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SQUARES OF DEGREES OF BRAUER CHARACTERS AND MONOMIAL BRAUER CHARACTERS

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Abstract

Let G be a finite group and let p be a prime factor of $|G|$. Suppose that G is solvable and P is a Sylow p-subgroup of G. In this note, we prove that $P \lhd G$ and G/P is nilpotent if and only if $\varphi(1)^2$ divides $|G : \ker\varphi|$ for all irreducible monomial p-Brauer characters φ of G.

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All groups throughout this note are finite. Gagola and the second author in [2] proved that a group G is nilpotent if and only if $\chi(1)^2$ divides $|G : \ker\chi|$ for all characters $\chi \in \mathrm{Irr}(G)$. Recently, under the hypothesis that G is solvable, Lu proved, in [5], that G is nilpotent if and only if $\chi(1)^2$ divides $|G : \ker\chi|$ for every monomial character $\chi \in \mathrm{Irr}(G)$. Also recently, the authors with Cossey and Tong-Viet proved, in [1], a Brauer version of the theorem of Gagola and the second author. In particular, the following result was obtained. Let p be a prime and let G be a group. Then $\phi(1)^2$ divides $|G : \ker\phi|$ for all $\phi \in \mathrm{IBr}(G)$ if and only if G has a normal Sylow p-subgroup P and G/P is nilpotent.

Inspired by these results, we consider monomial Brauer characters in this note. Our goal is to prove the following theorem.

Theorem 1. *Suppose that G is a solvable group and p is a prime divisor of $|G|$. Fix $P \in \mathrm{Syl}_p(G)$. Then P is normal in G and G/P is nilpotent if and only if $\varphi(1)^2$ divides $|G : \ker\varphi|$ for all monomial $\varphi \in \mathrm{IBr}(G)$.*

Notice that the hypothesis of G being solvable cannot be dropped. For example, let S_5 be the symmetric group of degree five and let $p = 2$. It is not difficult to see that

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S_5 has no subgroup of order 30, and it follows that all nonlinear irreducible 2-Brauer characters of G are not monomial since they are of degree four; however, S_5 has no normal Sylow 2-subgroup.

Now, we give the proof of Theorem 1.

Proof of Theorem 1. Suppose that P is normal in G and that G/P is nilpotent. Using [1, Lemma 3.4], $\varphi(1)^2$ divides $|G : \ker\varphi|$ for all $\varphi \in \mathrm{IBr}(G)$.

Conversely, assume that $\varphi(1)^2$ divides $|G : \ker\varphi|$ for every monomial Brauer character $\varphi \in \mathrm{IBr}(G)$. We work by induction on $|G|$. First, suppose that $\mathbf{O}_p(G) > 1$. Note that $G/\mathbf{O}_p(G)$ satisfies the induction hypothesis. Thus, by induction, $P/\mathbf{O}_p(G)$ is a normal Sylow p-subgroup of $G/\mathbf{O}_p(G)$. This implies that $P = \mathbf{O}_p(G)$ is a normal Sylow p-subgroup of G, and we have the result.

Thus, we may assume that $\mathbf{O}_p(G) = 1$. If we can prove that G is a p'-group, then we may apply Lu [5, Theorem 1.2] and that result gives the desired conclusion. With this in mind, we work to prove that G is a p'-group.

Let M be a minimal normal subgroup of G. Since $\mathbf{O}_p(G) = 1$, it follows that M is a p'-subgroup of G. We see that G/M satisfies the induction hypothesis. By induction, PM/M will be a normal Sylow p-subgroup of G/M and G/PM will be nilpotent. Write $PM = N$. The Frattini argument implies that

$$G = NN_G(P) = MPN_G(P) = MN_G(P).$$

Suppose that $M_1 \neq M$ is another minimal normal subgroup of G. We see that M_1 is also a p'-subgroup of G. We claim that $M_1 \cap MP = 1$. If not, then M_1 would be contained in MP. Since M is the normal p-complement of MP, we would have $M_1 \leq M$ and that is a contradiction. Applying the previous argument with M_1 in place of M, we see that $G = M_1N_G(P)$. Since M_1 and MP are normal subgroups that intersect trivially, they centralise each other and, in particular, M_1 centralises P. It follows that $G = N_G(P)$ and P is normal in G. Using the fact that $\mathbf{O}_p(G) = 1$, we obtain $P = 1$, and G is a p'-group, as desired.

Therefore, we may assume that M is the unique minimal normal subgroup of G. Since G is solvable, M is an elementary abelian q-group for some prime $q \neq p$. As $G = MN_G(P)$ and M is an abelian normal subgroup of G, we find that $M \cap N_G(P)$ is normal in G. Because M is minimal normal, either $M \cap N_G(P) = 1$ or $M \leq N_G(P)$. If $M \leq N_G(P)$, then $G = N_G(P)$ and P is normal in G. Since $\mathbf{O}_p(G) = 1$, this shows that G is a p'-group, as desired. Thus, $M \cap N_G(P) = 1$.

Observe that $N_G(P)$ normalises $C_P(M)$ and M normalises $C_P(M)$. Hence $C_P(M)$ is normal in $G = MN_G(P)$. As $\mathbf{O}_p(G) = 1$, we conclude that $C_P(M) = 1$. Therefore, P acts faithfully on M and thus it acts faithfully on $\mathrm{IBr}(M) = \mathrm{Irr}(M)$.

Applying Isaacs' large orbit result [4, Theorem B], we see that there exists a character $\lambda \in \mathrm{IBr}(M)$ so that $|\mathbf{C}_P(\lambda)| < \sqrt{|P|}$. This gives $|P : \mathbf{C}_P(\lambda)| > \sqrt{|P|}$ and so $|P : \mathbf{C}_P(\lambda)|^2 > |P|$. Write T for the inertia group of λ in G and write $S = M\mathbf{C}_P(\lambda)$. Observe that S is the stabiliser of λ in N and $S \leq T$. In particular, $S = T \cap N$. Since N/M is the Sylow p-subgroup of G/M, it follows that S/M is the Sylow p-subgroup

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of T/M. Thus, $|N : S| = |P : \mathbf{C}_P(\lambda)|$. We deduce that

$$|G : T|_p = |P : \mathbf{C}_P(\lambda)| > \sqrt{|P|} = \sqrt{|G|_p}.$$

In particular, $|G : T|^2$ does not divide $|G|$.

Observe that M is complemented in T. It follows from a result of Gallagher (see [3, Lemma 1] that there exists some Brauer character $\mu \in \mathrm{IBr}(T)$ such that $\mu_M = \lambda$. By the Clifford correspondence for Brauer characters [6, Theorem 8.9], $\varphi = \mu^G \in \mathrm{IBr}(G)$. Since μ is linear, this implies that φ is monomial and $\varphi(1) = |G : T|$. We have seen that $|G : T|^2$ does not divide $|G|$ and so this contradicts the hypothesis that the squares of the degrees of the monomial Brauer characters divide $|G|$. Therefore, we can conclude that G is a p'-group, and this gives the desired conclusion. □

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