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A representation theory for partially ordered and topological vector spaces

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In 1951, R.V. Kadison proved a general representation theorem for archimedean partially ordered vector spaces with order unit and deduced from this result the Stone Representation Theorem and the Stone-Kreĭn-Kakutani-Yosida Theorem. In the present work we show that Kadison's result may be applied in vector spaces which are not (initially) archimedean ordered vector spaces with order unit.

In Chapter II we show that, for a normed space E, Kadison's Theorem implies at least the Alaoglu representation of E as a subspace of the space of continuous functions on the closed unit ball of its dual space. In certain cases, more may be said. We are able to prove a result which, firstly, is as strong as Kadison's result and, secondly, is a generalisation of the theorems of Clarkson, Kreĭn-Kakutani (the representation of an (M)-space) and of the Gelfand-Naimark Theorem. Characterisations (some new) are given of $C(X)$ as a normed space. At the end of the chapter, a topological proof is given of Kadison's result. Because Kadison's result applies to ordered vector spaces we must introduce an order into the normed space E in order to apply it. This is the concern of Chapter I, which not only introduces the order but establishes some useful properties, many of which are needed for Chapter II.

In Chapter III we revert to partially ordered vector spaces. Recently Kung-Fu Ng has proved some theorems on the representation of "approximate

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order unit normed spaces". This class of spaces includes the archimedean partially ordered vector spaces with order unit and is in some sense the class of partially ordered vector spaces which "almost" have an order unit. The techniques Kung-Fu Ng used to prove these results are techniques within the theory of approximate order unit normed spaces. We show that these results may in fact be deduced from the corresponding results in the theory of partially ordered vector spaces with order unit. This is the content of the first part of Chapter III. The next part looks at a different sort of representation. The representing space is once again $C(X)$ (X compact, Hausdorff) as in the previous representations, but instead of carrying its usual pointwise order it now carries the order:

$$f > 0 \text{ if, and only if, } f(x) > 0 \text{ for all } x \in X .$$

We characterise the ordered subspaces of $(C(X), \leq)$ which contain the constant functions. In the last section of this chapter we look at the general problem of representing partially ordered vector spaces as subspaces of $C(X)$. Among those subspaces containing the constants, Kadison's result characterises those which have the usual order, and our result above characterises those which have the order $\leq$ defined above. In this last section we look at representations in terms of other orders on $C(X)$.

The last chapter, Chapter IV, examines the "Riesz Interpolation Property" in partially ordered vector spaces. The importance of this property has already been amply demonstrated in the work of others and also features in some of our representation theorems. Thus our concern is to examine it more closely. Results are demonstrated which link this property to lattice properties and to some other interpolation properties. The chapter ends with some characterisations of R^n as a partially ordered vector space and a discussion of orders in R^n.