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ON BALANCED ONE-SIDED IDEALS

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The relation between the subclass χ of balanced one-sided ideals, which contains all two-sided ideals, and weakly prime one-sided ideals is considered. If a member P of χ is prime, then it is prime in its idealizer $I(P)$. Furthermore, if the ring is left and right noetherian, then $I(P)$ modulo P is a prime Goldie ring.

Introduction.

In this paper we study aspects of a situation which is fairly common in for example matrix rings: L and R are a left and a right ideal respectively, the largest two-sided ideal contained in L equals that contained in R, and, modulo this two-sided ideal, L is the left annihilator of R and vice versa. We call such a pair (L,R) a balanced pair, and derive some elementary properties of balanced pairs in section 1.

In section 2 we consider briefly left ideals which satisfy a necessary condition to be one component of a balanced pair, namely those which satisfy a double annihilator condition modulo the largest two-sided ideal which they contain. Such left ideals are called right balanced

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weakly prime (see [3]) left ideals.

Section 3 is devoted to right balanced prime left ideals. It is shown that such a left ideal conforms to Fitting's [1] definition of a prime left ideal, namely it is a prime two-sided ideal of its idealizer. Furthermore, if the ring is left and right noetherian modulo the largest two-sided ideal contained in the right balanced prime left ideal L, then $I(L)/_L$ is a prime Goldie ring.

In all our considerations, the ring A is associative and possesses an identity.

1. Balanced Pairs of One-sided Ideals.

For a left ideal L and a right ideal R of A we define

$$L_R := \{x \in A \mid xR \subseteq L\},$$

$$L^R := \{x \in A \mid Lx \subseteq R\},$$

$$\rho(L) := \{x \in A \mid LxA \subseteq L\}$$

and

$$\lambda(R) := \{x \in A \mid AxR \subseteq R\}.$$

Some elementary facts about $\rho(L)$ (and, dually, $\lambda(R)$) are stated in the following

PROPOSITION 1.1.

(a) *$\rho(L)$ is the sum of all right ideals of A contained in the idealizer $I(L)$ of L,*

(b) *$\rho(L)$ is the largest right ideal of A contained in $I(L)$,*

(c) *$\rho(L)$ is a two-sided ideal of $I(L)$,*

(d) *$\rho(L)$ is the right annihilator of L modulo L_A,*

(e) *$I(L) \subset I(\rho(L))$,*

(f) *$L \subset \lambda(\rho(L))$.* □

DEFINITION. A pair (L,R) where L is a left and R a right ideal of A, is called a balanced pair if $\rho(L) = R$ and $\lambda(R) = L$.

The following result contain some easy deductions from Proposition 1.1. We recall that a left ideal L is weakly prime if for left ideals J and K, with $JK \subseteq L \subseteq J \cap K$ imply $J = L$ or $K = L$. See [3] for details.

PROPOSITION 1.2. *If* (L,R) *is a balanced pair, then*

(a) $I(L) = I(R)$,

(b) L *is a weakly prime left ideal,*

(c) R *is a weakly prime right ideal,*

(d) $L_A = {}_AR$,

(e) $\rho(L^n) = R$, $n \in \mathbb{N}$.

Proof. (a) follows from Proposition 1.1 (d) and its dual.
(b) and (c) follow from Proposition 1.1 (a) and its dual.
(d) Clearly $L_A \subset \rho(L) = R$, which implies that $L_A \subset {}_AR$. The dual statement completes the proof.
(e) By [3], $I(L) = I(L^n)$, because L is weakly prime. So, R is the largest right ideal of A contained in $I(L^n)$. □

PROPOSITION 1.3. *If* (L,R) *is a balanced pair, then* $L_R = L$ *and dually* $L^R = R$.

Proof. Clearly

$$\begin{aligned} L_R &= \{x \in A \mid xR \subseteq L\}, \\ &= \{x \in A \mid xR \subseteq L_A = {}_AR\} , \\ &= \{x \in A \mid xR \subseteq R\} = \lambda(R) = L . \end{aligned}$$ □

2. Balanced One-sided Ideals.

It is interesting to note that all two-sided ideals L satisfy the conclusion of Proposition 1.3, namely that $L_{\rho(L)} = L$. However, from Proposition 1.2 (b) it is clear that (L,A) is a balanced pair if and only if $L = A$.

In the sequel we consider left ideals L such that $L_{\rho(L)} = L$.

DEFINITION. A left ideal L is called right balanced (abbreviated as *rb*) if $L_{\rho(L)} = L$.

In view of Proposition 1.2, we have that

PROPOSITION 2.1. *If* L *is a rb left ideal of* A *, then:*

(a) $I(L) = I(\rho(L))$,

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(b) $\rho(L)$ *is a weakly prime right ideal.*

Proof. (a) If $x \in I(\rho(L))$, then $x.\rho(L) \subseteq \rho(L)$ and hence $Lx.\rho(L) \subseteq L$. But then $Lx \subseteq L$, that is, $x \in I(L)$. Proposition 1.1 (c) completes the proof.

(b) From [3], the dual of Proposition 1.1 (a) suffices as proof. □

In the following example it is shown that not all *rb* left ideals are weakly prime.

EXAMPLE 2.2.

Let $A = \begin{bmatrix} Z & Z_6 \\ 0 & Z_6 \end{bmatrix}$, where Z_6 is the integers modulo 6 ,

and $L = \begin{bmatrix} 2Z & 0 \\ 0 & 0 \end{bmatrix}$. It is easily verified that

$$\rho(L) = \begin{bmatrix} Z & 3Z_6 \\ 0 & Z_6 \end{bmatrix} \text{ and } \lambda(\rho(L)) = \begin{bmatrix} Z & 3Z_6 \\ 0 & 3Z_6 \end{bmatrix}.$$ □

The following result provides a criterion for a *rb* left ideal to be weakly prime.

PROPOSITION 2.3. *If* L *is a* *rb* *left ideal of* A, *then the following are equivalent:*

(a) $L_A = {}_A\rho(L)$,

(b) L *is weakly prime.*

Proof.

(a) ⟹ (b). If $a \in A$ is such that $L.Aa \subseteq L$, then

$$L.Aa\,\rho(L) \subseteq L.\,)L) \subseteq L_A = {}_A\rho(L)$$

and hence $Aa.\rho(L) \subseteq {}_A\rho(L) = L_A$. Since $L_{\rho(L)} = L$, we conclude that $a \in L$.

(b) ⟹ (a): Clearly $L_A \subseteq {}_A\rho(L)$. Therefore, if $x \in {}_A\rho(L)$, then $LAxA \subseteq L$. Since L is weakly prime, we have by [3] that $AxA \subseteq L_A$. □

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If any one of the above two conditions are satisfied, then $(L, \rho(L))$ is a balanced pair.

Another way in which a balanced pair can be obtained from a rb left ideal is

PROPOSITION 2.4. *If L is a rb left ideal of A such that $L.\lambda(\rho(L)) = L$, then $(\lambda(\rho(L)), \rho(L))$ is a balanced pair.*

Proof. It suffices to show that $I(\lambda(\rho(L)) = I(L)$. Thus, if $x \in I(\lambda(\rho(L))$, then

$$\lambda(\rho(L))x.\rho(L) \subseteq \lambda(\rho(L)).\rho(L) \subseteq \rho(L) ,$$

and hence, by our hypothesis,

$$Lx.\rho(L) = \{L.\lambda(\rho(L))\}x.\rho(L) \subseteq L.\rho(L) \subseteq L.$$

Since L is rb, $Lx \subseteq L_{\rho(L)} = L$. □

3. Balanced Prime One-sided Ideals.

Throughout this section P is a rb prime left ideal. Since P is prime it is weakly prime and hence $(P, \rho(P))$ is a balanced pair. Furthermore:

PROPOSITION 3.1. *If P is a rb prime left ideal, then:*

(a) $\rho(P)$ is a prime right ideal,

(b) P and $\rho(P)$ are both prime two-sided ideals of $I(P)$.

Proof. (a) If $xA.yA \subseteq \rho(P)$, then $(PxA).(PyA) \subseteq P$. But since P is prime, either $PxA \subseteq P$, whence $x \in \rho(P)$, or $PyA \subseteq P$, whence $y \in \rho(P)$.

(b) If $a,b \in I(P)$ such that $a.I(P).b \subseteq P$, then $a\rho(P).A.b\rho(P) \subseteq P$. Since P is prime in A, $a.\rho(P) \subseteq P$ or $b.\rho(P) \subseteq P$, that is $a \in P$ or $b \in P$.

Similarly, it is shown that $\rho(P)$ is prime in $I(P)$. □

COROLLARY 3.2. *$P \cap \rho(P)$ is the prime radical of P_A in $I(P)$.*

Proof. Clearly, the prime radical of P_A is contained in $P \cap \rho(P)$. Conversely, since $\{P \cap \rho(P)\}^2 \subseteq P_A$ it follows that $P \cap \rho(P)$ is contained in the prime radical of P_A in $I(P)$. □

PROPOSITION 3.3. *If A satisfies the ascending chain condition on right annihilators modulo P_A, then so does $I(P)$ modulo P.*

Proof. It suffices to show that if $P \subset \Lambda_1 \subset \Lambda_2$ are non-trivial right annihilators modulo P in $I(P)$, then $\Lambda_1.\rho(P) + P_A \subset \Lambda_2.\rho(P) + P_A$ are right annihilators of A modulo P_A.

If $\Lambda_1^{\perp} \supset \Lambda_2^{\perp} \supset P$ are the left ideals of $I(P)$ such that

$$\Lambda_i^{\perp}.\Lambda_i \subseteq P \ (i = 1,2) ,$$

then

$$[\Lambda_i^{\perp}). \ (\Lambda_i \ \rho(P) + P_A) \subseteq P_A \quad (i = 1,2) ,$$

where $[\Lambda_i^{\perp})$ is the left ideals of A generated by $\Lambda_i^{\perp}$. Furthermore, if $\Lambda_1.\rho(P) + P_A = \Lambda_2.\rho(P) + P_A$, then by Proposition 3.1 (b), $\Lambda_1^{\perp} \Lambda_2 \subseteq P$, that is, $\Lambda_2 \subseteq \Lambda_1$. □

Similarly it is shown that:

PROPOSITION 3.4. *If A satisfies the ascending chain condition on left annihilator ideals modulo P_A, then so does $I(P)$ modulo P.*

Proof. If Q is a left annihilator of ${}^{\perp}Q$ modulo P in $I(P)$, then AQ is the left annihilator of ${}^{\perp}Q.\rho(P)$ in A modulo P_A. □

With little adaptation, we can show that:

PROPOSITION 3.5. *If A modulo P_A satisfies the ascending chain condition on left (respectively, right) annihilator ideals, then so does $I(P)$ modulo $\rho(P)$.* □

With the help of the previous results, it is now possible to show that:

THEOREM 3.6. *Let P be a rb prime left ideal of A. If A is such that it is left and right noetherian modulo P_A, then $I(P)$ modulo P is a prime Goldie ring.*

Proof. In view of Propositions 3.3 and 3.4, we need show only that $I(P)$ modulo P is of finite left and right uniform dimension.

Let $\{U_i\}_{i\in I}$ be a family of left ideals of $I(P)$ containing P such that $U_i \cap (\sum_{i\neq j} U_j) \subseteq P$. Since A modulo P_A is left noetherian, there exists a finite subset F of I such that $A.U_i \subseteq \sum_{j\in F} AU_j$ for all i. But then $\rho(P).U_i \subseteq \sum_{j\in F} \rho(P).U_j$. Consequently, since U_i are left ideals of $I(P)$,

$$\rho(P).U_i \subseteq (\sum_{j\in F} U_j) \cap U_i \subseteq P ,$$

for all $i \in I\backslash F$. Since P is prime, we have that $U_i \subseteq \sum_{j\in F} U_j$, for all $i \in I\backslash F$.

On the other hand, let $\{V_i\}_{i\in I}$ be a family of right ideals of $I(P)$ containing P such that $P \supseteq V_i \cap (\sum_{i\neq j} V_j)$. Again, since A modulo P_A is right noetherian, we have that there exists a finite subset T of I such that $V_i.\rho(P) \subseteq \sum_{t\in T} V_j.\rho(P) + P_A$, for all $i \in I$. Furthermore, we have that

$$V_i.\rho(P) \subseteq (\sum_{t\in T} V_t) \cap V_i \subseteq P ,$$

for all $i \in I\backslash T$. Again, using the primeness of P in $I(P)$, we conclude that $V_i \subseteq P$, for all $i \in I\backslash T$. □

Dually:

THEOREM 3.7. *Let P be a rb prime left ideal of A. If A is such that it is left and right noetherian modulo P_A, then $I(P)$ modulo $\rho(P)$ is a prime Goldie ring.* □

Another important result related to P is:

PROPOSITION 3.8. *If P is a rb prime left ideal of A, then $I(P) = I(P \cap \rho(P))$, where $I(P \cap \rho(P))$ is the largest subring of A which contains $P \cap \rho(P)$ as a two-sided ideal.*

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Proof. If $x \in A$ is such that $x(P \cap \rho(P)) \subseteq P \cap \rho(P)$, then

$$x.\rho(P).P \subseteq x(P \cap \rho(P)) \subseteq P \cap \rho(P) \subseteq \rho(P) .$$

By Proposition 3.1 (b), it follows that $x.\rho(P) \subseteq \rho(P)$, that is, $x \in I(\rho(P)) = I(P)$, by Proposition 1.2 (a).

Similarly, it follows that if $x \in A$ such that $(P \cap \rho(P)).x \subseteq P \cap \rho(P)$, then $x \in I(P)$. □

Since the subdirect product of prime Goldie rings is Goldie, we have that:

THEOREM 3.9. *Let* P *be a rb prime left ideal of* A. *If* A *is such that* A *modulo* P_A *is left and right noetherian, the* $I(P)$ *modulo* $P \cap \rho(P)$ *is a semiprime Goldie ring.*

Proof. We need only recognize $I(P)$ modulo $P \cap \rho(P)$ as a subdirect product of $I(P)$ modulo P and $I(P)$ modulo $\rho(P)$. □

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