We might dispense with the inequalities (3) by writing

$$\left(\cos\frac{x}{n}\right)^n = \left(1 - 2\sin^2\frac{x}{2n}\right)^n$$

and putting a equal to $2\sin^2\dfrac{x}{2n}$; it is obvious that for large values of n the function $2\sin^2\dfrac{x}{2n}$ differs but little from $\dfrac{x^2}{2n^2}$, and therefore na but little from $\dfrac{x^2}{2n}$.

So far as the limit of $\left(\cos\dfrac{x}{n}\right)^n$ is concerned, we may, in place of the inequalities (1), put the single theorem

$$\mathop{\mathrm{L}}_{n \to \infty} (1-a)^n = 0$$

if na is a positive proper fraction. The restriction that n should tend to infinity by integral values is easily removed, for n will lie between two integers, m and $m+1$ say. Then

$$(1-a)^{m+1} < (1-a)^n < (1-a)^m,$$

and therefore, by applying the inequalities (1),

$$1-(m+1)a < (1-a)^n < \frac{1}{1+ma}.$$

G. A. GIBSON.

Centre of Curvature.

Professor Bryan's article on Curvature, etc., in your last issue (p. 219) ends with a challenge. I wonder whether he would be satisfied with the following reasoning to prove that the intersection of "consecutive" normals and the centre of the circle through these "consecutive" points of a curve have the same point as limiting position.

The circumcentre of the triangle formed by three points on a curve is the intersection of the mid-normals of the sides PQ, QR, and these are distant by infinitesimal amounts of higher order than PQ, QR from the normals to the curve at the points of the arcs PQ and QR where they are touched by tangents parallel to the chords PQ and QR respectively.

Hence the limiting positions of the intersections of the mid-normals of PQ, QR on the one hand, and of these two normals to the curve on the other, are coincident. But the former, as we see, is also the limiting position of the circumcentre of PQR when these points approach coincidence.

This reasoning is no doubt somewhat incomplete, but it has the advantage of being geometrical rather than analytical, and therefore applicable when Professor Bryan's proof, depending as it does on the assumption that y is expressible as a power series of x, might not be available.

But to me it seems that the most fundamental and satisfactory definition of curvature is the rate of change of direction per unit-length of arc $\left(\text{having as its measure } \dfrac{d\theta}{ds}\right)$. The corresponding definition of circle of curvature is that circle which touches the curve at the point considered, and has the same curvature. The radius of this circle is the radius of curvature ρ, and its centre is the centre of curvature. From the property "radian measure of angle at centre = arc : radius," it follows that the curvature of the circle, and therefore of the curve at the point considered, is $\dfrac{1}{\rho}$.

To prove that the centre of curvature thus defined is the intersection of consecutive normals, we might proceed thus:

Let the normals at points P and Q of a curve intersect in O, and let the circle, centre O, radius OP, cut OQ in q.

Then if θ be the radian measure of $P\widehat{O}q$, we have $OP = \text{Arc } Pq : \theta$.

Hence $$\lim_{PQ \to 0} OP = \text{Lim.} \left(\frac{\text{Arc } Pq}{\text{Arc } PQ} \cdot \frac{\text{Arc } PQ}{\theta}\right).$$

Now, by definitions, $\text{Lim.} \dfrac{\text{Arc } PQ}{\theta} = \rho.$

And by intuition $\text{Lim.} \dfrac{\text{Arc } Pq}{\text{Arc } PQ} = 1.$

Hence $\text{Lim. } OP = \rho.$

Thus the limiting position of the intersection of consecutive normals is the centre of curvature as defined above.

Here the only step depending on an intuitive postulate is where we say $\text{Lim.} \dfrac{\text{Arc } Pq}{\text{Arc } PQ} = 1.$

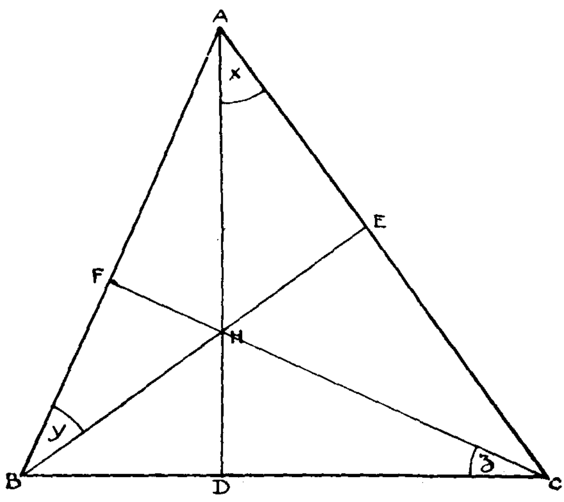
Of course the complete discussion of the restrictions under which this postulate could be proved would open up the whole thorny question of the nature of a curve in general; but I think there would be no great harm in admitting that, unless the curve has the property $\text{Lim.}\ \frac{\text{Arc}\ Pq}{\text{Arc}\ PQ} = 1$, the proposition must be regarded as unproved.

It might not be difficult to show that this postulate must hold good in every case where the arc has a definite centre of curvature.

R. F. MUIRHEAD.

Geometrical proof that $\tan x \tan y + \tan y \tan z + \tan z \tan x = 1$ when $x + y + z = 90°$.

H being the orthocentre of a triangle ABC, we may call the angles HAC, HBA, $HCB = x$, y, z respectively, for their sum is 90°.



Now $\tan x \tan z = \frac{DC}{DA} \cdot \frac{HD}{DC} = \frac{HD}{DA} = \frac{\triangle BHC}{\triangle ABC}$,

$$\therefore \quad \tan x \tan z + \tan z \tan y + \tan x \tan y = 1.$$

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