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ON GROUPS GENERATED BY THREE-DIMENSIONAL SPECIAL UNITARY GROUPS I

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Introduction

This paper is the result of our attempt to carry over the method of our characterization of the linear groups $PSL(n, q)$ (Phan (1972)) to that of the unitary groups $PSU(n, q)$. It appears desirable to have available a result on the generation of the unitary groups more closely related to the splitting of the underlying hermitian spaces of these groups into orthogonal direct sum of anisotropic lines. Another motivation comes also from our work on the unitary groups in which we obtained a fusion pattern different from that of the unitary group when the dimension of the underlying space is eight. This fusion pattern produces a group whose structure is not immediately apparent. In order to identify this group, the result of this paper appears to be a necessary step.

To discuss the type of problems that we shall consider, it is convenient to introduce the language of graphs. A graph Γ on a set Δ is the couple (Δ, E) where E is a set of 2-element subsets of Δ. The elements of Δ (resp. E) are called vertices (resp. edges). An ordered subset Δ' of Δ is a chain if the consecutive members of Δ' are the only edges in Γ. We shall also use pictorial representation of a graph with points corresponding to vertices and line segment joining two points if the corresponding vertices form an edge. We are interested in graphs whose pictorial representations are the Dynkin diagrams of types A_n, D_n, E_6, E_7, E_8. We say a graph is of type X if its pictorial representation is a Dynkin diagram of type X.

Let $\Gamma = (\Delta, E)$ be a graph of type X on a finite set Δ. We want to investigate the structure of groups G such that there is a mapping which takes i in Δ to a pair of subgroups (L_i, H_i) of G where L_i is isomorphic to $SU(2, q)$ and H_i is a subgroup of order $q+1$ in L_i (q finite). Furthermore these subgroups satisfy the following conditions

(a) $G = \langle L_i \mid \forall i \in \Delta \rangle$;
(b) $[L_i, L_j] = 1$ if $\{i, j\} \notin E$;
(c) $\langle L_i, L_j \rangle \cong SU(3, q)$ and $\langle L_i, H_j \rangle \cong GU(2, q) \cong \langle H_i, L_j \rangle$ if $\{i, j\} \in E$;
(d) $\langle H_i, H_j \rangle = H_i \times H_j \quad i \neq j, \quad \forall i, j \in \Delta$.

DEFINITION. *A group G together with a mapping from a graph Γ of type X into pairs of subgroups of G satisfying the above conditions is called a group generated by $SU(3,q)$'s of type X.*

Our aim is to prove the following

THEOREM. *Let G be a group generated by $SU(3,q)$'s of type A_n. Assume that q is greater than 4. Then G is a homomorphic image of $SU(n+1,q)$.*

The result holds trivially when $n = 1$ or 2. We may assume that $n > 2$. Among the groups of the same type, we show that there is a 'universal' group G such that every group of the same type is a homomorphic image of G. Then we prove that $SU(n+1,q)$ is of type A_n. Consequently there exists a homomorphism $\theta : G \to SU(n+1,q)$. The theorem then follows at once because the kernel of θ is 1. This fact results from a factorization of G involving the subgroups L_i. We are able to obtain this factorization because of the identity $\langle L_i, L_j \rangle = L_i L_j L_i L_j$ for any edge $\{i, j\}$.

1. $SU(n+1,q)$ as group of type A_n

Let V be a vector space of dimension m over the field F_{q^2} of q^2 elements equipped with a non degenerate hermitian form $(\,,)$. Throughout this paper we shall assume $q > 4$. The space V has an orthonormal basis $B = \{v_i \mid 1 \leqq i \leqq m\}$. The set of all non singular linear transformations x of V which are isometries (i.e. $(x(v), x(v')) = (v, v') \forall v, v' \in V$) forms a group $GU(V)$, the general unitary group of V. The special unitary group $SU(V)$ of V is the subgroup of determinant 1 in $GU(V)$. We note that for a linear transformation x to belong to $GU(V)$, it is necessary and sufficient that the matrix $[x]$ of x relative to the basis B satisfies $[x]^t[\bar{x}] = 1$ where $[\bar{x}]$ is the matrix obtained from $[x]$ by replacing each of its entry x_{ij} by $\overline{x_{ij}} = x_{ij}^q$.

For $1 \leqq i \leqq m-1$, let L_i^* be the elements of $SU(V)$ which fix all v_j $j \neq i$, $i+1$. Since $V_i = \langle v_i, v_{i+1} \rangle = \langle v_j \mid j \neq i, i+1 \rangle^\perp$, L_i^* leaves V_i invariant and hence $L_i^* \cong SU(2,q)$. We compute that each element in L_i^* when restricted to V_i has a matrix of the form

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

where $\alpha, \beta \in F_{q^2}$ and $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$. We shall denote such an element by $(\alpha, \beta)_i^*$. The set $(\alpha, 0)_i^*$ forms a diagonal subgroup H_i^* of order $q + 1$.

We want to show that $SU(V)$ is generated by L_i^*. First we note the following elementary facts. The set of elements in F_{q^2} fixed by the map $x \mapsto \bar{x}$ is a subfield of order q and for each non zero element α of this subfield, there are exactly $q + 1$ elements x such that $x\bar{x} = \alpha$.

LEMMA 1.1. *Let α, β, γ be in F_{q^2} such that $\beta\gamma \neq 0$ and $\alpha\bar{\alpha} + \beta\bar{\beta} = 0$. Then there are at least $(q^2 - 2q - 1)(q + 1)$ ordered pairs (ξ, η) in F_{q^2} such that $\xi\bar{\xi} + \eta\bar{\eta} = 1$ and*

$$\alpha\bar{\alpha} + (\beta\xi + \gamma\eta)\overline{(\beta\xi + \gamma\eta)} \neq 0.$$

PROOF. Any pair (ξ, η) such that $\xi\bar{\xi} = 1$ and $\eta = 0$ clearly does not have the required property. We may suppose $\eta \neq 0$ and set $\xi = x\eta$. We required that

$$(*) \qquad \eta\bar{\eta}(1 + x\bar{x}) = 1.$$

Now suppose

$$\alpha\bar{\alpha} + (\beta\xi + \gamma\eta)\overline{(\beta\xi + \gamma\eta)} = 0.$$

This implies that

$$(**) \qquad \alpha\bar{\alpha} + \gamma\bar{\gamma} + \beta\bar{\gamma}x + \bar{\beta}\gamma\bar{x} = 0.$$

Equation (**) has at most q solutions in x. For (*) to hold, we have to exclude the $(q + 1)$ members x of F_{q^2} such that $x\bar{x} = -1$. Hence there are $q^2 - 2q - 1$ possible choices of x not satisfying (**) and $x\bar{x} = -1$. For each of these x's, we can choose $q + 1$ different η's satisfying (*). So there are altogether $(q^2 - 2q - 1)(q + 1)$ ordered pairs (ξ, η) with the required properties.

PROPOSITION 1.2. *We have*

(a) $\langle L_i^*, L_{i+1}^* \rangle = L_i^* L_{i+1}^* L_i^* L_{i+1}^* = L_{i+1}^* L_i^* L_{i+1}^* L_i^* \cong SU(3, q)$.

(b) $SU(V) = \langle L_i^* \mid 1 \leqq i \leqq m - 1 \rangle$.

PROOF. (a) We may assume dim $V = 3$ and $i = 1$. It suffices by symmetry to show that $SU(V) = L_2^* L_1^* L_2^* L_1^*$.

Let $g \in SU(V)$. Then $g(v_3) = \alpha v_1 + \beta v_2 + \gamma v_3$. If $\alpha = \beta = 0$ or if $\alpha v_1 + \beta v_2$ is a non-isotropic vector, set $g_2 = 1$. Otherwise we use (1.1) to find an element $g_2 = (\xi, \eta)_2^* \in L_2^*$ such that the projection of $g_2 g(v_3)$ into $\langle v_1, v_2 \rangle$ is not a non zero isotropic vector. In all cases we can now find $g_1 \in L_1^*$ such that

$$g_1 g_2 g(v_3) = \beta' v_2 + \gamma v_3$$

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since L_1^* is transitive on non isotropic vector of the same length and the subspace contains non isotropic vector of any length as well as the zero vector. For the same reason, there exists $h_2 \in L_2^*$ such that

$$h_2 g_1 g_2 g(v_3) = v_3.$$

The assertion now follows because the stabilizer of v_3 in $SU(V)$ is L_1^*.

(b) We may assume $m \geqq 4$. Let $V_1 = \langle v_i \mid 1 \leqq i \leqq m-1 \rangle$, and $V_2 = \langle v_{m-2}, v_{m-1}, v_m \rangle$. Set $S = \langle L_i^* \mid 1 \leqq i \leqq m-2 \rangle$; $T = \langle L_{m-2}^*, L_{m-1}^* \rangle$. By induction, $S \cong SU(V_1)$ and $T \cong SU(V_3)$.

Let $g \in SU(V)$. Then $g(v_m) = v + v'$ where $v \in V_1$ and $v' \in \langle v_m \rangle$. Since v_2 contains non zero isotropic vectors and $\dim V_1 \geqq 3$, we can apply Witt's Theorem (Dieudonné (1955)) to obtain an element $s \in S$ and $t \in T$ such that $sg(v_m) \in V_2$ and $tsg(v_m) = v_m$. The result follows since the stabilizer of v_m in $SU(V)$ is S.

We next investigate the embedding of L_i, L_j in $\langle L_i, L_j \rangle$ if $\{i, j\}$ is an edge.

LEMMA 1.3. *Suppose* $\dim V = 3$. *Let* $\{i, j\}$ *be an edge and* θ *the isomorphism from* $\langle L_i, L_j \rangle$ *onto* $SU(V)$. *Then replacing* θ *by its composition with an inner automorphism of* $SU(V)$ *we may assume* $\theta(L_i) = L_1^*$; $\theta(H_i) = H_1^*$; $\theta(L_j) = L_2^*$ *and* $\theta(H_j) = H_2^*$.

PROOF. We shall regard $SU(V)$ as the set of fixed points of a suitable algebraic endomorphism σ on the algebraic group $SL(\bar{V})$ where $\bar{V}$ is an extension of V by an algebraically closed field (Steinberg and Springer (1970) §2). Then $\theta(H_i H_j)$ is a set of commuting semisimple elements and hence lie in a maximal torus of $SL(\bar{V})$ which is fixed by σ (II.5.10(a) of Springer–Steinberg). Since any two σ-fixed maximal tori are conjugate by an element of $SL(\bar{V})$, we conclude that $\theta(H_i H_j) = H_1^* H_2^*$ after adjusting θ by an inner automorphism (Steinberg and Springer (1970); II.5.8; I.2.9).

Since $\langle L_i, H_j \rangle \cong GU(2, q)$ by (c) and $q > 4$, we have $\langle L_i, H_j \rangle' = L_i$ as L_i is perfect. So $\langle L_i, H_j \rangle = L_i H_j$ and $C_{L_i H_j}(H_i) = H_i H_j$. Therefore $Z(L_i H_j)$ which has order $q+1$ is contained in $H_i H_j$. An easy computation shows that every subgroup in $H_1^* H_2^*$ of order $q+1$ has its centralizer in $SU(V)$ equal to $H_1^* H_2^*$ except the following three subgroups $\langle (\alpha, 0)_1^* (\alpha', 0)_2^* \rangle$ where $\alpha' = \alpha^2$; $(\alpha')^2 = \alpha$ or $\alpha' = \alpha^{-1}$ where $0(\alpha) = q+1$ in the first case and $0(\alpha') = q+1$ in the remaining cases. Moreover these three subgroups are conjugate in $N_{SU(V)}$ $H_1^* H_2^*)$. Therefore after adjusting θ by an inner automorphism we may suppose $\theta(Z(L_i H_j)) = \langle (\alpha, 0)_1^* (\alpha^2, 0)_2^* \rangle$. It follows then $\theta(L_i) = L_1^*$. Finally because the remaining two subgroups of order $q+1$ are conjugate by an element of L_1^*, we may assume $\theta(L_j) = L_2^*$ after adjusting θ by an inner automorphism. It now follows easily that $\theta(H_i) = H_1^*$ and $\theta(H_j) = H_2^*$.

LEMMA 1.4. *There exists a group $\tilde{G}$ of type X with the map $i \mapsto (\tilde{L}_i, \tilde{H}_i)$, i a vertex of X such that whenever G is a group of the same type with the map $i \mapsto (L_i, H_i)$, then there exists a homomorphism θ mapping $\tilde{H}_i$ on H_i and $\tilde{L}_i$ on L_i.*

PROOF. For each vertex i, let F_i be a set and R_{ii} be words generated by elements of F_i such that the group $\tilde{L}_i$ generated by F_i and presented by R_{ii} is isomorphic to $SU(2,q)$. If $\{i,j\}$ is not an edge, let R_{ij} be the words $[x,y]$ where $x \in F_i$ and $y \in F_j$. If $\{i,j\}$ is an edge, we use (1.2), (1.3) to obtain a set of words R_{ij} each involving both elements of F_i and F_j such that the group generated by $F_i \cup F_j$ and presented by the relations $R_{ii} \cup R_{jj} \cup R_{ij}$ is isomorphic to $SU(3,q)$.

Let $\tilde{G}$ be the group generated by $F = \bigcup_i F_i$ defined by relations $\bigcup_{i,j} R_{ij}$. Each $\tilde{L}_i$ defined earlier can be naturally identified with a subgroup of $\tilde{G}$. As $\tilde{G}$ is a free product with certain amalgamations of groups isomorphic to $SU(3,q)$ or $SU(2,q) \times SU(2,q)$; $\tilde{L}_i \neq 1$. By means of (1.3), we can now choose subgroups $\tilde{H}_i$ in $\tilde{L}_1$ such that all conditions (a), (b), (c), (d) are satisfied and so $\tilde{G}$ is a group of type X.

It is now obvious that the mapping θ in the lemma has the required property.

DEFINITION. *The group $\tilde{G}$ in (1.4) will be called a universal group of type X. It is clear that up to isomorphism there is only one universal group of type X.*

LEMMA 1.5. *Let $\tilde{G}$ be a universal group of type X with the map $i \mapsto (\tilde{L}_i, \tilde{H}_i)$ and Δ' be a subgraph of type Y. Then $\langle \tilde{L}_i \mid i \in \Delta' \rangle$ is a universal group of type Y.*

PROOF. This is obvious from our construction of $\tilde{G}$ in (1.4).

LEMMA 1.6. *The group $SU(V)$ where $\dim V = n + 1$ is a group of type A_n.*

PROOF. If the vertices of the graph of type A_n are naturally ordered by the set of integers $\{1, 2, \cdots, n\}$ such that the consecutive integers form an edge, then the map $i \mapsto (L_i^*, H_i^*)$ satisfies conditions (a)–(d) by (1.2) and direct verification.

We need the following result at the end of our proof.

LEMMA 1.7. *Let g, g' be elements of $SU(V)$ and set $S = \langle L_i^* \mid 1 \leqq i \leqq m-2 \rangle$. Then there exist elements $a \in L_{m-1}^*$ such that both ag and ag' belong to $SL_{m-1}^* S$.*

PROOF. Suppose $g(v_m) = v + \alpha v_{m-1} + \beta v_m$ and $g'(v_m) = v' + \alpha' v_{m-1} + \beta' v_m$ where $v, v' \in \langle v_i \mid 1 \leqq i \leqq m-2 \rangle$. Let $a = (\xi, \eta)_{m-1}^* \in L_{m-1}^*$. Then

$$ag(v_m) = v + (\alpha\xi + \beta\eta)v_{m-1} + (-\alpha\bar{\eta} + \beta\bar{\xi})v_m$$

and

$$ag'(v_m) = v' + (\alpha'\xi + \beta'\eta)v_{m-1} + (-\alpha'\bar{\eta} + \beta'\bar{\xi})v_m.$$

If the projections of $ag(v_m)$ and $ag'(v_m)$ into $\langle v_i \mid 1 \leqq i \leqq m-1\rangle$ are not non zero isotropic vectors, then we use the argument of (1.2) to show that ag, $ag' \in SL^*_{m-1}S$.

We shall assume $\eta \neq 0$ and set $\xi = x\eta$ and suppose that the projections of $(ag)(v_m)$ and $ag'(v_m)$ into $\langle v_i \mid 1 \leqq i \leqq m-1\rangle$ are non zero isotropic vectors. Then we have

$$(v, v)(1 + x\bar{x}) + \alpha\bar{\alpha}x\bar{x} + \alpha\bar{\beta}x + \bar{\alpha}\beta x + \beta\bar{\beta} = 0$$

and

$$(v', v')(1 + x\bar{x}) + \alpha'\bar{\alpha}'x\bar{x} + \alpha'\bar{\beta}'x + \bar{\alpha}'\beta'x + \beta'\bar{\beta}' = 0.$$

For the existence of a with the required property, we must have $x \in F_{q^2}$ not satisfying any one of above non-trivial equations of degree $q+1$ as well as $1 + x\bar{x} = 0$. Hence a exists provided $q^2 - 3(q+1) > 0$, ie $q > 3$.

Remark. When q is odd, it is not necessary to assume second part of (c) as this follows from the other assumptions.

2. Factorization of G

Let G be a group of type A_n with the set of pairs of subgroups (L_i, H_i). In view of (1.3) we can denote each element of L_i as $(\alpha, \beta)_i$ where $\alpha; \beta \in F_{q^2}$ and $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$ and in case $\{i, j\}$ is an edge, we may perform the multiplication $(\alpha, \beta)_i(\gamma, \delta)_j$ in $SU(V)$ with $\dim V = 3$ by setting $(\alpha, \beta)_i = (\alpha, \beta)^*_1$ and $(\gamma, \delta)_j = (\gamma, \delta)^*_2$. Thus $(\alpha, \beta)_i \in H_i$ if $\beta = 0$.

Lemma 2.1. *Suppose that $\langle L_i, L_j\rangle$ is isomorphic to $SU(3, q)$. Let $a = (\alpha, \beta)_j$, $c = (\xi, \eta)_j$ belong to L_j and $b = (\gamma, \delta)_i$ belong to L_i. Set $\sigma = \alpha\bar{\gamma}\eta + \beta\bar{\xi}$ and $\rho = \delta\eta$. Then abc belongs to $L_iL_jL_i$ if and only if either $\beta\delta\eta = 0$ or $\sigma\bar{\sigma} + \rho\bar{\rho} \neq 0$.*

In case $\beta\delta\eta \neq 0$ and $\sigma\bar{\sigma} + \rho\bar{\rho} \neq 0$, let $\mu \in F_{q^2}$ such that $\mu\bar{\mu} = \rho\bar{\rho}/\sigma\bar{\sigma} + \rho\bar{\rho}$ and let $\lambda = -\sigma\mu/\rho$. Then

$$abc = (\bar{\lambda}, -\mu)_i(-\beta\gamma\bar{\eta} + \alpha\xi, -\rho/\mu)_j(-(\beta\gamma\bar{\xi} + \alpha\eta)\mu/\rho, -\beta\mu/\eta)_i.$$

Proof. A tedious verification.

LEMMA 2.2. *Let* $\{i, j, k\}$ *be a chain. Then*

$$L_k L_j L_i L_j L_k \subseteq L_j L_i L_k L_j L_k L_i L_j.$$

PROOF. Let $a = (\alpha, \beta)_k$; $b = (\gamma, \delta)_j$; $c = (\sigma, \tau)_i$; $d = (\nu, \rho)_j$ and $e = (\lambda, \mu)_k$. If one of these elements lies in $H_i H_j H_k$, then their product $abcde$ belongs to right-hand side of the inequality since $H_i \subseteq N_G(L_j)$; $H_j \subseteq N_G(L_i) \cap N_G(L_k)$ and $H_k \subseteq N_G(L_j)$ and $[L_i, L_k] = 1$. Hence we may assume none of these elements lie in $H_i H_j H_k$; that is

$$\beta\delta\tau\rho\mu \neq 0. \tag{1}$$

Also we have

$$\alpha\bar{\alpha} + \beta\bar{\beta} = \gamma\bar{\gamma} + \delta\bar{\delta} = \sigma\bar{\sigma} + \tau\bar{\tau} = \nu\bar{\nu} + \rho\bar{\rho} = \lambda\bar{\lambda} + \mu\bar{\mu} = 1. \tag{2}$$

Next we may assume that $bcd \notin L_i L_j L_i$, otherwise we are done as $[L_i, L_k] = 1$. Therefore we have by (2.1)

$$\tau\bar{\tau}\rho\bar{\rho} + \gamma\bar{\gamma}\sigma\bar{\sigma}\rho\bar{\rho} + \delta\bar{\delta}\nu\bar{\nu} + \gamma\bar{\delta}\bar{\sigma}\nu\rho + \bar{\gamma}\delta\sigma\bar{\nu}\bar{\rho} = 0. \tag{3}$$

We shall now show that it is possible to choose a suitable element $f = (\xi, \eta)_k$ such that

$$(*) \qquad \begin{aligned} abf &= a_j b_k c_j \\ f^{-1} de &= d_j e_k f_j \\ c_j c d_j &\in L_i L_j L_i \end{aligned}$$

where $a_j, c_j, d_j, f_j \in L_j$ and $b_k, e_k \in L_k$. Then the lemma follows since $[L_i, L_k] = 1$.

We shall choose $f = (\xi, \eta)_k$ such that

$$\eta \neq 0 \tag{4}$$

and set

$$\xi = x\eta. \tag{5}$$

Thus

$$(1 + x\bar{x})\eta\bar{\eta} = 1. \tag{6}$$

If there is f satisfying (*), we can then find ζ, ϕ in F_{q^2} such that

$$\zeta = -(\alpha\bar{\gamma} + \beta\bar{x})\phi/\delta \tag{7}$$

$$\phi\bar{\phi} = \delta\bar{\delta}\eta\bar{\eta}/\omega_1 \tag{8}$$

and

$$(9) \qquad c_j = (-(\alpha + \beta\gamma\bar{x})\phi/\delta, -\beta\phi/\eta)_i$$

$$(**) \qquad \omega_1 = (\delta\bar{\delta} + \alpha\bar{\alpha}\gamma\bar{\gamma} + \alpha\bar{\beta}\bar{\gamma}x + \bar{\alpha}\beta\gamma\bar{x} + \beta\bar{\beta}x\bar{x})\eta\bar{\eta} \neq 0.$$

Similarly if $f^{-1}de \in L_jL_kL_j$, there exist χ, ψ in F_{q^2} such that

$$(10) \qquad \chi = (\bar{\lambda} - \bar{\nu}\mu\bar{x})\eta\psi/\rho\mu$$

$$(11) \qquad \psi\bar{\psi} = \rho\bar{\rho}\mu\bar{\mu}/\omega_2$$

and

$$(12) \qquad dj = (\bar{\chi}, -\psi)_j$$

$$(***) \qquad \omega_2 = (\lambda\bar{\lambda} + \rho\bar{\rho}\mu\bar{\mu} - \nu\overline{\lambda\mu}x - \bar{\nu}\lambda\mu\bar{x} + \mu\bar{\mu}x\bar{x})\eta\bar{\eta} \neq 0.$$

We also want c_jed_j to belong to $L_iL_jL_i$. This will be so if

$$(****) \qquad \begin{aligned} \omega_3 = {} & \tau\bar{\tau}\psi\bar{\psi} + (\alpha + \beta\gamma\bar{x})\overline{(\alpha + \beta\gamma\bar{x})}\sigma\bar{\sigma}\phi\bar{\phi}\psi\bar{\psi}/\delta\bar{\delta} \\ & + \beta\bar{\beta}\phi\bar{\phi}\chi\bar{\chi}/\eta\bar{\eta} - (\alpha + \beta\gamma\bar{x})(\bar{\beta}\bar{\sigma}\bar{\phi}\bar{\chi}\psi)/\bar{\eta} \\ & - \overline{(\alpha + \beta\gamma\bar{x})}(\beta\sigma\phi\chi\bar{\psi})/\eta \neq 0. \end{aligned}$$

We note that for the first two equations of (*) to hold, we must choose x which are not solutions of any of the following equations

$$1 + x\bar{x} = 0$$

$$\omega_1 = 0$$

$$\omega_2 = 0.$$

There are at most $3(q + 1)$ elements which satisfy one of above equations. We shall now assume in the rest of the proof x is not a solution of one of these equations.

Using (7), (8), (10), (11), (**) and (***), we compute that the coefficient of $x\bar{x}$ in ω_3 is

$$\frac{\eta\bar{\eta}}{\omega_1\omega_2}\{\beta\bar{\beta}(\tau\bar{\tau}\rho\bar{\rho} + \gamma\bar{\gamma}\sigma\bar{\sigma}\rho\bar{\rho} + \delta\bar{\delta}\nu\bar{\nu} + \gamma\overline{\delta\sigma}\nu\rho + \bar{\gamma}\delta\sigma\bar{\nu}\bar{\rho})\mu\bar{\mu}\} = 0 \quad \text{by (3).}$$

The constant term in ω_3 is

$$(13) \qquad \frac{\eta\bar{\eta}}{\omega_1\omega_2}\{(\alpha\bar{\alpha} + \beta\bar{\beta}\delta\bar{\delta}\tau\bar{\tau})\rho\bar{\rho}\mu\bar{\mu} + \beta\bar{\beta}\delta\bar{\delta}\lambda\bar{\lambda} - \alpha\bar{\beta}\bar{\delta}\bar{\sigma}\rho\lambda\mu - \bar{\alpha}\beta\delta\sigma\bar{\rho}\bar{\lambda}\bar{\mu}\}$$

and the coefficient of x in ω_3 is

$$(14)\qquad \frac{\eta\bar{\eta}}{\omega_1\omega_2}\{\alpha\overline{\beta\gamma}\rho\bar{\rho}\mu\bar{\mu} + \alpha\bar{\beta}\bar{\delta}\bar{\sigma}\nu\rho\mu\bar{\mu} - \beta\bar{\beta}\delta\bar{\delta}\nu\bar{\lambda}\bar{\mu} - \beta\overline{\beta\gamma}\delta\sigma\bar{\rho}\bar{\lambda}\bar{\mu}\}.$$

We shall now show that (13) and (14) cannot both be zero. We note that

$$(\bar{\delta}\nu + \bar{\gamma}\bar{\rho}\sigma) \neq 0$$

otherwise from $(\bar{\delta}\nu + \overline{\gamma\rho}\sigma)(\delta\bar{\nu} + \gamma\rho\bar{\sigma}) = 0$ we get $\tau\bar{\tau}\rho\bar{\rho} = 0$ by (3) in contradiction to (1). Now set $\lambda = \varepsilon\bar{\mu}$. Then setting (14) to zero, we obtain

$$(15)\qquad \bar{\varepsilon} = \frac{\alpha\rho(\bar{\gamma}\bar{\rho} + \bar{\delta}\bar{\sigma}\nu)}{\beta\delta(\bar{\delta}\nu + \bar{\gamma}\sigma\bar{\rho})}$$

and

$$(16)\qquad \varepsilon\bar{\varepsilon} = \alpha\bar{\alpha}/\beta\bar{\beta}.$$

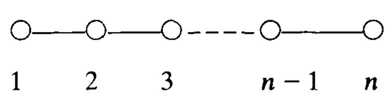
We apply (2), (3), (13), (14), (15), and (16) to find that the constant term is

$$(\delta\tau\rho\mu\eta)\overline{(\delta\tau\rho\mu\eta)}/\omega_1\omega_2$$

which is not zero by (1) and (4). Thus ω_3 is not identically zero. Hence there are at most q values of x such that $\omega_3 = 0$. Therefore the assertion is proved provided $q^2 - (3q + 3) - q > 0$ ie $q > 4$.

THEOREM 2.3. *Let G be a universal group of type A_n. Then G is isomorphic to $SU(n+1, q)$.*

PROOF. We may suppose $n \geqq 3$ and let the graph be



Let $S = \langle L_i \mid 1 \leqq i \leqq n-1\rangle$ and $T = \langle L_i \mid 1 \leqq i \leqq n-2\rangle$. Clearly each element of G belongs to $(SL_n)^k S$ for some positive integer k. We shall show by induction on k that $g \in SL_n L_{n-1} L_n S$. This is obviously true when $k = 1$.

We deal first with the case $n = 3$. Using the relation $[L_1, L_3] = 1$ and (1.2) we clearly have

$$(SL_3)^k S = S(L_3 L_2 L_1 L_2)^k S.$$

Now

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$$
\begin{aligned}
L_3L_2L_1L_2(L_3L_2L_1L_2L_3)& \\
&\subseteq L_3(L_2L_1L_2L_2L_1)L_3L_2L_3L_1L_2 && \text{by (2.2)}\\
&= L_1L_3L_2L_1(L_2L_3L_2L_3)L_1L_2 && \text{by (1.2)}\\
&= L_1(L_3L_2)L_1(L_3L_2L_3)L_2L_1L_2 && \text{by (1.2)}\\
&\subseteq SL_3L_2L_1L_2L_3S && \text{by (1.7).}
\end{aligned}
$$

Therefore we can now apply induction to show $g \in SL_3L_2L_3S$.

By (1.4) and (1.5), there is a homomorphism $\theta: G \to SU(4,q)$ mapping L_i to L_i^*. Let $g \in$ kernel of θ and let $g = s_1cbc's_2$ where $s_1, s_2 \in S$; $c, c' \in L_3$ and $b \in L_2$. Now $s_2gs_2^{-1} \in$ kernel of θ and $s_2gs_2^{-1} = scbc'$ where $s = s_2s_1$. Therefore $\theta(cbc') = \theta(s)^{-1}$. So $\theta(cbc')$ leaves v_4 fixed and therefore $\theta(cbc') \in L_2^*$ as the stabilizer of v_4 in $\langle L_2^*, L_3^*\rangle$ is L_2^*. As θ restricted to $\langle L_2, L_3\rangle$ is an isomorphism, there exists $b' \in L_2$ such that $b'cbc' = 1$. So $\theta(b's^{-1}) = 1$. As θ restricted to $\langle L_1, L_2\rangle$ is again an isomorphism, $b's^{-1} = 1$. Therefore $g = 1$ and θ is an isomorphism.

We next suppose $n > 3$ and use induction on n to show that

$$G = SL_nL_{n-1}L_nS.$$

We have by induction,

$$
\begin{aligned}
L_nSL_n &= L_nTL_{n-1}L_{n-2}L_{n-1}TL_n\\
&= TL_nL_{n-1}L_{n-2}L_{n-1}L_nT\\
&\subseteq SL_nL_{n-1}L_nS \quad \text{by (2.2).}
\end{aligned}
$$

Hence

$$
\begin{aligned}
(L_nSL_n)SL_n &\subseteq SL_nL_{n-1}L_nSL_n\\
&= SL_nL_{n-1}L_nTL_{n-1}L_{n-2}L_{n-1}TL_n\\
&= S(L_nL_{n-1}L_n)T(L_{n-1}L_{n-2}L_{n-1}L_n)T\\
&\subseteq SL_nSL_nS
\end{aligned}
$$

by (1.7), (1.5) and by what we just proved. So $(SL_n)^kS \subseteq (SL_n)^{k-1}S$ for $k \geqq 3$ and thus $G = (SL_n)^2S = SL_nL_{n-1}L_nS$ by the above. Because of (1.5), θ is an isomorphism by induction and we apply the same argument for the case $n = 3$.

The theorem is now an immediate consequence of (2.3) and (1.4).

References

Jean Dieudonné (1955), *La Géométrie des Groupes Classiques* (Ergebnisse der Mathematik und ihrer Grenzgebiete, 5. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1955).

Kok-Wee Phan (1972), 'A characterization of the finite groups $PSL(n, q)'$', *Math. Z.* **124**, 169–185.

T. A. Springer and R. Steinberg (1970), 'Conjugacy classes', *Seminar on Algebraic Groups and Related Finite Groups*, EI–E100 (Lecture Notes in Mathematics, 131. Springer-Verlag, Berlin, Heidelberg, New York, 1970).

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