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PARAMETERISATION OF DEVELOPABLE SURFACES BY ASYMPTOTIC LINES

VITALY USHAKOV

An example of a "non-developable" surface of vanishing Gaussian curvature from W. Klingenberg's textbook is considered and its place in the theory of 2-dimensional developable surfaces is pointed out. The surface is found in explicit form. Other examples of smooth developable surfaces not allowing smooth asymptotic parametrisation are analysed. In particular, Hartman and Nirenberg's example (1959) is incorrect.

In Klingenberg's textbook [**4**, pp.68–69] an example of a "surface with vanishing curvature which is not a developable surface" is described. Unfortunately, this example is rather vague and several questions remain open.

1. The surface itself does not appear in the example, but the fundamental theorem of surface theory about the existence of the surface with prescribed first and second fundamental forms is employed.

2. It remains unclear in which sense the surface is not developable. The definition of a developable surface in the textbook is not very apt: a ruled surface whose normal vector is a constant along generators. In fact, such a surface has its own classical name—a *torse*—due to Euler. The classical meaning of "developable" is that the surface can be developed, bent onto the plane: "a 'development' of one surface on another is the very classical name for an isometry" [**9**, p.212]. A torse and a developable surface are very similar to each other [**2**, Theorem 5], but not exactly the same, as the example demonstrates. Thus, Klingenberg appears to be claiming that the surface either

(1) is not ruled; or
(2) is ruled, but the normal is not a constant along generators.

It turns out, however, that both properties are satisfied. But the surface is ruled only from the geometrical point of view (there exists a unique rectilinear generator passing through every point), but is not quite ruled from the analytical point of view: the

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direction vector of the generators is continuous but not differentiable. (Klingenberg claims, with a rather vague proof, that there is no suitable change of variables; in fact, such a change exists but is only C^0.) The wonderfulness of this example is precisely in the fact that despite C^∞ smoothness of the surface, its asymptotic parametrisation is only continuous.

In our opinion the name "a surface of vanishing curvature not allowing a smooth asymptotic parametrisation" would be more apt for the example.

The aim of this paper is to sort out the geometric picture of Klingenberg's example. For this purpose we include two classical theorems about the structure of surfaces of vanishing Gaussian curvature (Section 1) and a theorem about the smoothness of asymptotic parametrisation for such surfaces (Section 2). In Section 3 we describe carefully Klingenberg's example. In Section 4 we investigate the appearance of the surface and even give a drawing of it. In Section 5 we shall discuss other examples of developable surfaces not allowing a smooth asymptotic parametrisation.

1. The structure of the surfaces of vanishing Gaussian curvature

By a *surface* is meant a 2-dimensional manifold in 3-dimensional Euclidean space parametrised over a simply-connected domain in $\mathbb{R}^2$. A *developable surface* is a surface isometric to a part of the plane, or a surface of vanishing Gaussian curvature. (The equivalence of these definitions is proved in [**2**, Theorem 4].)

The vanishing of the Gaussian curvature at some point Q implies vanishing of one of the two principal curvatures. A point at which both principal curvatures vanish is called a *planar* point. Thus, a developable surface consists of planar and non-planar points. (The latter have just one of the principle curvatures non-vanishing.)

In order to calculate the Gaussian curvature one needs C^2 smoothness of a surface. This provides the existence of the second fundamental form. A.V. Pogorelov studied developable surfaces [**6**, **7**] (an exposition of [**7**] was given in chapter IX of a book [**8**]), considering instead of C^2 smooth surfaces the more general *surfaces of bounded exterior curvature*—surfaces of C^1 smoothness with bounded total variation of the spherical Gauss map. (The notion of *variation of a function* can be found in [**10**, p.366], for example.) Under such an approach the generalisation of developable surfaces is *surfaces of vanishing exterior curvature*—C^1 smooth surfaces with vanishing total variation of the spherical Gauss map.

Theorem 1. [**8**, pp.694–695] *Let F^2 be a surface of vanishing exterior curvature and Q be a point such that there is no neighbourhood of Q entirely lying in a plane. Then there exists a unique segment $l \subset F^2$ passing through Q with ends on the boundary of F^2 (or at ∞). The normal direction to the tangent plane is constant along the segment l.*

Therefore, the surface consists of flat and ruled parts; every generator (rectilinear segment) of a ruled part has tangent plane stable along it and both ends of the generator lie on the boundary of F^2 (or at ∞); a flat part is a piece of a plane and can go out to the boundary of F^2 or be bounded by rectilinear segments, each of which joins two points of the boundary of F^2 (or goes to ∞).

In other words, by moving along a generator of a nonflat part of the surface we can either reach the boundary of F^2 or go to infinity, but can never find ourselves on a flat part. While moving along a flat part, we can add to the previous two possibilities another one: we can go out on a rectilinear generator transverse to the direction of the movement.

The simple corollary of this theorem is Pogorelov's well-known theorem about complete surfaces of vanishing curvature:

THEOREM 2. [**8**, p.696] *A complete surface of vanishing exterior curvature is a cylinder.*

Several proofs of this theorem (for C^∞ surfaces) can be found in [**9**, pp.363–367].

In 1959 Hartman and Nirenberg rediscovered Theorems 1 and 2 for C^2-smooth surfaces [**1**]. In 1962 W. Massey published a paper [**5**], where he stated that Theorem 2 was "announced by Pogorelov without proof in 1956". And in a footnote Massey noticed: "To the best of my knowledge, Pogorelov has not as yet published a proof". Since then it is generally believed that Pogorelov's proof was published only in 1969 [**8**]. But in fact Chapter IX of the book [**8**] is only an exposition of the book [**7**] as we mentioned before.

Besides that, in [**1**] Hartman and Nirenberg investigated the following important aspect of developable surfaces.

2. Parametrisation by asymptotic lines

Every ruled surface has *a standard parametrisation* (by asymptotic lines)

$$r(u,v) = \rho(u) + v \cdot s(u), \tag{1}$$

where the vector-valued function $\rho(u)$ gives a directrix, and the non-vanishing vector-valued function $s(u)$ gives the directions of the generators. Given a smooth surface of vanishing Gaussian curvature one can try to reparametrise its ruled parts with a standard parametrisation. What can the smoothness of such a change of variables be?

THEOREM 3. [**1**, pp. 916–917] *Let F^2 be a C^2-surface $z = z(x,y)$ given over the unit disk $D = \{x^2 + y^2 < 1\}$. Let F^2 have vanishing Gaussian curvature and let the set of non-planar points be dense on D. Then there exists a standard reparametrisation* *such that $\rho(u)$ is C^2 and $s(u)$ is C^0.*

(1) *If there are no planar points, then $s(u)$ is C^1, but for a general surface, the vector $s(u)$ is not C^2.*

(2) *If there exists at least one planar point (hence, a line segment of planar points), then in general $s(u)$ is not C^1.*

The following example describes precisely the case (2): even for a C^∞ developable surface the direction vector of the generators need not be C^1 smooth.

3. Klingenberg's example

Let us consider two fundamental forms over the strip $\mathbb{R} \times (-1,1)$:

$$(2) \qquad \begin{cases} (g_{ij}) &= (\delta_{ij}) \\ (h_{ij}) &= \exp\Big(-\Big(\dfrac{1+y\cdot\operatorname{sgn}(x)}{x}\Big)^2\Big)\cdot P_{ij}(x,y) \end{cases} \qquad (i,j=\overline{1,2}),$$

where

$$P_{11} = \frac{1}{1+y\cdot\operatorname{sgn}(x)}; \qquad P_{12} = \frac{-|x|}{(1+y\cdot\operatorname{sgn}(x))^2}; \qquad P_{22} = \frac{x^2}{(1+y\cdot\operatorname{sgn}(x))^3}.$$

It is not difficult to see that

(i) h_{ij} is C^∞;

(ii) $h_{11}\cdot h_{22} - (h_{12})^2 = 0$;

(iii) $h_{11,2} = h_{12,1}$ and $h_{12,1} = h_{22,1}$;

(iv) from (ii) and (iii) one can derive that the forms (g) and (h) satisfy the Gauss and Codazzi-Mainardi equations; therefore, by the fundamental theorem of surface theory, there exists a C^∞ smooth surface $F^2 \subset E^3$ with the fundamental forms (g) and (h) (let us denote its radius vector by $r : \mathbb{R}\times(-1,1) \to E^3$);

(v) the Gaussian curvature (of the surface) is $K = h_{11}\cdot h_{22} - (h_{12})^2 = 0$;

(vi) the planar points, that is, ones with $h_{11} = h_{12} = h_{22} = 0$, fill out the interval $(-1,1)$ of the y-axis; the strip $\mathbb{R}\times(-1,1)$ is divided into two parts $x > 0$ and $x < 0$; so, the surface F^2 is glued from two ruled parts;

(vii) the preimages of the generators are the rays emanating from the point

$$\begin{cases} (0,-1) & \text{for the part } x > 0, \\ (0,1) & \text{for the part } x < 0; \end{cases}$$

these rays are the integral lines of the line field annihilating the matrix (h_{ij}) of the second fundamental form;

(viii) if one attempts to reparametrise the surface in the standard way (1) there is an irregularity.

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The last statement can be seen from the vector point of view. Indeed, let us introduce in the strip $\mathbb{R} \times (-1,1)$ a new coordinate system (u,v) in the form:

$$\begin{pmatrix} x \\ y \end{pmatrix} = a(u) + v \cdot b(u),$$

where $a(u) = (r_*)^{-1}(\rho(u))$ is the preimage of a directrix and $b(u) = (r_*)^{-1}(s(u))$ gives the directions of the preimages of the generators. As a directrix we can take the image of the x-axis and choose a direction vector $b(u)$ such that its projection onto the y-axis equals 1. Then from (vii)

$$a(u) = \begin{pmatrix} u \\ 0 \end{pmatrix}, \qquad b(u) = \begin{pmatrix} |u| \\ 1 \end{pmatrix} \qquad (u \in \mathbb{R})$$

and the change of variables is:

(3) $$\begin{cases} x &= u + v \cdot |u| \\ y &= v \end{cases} \iff \begin{cases} u &= \dfrac{x}{1 + y \cdot \operatorname{sgn}(x)} \\ v &= y\,. \end{cases}$$

It is easy to see that $b(u)$ is C^0, but $b(u)$ is not C^1:

$$b'(+0) - b'(-0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Therefore, the change of variables (3) is not C^1 smooth. And since the radius vector $r(x,y)$ is C^∞, the composition $\widetilde{r}(u,v) = r(x(u,v), y(u,v))$ is not C^1. Any other standard parametrisation of the surface has the same sort of irregularity—a jump of the direction vector—since all the standard parametrisations are connected to each other in a smooth way.

4. Geometric structure of the surface

4.1. The symmetry of the surface. As we already know, the surface is glued from two ruled parts along their common rectilinear generator $r(0 \times (-1,1))$. It is not difficult to show that these two parts are symmetric with respect to the normal of the surface at the point $r(0,0)$. Indeed, let us put $F_-^2 = r((-\infty,0] \times (-1,1))$, $F_+^2 = r([0,\infty) \times (-1,1))$ and consider the map

$$\phi : (-\infty, 0] \times (-1,1) \to [0,\infty) \times (-1,1),$$

given by the formula $\phi(x,y) = (-x,-y)$. Then there is the induced map

$$\widetilde{\phi} = r \circ \phi \circ r^{-1} : F_-^2 \to F_+^2.$$

And since the first and second fundamental forms are invariant with respect to $\widetilde{\phi}$, the surfaces F^2_+ and F^2_- can be superposed by a movement (due to the fundamental theorem of surface theory). In order to understand the nature of this movement, it is sufficient to consider the point $Q = r(0,0)$, which remains invariant under the map $\widetilde{\phi}$. The differential $\widetilde{\phi}_* : T_Q F^2_- \to T_Q F^2_+$ maps $e_1 \mapsto -e_1$, $e_2 \mapsto -e_2$. $\Big($Here $e_1 = r_*\Big(\dfrac{\partial}{\partial x}\Big)$, $e_2 = r_*\Big(\dfrac{\partial}{\partial y}\Big).\Big)$ The normal n is usually chosen so that the triple (e_1, e_2, n) is right-handed. So the normal at the point Q has the same direction for both parts. Therefore, the movement superposing F^2_+ and F^2_- is the reflection in the normal n.

4.2. F^2_+ IS A CONE. Since the preimages of all the generators of F^2_+ pass through the point $(0,-1)$ (see (vii)) F^2_+ is a cone. This can be rigorously proved by the following reasoning. The geodesic curvature of any trajectory on F^2_+ orthogonal to the generators is constant (the geodesic curvature and orthogonality are invariant under bendings, and F^2_+ can be bent into its parameter domain $[0,\infty) \times (-1,1)$; orthogonal trajectory is mapped into an arc of a circle with the centre $(0,-1)$). This property uniquely identifies the cone among the flat ruled surfaces.

Therefore, the surface is glued from two identical cones F^2_+ and F^2_-.

4.3. A DIRECTRIX OF F^2_+. In order to find the surface in explicit form it remains to understand how a directrix of, say, F^2_+ is imbedded in the ambient E^3. As a directrix in the parameter domain let us take the arc of the circle with the centre at the point $(0,-1)$, radius 1, and lying in the semistrip $[0,\infty) \times (-1,1)$. For investigating this directrix it is convenient to introduce the polar coordinate system (v,ϕ) such that its vertex is at the point $(0,-1)$, the polar axis is the y-axis and the measurement of the angles is clockwise:

$$\left\{\begin{array}{ll} x & = v \cdot \sin\phi \\ y & = v \cdot \cos\phi - 1 \end{array}\right. \qquad \left(v \in \left[0; \frac{2}{\cos\phi}\right), \quad \phi \in \left[0; \frac{\pi}{2}\right)\right). \tag{4}$$

In this coordinate system the directrix is given by the equation $v = 1$ $(\phi \in (0, \pi/2))$. Let us denote its radius vector by $\rho(\phi)$. Setting the origin of a Cartesian coordinate system in the ambient E^3 at the point $r(0,-1)$, the directrix $\rho(\phi)$ lies in the unit sphere S^2. Let us show that the geodesic curvature $\rho(\phi) \subset S^2$ is given by the formula

$$k_g = \frac{\exp\{-\cot^2\phi\}}{\cos^3\phi} \qquad \left(\phi \in \left(0, \frac{\pi}{2}\right)\right). \tag{5}$$

Indeed, since ϕ is the natural parameter on $\rho(\phi)$, the geodesic curvature k_g equals the length of the projection of $\ddot{\rho}$ onto the tangent plane of the sphere S^2. At the same time the curve $\rho(\phi)$ is the line of intersection of two orthogonal surfaces—the cone F^2_+

and the sphere S^2. So the projection of $\ddot{\rho}$ onto TS^2 equals the projection of $\ddot{\rho}$ onto the normal of the cone F_+^2, that is, the coefficient $h_{\phi\phi}$ of the second fundamental form of the cone F_+^2 in the coordinate system (v, ϕ). Substituting the values of x and y from (4) in the formulae (2) the matrix $H(x,y)$ becomes the matrix

$$H(v,\phi) = \frac{\exp(-\cot^2\phi)}{v \cdot \cos\phi} \cdot \begin{pmatrix} 1 & -\tan\phi \\ -\tan\phi & \tan^2\phi \end{pmatrix}.$$

It is not as yet the matrix of the second fundamental form in the coordinate system (v, ϕ) as long as we have not changed the variables in the tangent space: the above matrix is the matrix $H(v; \phi, dx, dy)$ of the bilinear form in dx, dy; its elements are h_{xx}, h_{xy} and h_{yy}, while we need h_{vv}, $h_{v\phi}$ and $h_{\phi\phi}$. The entries of the desired matrix can be computed from

$$h_{vv} = h_{xx} \cdot \left(\frac{\partial x}{\partial v}\right)^2 + 2 \cdot h_{xy} \cdot \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial v} + h_{yy} \cdot \left(\frac{\partial y}{\partial v}\right)^2,$$

et cetera. Thus

$$H(v,\phi; dv, d\phi) = \frac{v \cdot \exp\{-\cot^2\phi\}}{\cos^3\phi} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

And since the directrix is given by the equation $v = 1$, then $k_g = h_{\phi\phi}(1, \phi)$, which proves formula (5).

Therefore, the directrix $\rho(\phi)$ is a curve inside the unit sphere parametrised by the natural parameter $\phi \in (0, \pi/2)$. At the initial point $\phi = 0$ the curve is very close to its tangent (with infinite order of osculation—like the function $\exp\{-1/x^2\}$ at 0). Then the curve begins to twist into a spiral (the geodesic curvature monotonically increases) and as ϕ approaches $\pi/2$ it twists more and more tightly (the geodesic curvature goes to infinity).

4.4. THE APPEARANCE OF THE SURFACE. The surface consists of two cones glued together along their common generator. The cones are symmetric with respect to the normal at the point $r(0,0)$. Each of the cones osculates the tangent plane at the points of this generator with infinite order of osculation. Each of the cones can be parametrised by its asymptotic lines (generators) and such a parametrisation has C^∞ smoothness. Nevertheless, the asymptotic parametrisation of the whole surface is only continuous.

Therefore, the surface is fibered with rectilinear generators, that is, for every point of the surface there exists a unique rectilinear generator passing through this point. The tangent plane is constant along these generators. But the family of the generators has a jump in the first derivative of the direction vector—the irregularity. And if we picture a ruled surface as a surface formed by a movement of a straight line (one of the definitions of a ruled surface), then our surface is ruled, but the movement is only continuous and is not C^1.

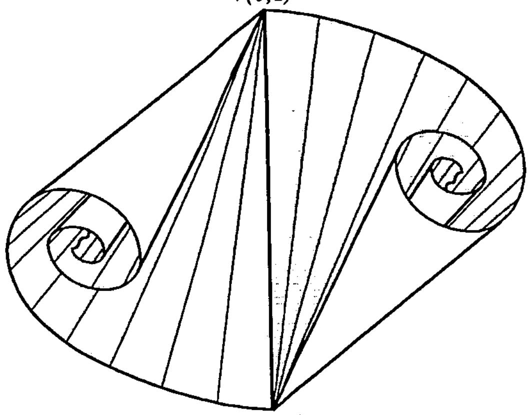

Figure 1. The surface of Klingenberg's example

It is remarkable that the surface itself is C^∞-smooth. The reason for this is that the jump in the derivative of the direction vector of the generators belongs to the tangent space of the surface and does not influence the exterior geometry of the surface.

5. Other examples of developable surfaces not allowing a smooth parametrisation by asymptotic lines

5.1. Analytic surfaces. In the work [1] Hartman and Nirenberg state the following example of an analytic surface not allowing even C^1 standard parametrisation.

Hartman and Nirenberg's example: Let a surface be given over the unit disk D by the formulae:

$$r(x,y) = \begin{pmatrix} x \\ y \\ z(x,y) \end{pmatrix}, \qquad z(x,y) = \frac{y^4}{(2-x)^3}. \tag{6}$$

Let us introduce a new coordinate system:

$$\begin{cases} x &= v \\ y &= (2-v)\cdot(\frac{u}{4})^{1/3} \end{cases} \iff \begin{cases} u &= 4\cdot\left(\frac{y}{2-x}\right)^3 \\ v &= x. \end{cases}$$

Then the surface has a standard parametrisation (1) with

$$\rho(u) = 2\cdot\left(\frac{u}{4}\right)^{1/3}\cdot\begin{pmatrix} 0 \\ 1 \\ u/4 \end{pmatrix}, \qquad s(u) = \begin{pmatrix} 1 \\ -(u/4)^{1/3} \\ -(u/4)^{4/3} \end{pmatrix}.$$

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The following is a quotation from [**1**, p.917]:

> This parametrisation is continuous but not of class C^1. An argument similar to that of [**3**, pp.169–170] shows that the surface has no C^1 parametrisation of the desired type.

Unfortunately, this is an unsuccessful example, since the surface even has an analytic standard parametrisation. Indeed, the preimages of the generators in the (x, y)-plane form a pencil of straight lines, passing through the point $(2, 0)$: the simplest way to find the preimages of the generators is to use the fact that z_x and z_y are constant along them (of course, we consider an explicit given surface $(x, y, z(x, y))$ only); this condition provides

$$y = C \cdot (2 - x).$$

The set of the planar points is given by $z_{xx} = z_{xy} = z_{yy} = 0$, hence the set is the x-axis. Now we can introduce a good analytic standard parametrisation of the surface (6) by the change of variables

$$\begin{cases} x &= \widetilde{v} \\ y &= (2 - \widetilde{v}) \cdot \widetilde{u} \end{cases} \iff \begin{cases} \widetilde{u} &= \dfrac{y}{2 - x} \\ \widetilde{v} &= x. \end{cases}$$

Then

$$r(\widetilde{u}, \widetilde{v}) = \rho(\widetilde{u}) + \widetilde{v} \cdot s(\widetilde{u}) = 2 \cdot \widetilde{u} \cdot \begin{pmatrix} 0 \\ 1 \\ \widetilde{u}^3 \end{pmatrix} + \widetilde{v} \cdot \begin{pmatrix} 1 \\ -\widetilde{u} \\ -\widetilde{u}^4 \end{pmatrix}.$$

The generator with the planar points (the x-axis) has equation $\widetilde{u} = 0$. The function $s(\widetilde{u})$ is analytic at the point 0 (in fact, everywhere).

Moreover, we claim that every analytic surface of vanishing curvature has an analytic standard parametrisation. Indeed, the only obstruction for such a parametrisation can be a generator consisting of planar points and such that in every vicinity of it there is a non-planar point. Let us introduce a Cartesian rectilinear coordinate system (x, y, z) in the ambient space so that the y-axis coincides with this generator, the x-axis lies in the tangent plane of the surface at a fixed point O of the generator. Then in some neighbourhood of O the surface has an explicit parametrisation $(x, y, z(x, y))$ and the function $z(x, y)$ is analytic. Let us take the curve $\rho(x) = (x, 0, z(x, 0))$ above the x-axis as a directrix and consider the direction vector $s(x)$ of the generators. It is completely defined by its projection $\widetilde{s}(x)$ on the (x, y)-plane, and the smoothness of $\widetilde{s}(x)$ and $s(x)$ is the same (as $s(x) = \widetilde{s}(x) + \langle \widetilde{s}(x), \nabla z\rangle \cdot e_3$). Without loss of generality one can assume

$$\widetilde{s}(x) = \begin{pmatrix} s_1(x) \\ 1 \end{pmatrix}$$

in the neighbourhood of O. And as long as $\tilde{s}(x)$ is defined as the vector annihilating the second fundamental form, then

$$s_1(x) = -z_{xy} : z_{xx} = -z_{yy} : z_{xy}.$$

Using $s_1(0) = 0$ (the y-axis is a generator), the analyticity of the function z and the existence of non-planar points in the neighbourhood of O, we obtain that s_1 is analytic at 0. This implies the analyticity of the direction vector of generators $s(x)$, and then the analyticity of the standard parametrisation itself.

5.2. C^∞ SURFACES. We are going to describe a general method of constructing C^∞ surfaces not allowing smooth asymptotic parametrisation. As we know, a problem may arise only when one passes from one ruled part of a surface to another ruled part through a generator consisting of planar points. (One can think about more complicated examples with clustering planar generators, but we prefer to deal here with rather simple examples.) Let us consider such a situation as a limiting one.

5.2.1. COLLAPSE OF A FLAT DOMAIN. At the beginning let us take instead of the planar generator a flat strip and glue two ruled parts to its boundaries. Then

(1) inside the flat strip one can arrange (at least locally) a ruled parametrisation, which smooths out the passing from one ruled part to the other one;
(2) the smoothness of the surface is determined by the smoothness of the ruled parts and by the degree of their osculation to the tangent plane in the locus of gluing.

Now if we shrink the flat strip into the straight line there may arise an obstruction for the smoothness of an asymptotic parametrization—a jump of the derivative of the direction vector of the generators: now the vector has "no time" to pass smoothly from one ruled part to the other one.

5.2.2. TOOL: THE MOLLIFIER. The function $\exp\{-1/u^2\}$, which has been used in Klingenberg's example (see (5)), mollifies the passage from a ruled part of the surface to the plane (which is reduced to the straight line). And thanks to this it was possible to obtain C^∞-gluing, while if one had glued two pieces of circular cones in a similar way, the resulting surface would have been only C^1.

5.2.3. GENERAL CONSTRUCTION. To construct any similar example one needs two arbitrary pieces of developable surfaces without planar points each of which is cut along a generator. Then one glues them together along these generators so that the direction vector has a jump in its derivative (for which it is sufficient to look after noncoincidence of the "key" points on the left and right: the point of the edge of regression for a tangent developable; the vertex for a cone; the point at infinity for a cylinder). Then the surface obtained could be mollified to any desirable degree (up to C^∞) by means of multiplication by a mollifier which is a function of the directrix parameter.

The surface from Klingenberg's example can be obtained in this way. One can easily construct another surface from a cone and a cylinder, for example.

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Department of Mathematics
The University of Melbourne
Parkville, Vic 3052
Australia
e-mail: vit@mundoe.maths.mu.oz.au