta(T_2)$ are linearly independent and that no nontrivial linear combination of them lies in $TC(A)$.

Case III. $x \in A - \overline{f(D)}$. Let $N_x = X - \overline{f(D)}$ and $\varepsilon_x = 1$.

Since $\overline{f(D)}$ is compact we can choose a finite number of points $x_1, \cdots, x_n \in X$ such that the corresponding neighborhoods N_{x_i} cover $f(D)$. We can discard any points in $A - \overline{f(D)}$ and still have the same property. Let $\varepsilon = \min(\varepsilon_{x_1}, \cdots, \varepsilon_{x_n})$. It is clear that ε has the desired property by construction.

Let $\tilde{\varphi}$ be the restriction of φ' to $D \times V_\varepsilon \times (D_\varepsilon - \{0\})$. Applying the transversality lemma we conclude that $\tilde{\varphi}^{-1}(A)$ is of codimension at least two, i.e., $\dim \tilde{\varphi}^{-1}(A) \leq \dim V_\varepsilon =$ vector space dimension of V. Let $\pi : \tilde{\varphi}^{-1}(A) \to V_\varepsilon$ be the projection map and let Z be the "real section of $\tilde{\varphi}^{-1}(A)$", i.e., $Z = \tilde{\varphi}^{-1}(A) \cap (D \times V \times \boldsymbol{R})$. We claim that $\pi(Z)$ is of the first category in V_ε. Before proving this let us see how we can use it to complete the proof of the Theorem. If $\pi(Z)$ is of the first category in V_ε, then certainly $\pi(Z) \neq V_\varepsilon$. Choose $T \in V_\varepsilon$ which is not in the image of π. $T \notin \pi(Z)$ says precisely that $\varphi'(d, T, t) \notin A$ for $d \in D$ and $t \in \boldsymbol{R}$, $0 < |t| < \varepsilon$. Thus

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$H(d, t) = \varphi'(d, T, t)$, $0 \leq t \leq \varepsilon/2$, provides the desired homotopy. Finally, to show that $\pi(Z)$ is of the first category, observe that since $\tilde{\varphi}^{-1}(A)$ has at most countably many irreducible components, it suffices to show that $\pi(W \cap Z)$ is of the first category for every irreducible component W of Z. Let $t: W \subset D \times V \times \boldsymbol{C} \to \boldsymbol{C}$ be the third projection. Since W is irreducible and t is holomorphic, t is either an open mapping or constant. If t is an open mapping then, since $\boldsymbol{R}$ is nowhere dense in $\boldsymbol{C}$, $W \cap Z = t^{-1}(\boldsymbol{R})$ is nowhere dense in W, and we may apply lemma 1 of § 2 to conclude that $\pi(W \cap Z)$ is nowhere dense in V_ε. If t is constant, let t_0 be its unique value. Then W is contained in the inverse image of A under the map $D \times V_\varepsilon \times \{t_0\} \to X$ induced by φ'. The argument given above to show that $\tilde{\varphi}$ satisfies the hypotheses of the transversality lemma also shows that this map does as well (since the "same" partial derivative produces the required tangent vectors in either case). Hence $\dim W \leq 1 + \dim V - 2 < \dim V_\varepsilon$ and we again conclude by applying lemma 1 of § 2. End of proof.

Remark. One can see why $\pi(Z)$ is of the first category by observing that $(D \times V \times \boldsymbol{R}) \cap \Omega'$ is a *real* analytic manifold and that an application of transversality and a count of *real* dimensions show that $\pi(Z)$ should be lower dimensional than V_ε.

As a corollary we obtain

THEOREM 2. *Let X be a complex manifold whose tangent bundle is spanned by its global sections. If A is any analytic subset of X of codimension ≥ 2 then A has property* III *as a closed subset of X. In particular* $\mathrm{Hol}(D, X - A)$ *is dense in* $\mathrm{Hol}(D, X)$ *and the restriction to $X - A$ of the Kobayashi pseudo-distance on X is the Kobayashi pseudo-distance on $X - A$.*

Proof. We will show that $\Theta(X)$ is sufficiently disjoint from the tangent cone to A. Let $a \in A$. If a is a nonsingular point of A, choose T_1 and T_2 so that $T_1(a)$ and $T_2(a)$ span a two dimensional subspace of $T_a(X)$ complementary to $T_a(A)$. If a is a singular point, $TC_a(A)$ is an algebraic cone in $T_a(X)$ of dimension equal to $\dim_a(A)$. By one well known definition of dimension there is a linear subspace, of codimension equal to $\dim_a(A)$, lying $T_a(X)$ and whose intersection with $TC_a(A)$ has $\{0\}$ as an isolated point. But then, since $TC_a(A)$ is a cone, the intersection of the linear subspace and $TC_a(A)$ reduces to $\{0\}$. Simply choose

T_1 and T_2 so that $T_1(a)$ and $T_2(a)$ are linearly independent and lie in that linear subspace. End of proof.

Remarks. 1) The reason for requiring X to be nonsingular does *not* lie in the proof given above. The reason is the fact, due to Rossi ([5]), that if X is reduced and its tangent spaces are spanned by the values of vectorfields, then X *must* be nonsingular.

2) Any *compact* manifold whose tangent bundle is spanned by its global sections necessarily has trivial Kobayashi pseudo-distance. For, if X is compact, $\Theta(X)$ is finite dimensional and vectorfields generate complete one parameter groups and the flow becomes a map $\varphi: X \times \Theta(X) \times \boldsymbol{C} \to X$. If $x_0 \in X$ then $(T, t) \to \varphi(x_0, T, t)$ is holomorphic and its image contains a neighborhood of x_0. Since $x_0 \times \Theta(X) \times \boldsymbol{C}$ has trivial pseudo-distance, so does that neighborhood. Since x_0 was arbitrary, it follows that X has trivial pseudo-distance.

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