# CONCEPTUAL DISTANCE AND ALGEBRAS OF CONCEPTS

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**Abstract.** We show that the conceptual distance between any two theories of first-order logic is the same as the generator distance between their Lindenbaum–Tarski algebras of concepts. As a consequence of this, we show that, for any two arbitrary mathematical structures, the generator distance between their meaning algebras (also known as cylindric set algebras) is the same as the conceptual distance between their first-order logic theories. As applications, we give a complete description for the distances between meaning algebras corresponding to structures having at most three elements and show that this small network represents all the possible conceptual distances between complete theories. As a corollary of this, we will see that there are only two non-trivial structures definable on three-element sets up to conceptual equivalence (i.e., up to elementary plus definitional equivalence).

**§1. Introduction.** The question how to compare scientific theories became of central interest recently. There are great many papers discussing what is the most adequate notion for determining if two theories are essentially the same, comparing the possible notions, or proving some concrete theories to be equivalent according to some of those notions (see, e.g., [2, 3, 14, 15]).

We believe that it is also interesting to compare nonequivalent theories by investigating the reason and degree of their non-equivalence. In [7], a new quantitative method was introduced to measure this degree of non-equivalence of theories by various notions of distances between theories. This opens new opportunities to get a better understanding of the relation between nonequivalent theories. The central notion of that paper is *conceptual distance*, that refines the notion of definitional equivalence and measures the minimal number of concept adding/removing steps that are needed to be taken to reach a theory from another.

In general it is rather difficult to determine the concrete conceptual distance between two theories. For example, the main PhD result of Lefever [9, 10] states, in terms of conceptual distance, that classical and relativistic kinematics are of conceptual

distance 1; in other words, these two theories are distinguished from each other by only one concept. If we keep track of the concrete concepts distinguishing the two theories in hand, we also get a qualitative comparison. For example, in the case of classical and relativistic kinematics the distinguishing concept is "being stationary" for a coordinate system.

In the present paper, after recalling some notions related to concept algebras of algebraic logic and the notion of generator distance between any two algebras of the same similarity type (see [1]), we show that the conceptual distance between any two theories of first-order logic is the same as the generator distance between their Lindenbaum–Tarski algebras of concepts (see Theorem 4.3). As a consequence of this, we show that, for any two arbitrary mathematical structures, the generator distance between their meaning algebras (also known as cylindric set algebras) is the same as the conceptual distance between their first-order logic theories (see Corollary 4.4). In Corollary 4.5, we show that it does not matter if this distance is calculated in the class of meaning algebras (corresponding to models) or in the bigger class of Lindenbaum–Tarski algebras (corresponding to not necessarily complete theories).

This connection between the conceptual distance of complete theories and the generator distance of concept algebras can give an effective algebraic method to determine the conceptual distance between arbitrary theories, which seems to be a quite difficult task in general.

Finally, in Section 5, we show some examples and applications. Among others, we give a complete description for the distances between meaning algebras corresponding to structures having at most three elements (see Figure 2). Then we show that this small network represents all the possible conceptual distances between complete theories (see Theorem 5.1). As a corollary of this, we will see that there are only two non-trivial structures definable on three-element sets up to conceptual equivalence (i.e., up to elementary plus definitional equivalence).

§2. Algebras of concepts. There are mainly two types of algebras of concepts: Lindenbaum–Tarski algebras of theories, and meaning algebras of models. The later corresponds to concepts that one can define on a given model. These algebras are also known in the literature as cylindric set algebras. Lindenbaum–Tarski algebras can be viewed as algebras of concepts that can be defined in a given theory (that is not necessarily complete). In this section, we are going to recall the definitions of these two types of concrete concept algebras.

The understanding of the term *concept* depends on where we are. In a Lindenbaum–Tarski algebra, concepts are equivalence classes of formulas modulo the theory, in a meaning algebra, they are the definable relations of the model, and in general, in an arbitrary cylindric algebra, viewed as abstract algebra of concepts, they are just elements of the algebra.

For simplicity, we assume that languages contain only predicates (relation symbols) and fix an enumeration of the variables $\{v_i : i < \omega\}$.

The *meaning* $[\![\varphi]\!]^{\mathcal{M}}$ *of formula $\varphi$ in model $\mathcal{M}$* is the set of infinite sequences from $\mathcal{M}$ satisfying $\varphi$ when the $i$-th variable $v_i$ is evaluated to the $i$-th element of the sequence in hand, i.e.,

$$[\![\varphi]\!]^{\mathcal{M}} \stackrel{\text{def}}{=} \{\bar{a} \in M^{\omega} : \mathcal{M} \models \varphi[\bar{a}]\}.$$

These meanings form a Boolean algebra with set operations $+$ (union), $\cdot$ (intersection) and $-$ (complement) corresponding, respectively, to disjunction, conjunction and negation. The operation $\mathsf{C}_i$ connecting the meanings of formulas $\exists v_i \varphi$ and $\varphi$ is called *cylindrification* because of its geometrical meaning: it is the smallest cylinder that contains $[\![\varphi]\!]^{\mathcal{M}}$ and whose axis is parallel to the $i$-th coordinate axis of $M^\omega$. Formally,

$$\mathsf{C}_i [\![\varphi]\!]^{\mathcal{M}} \stackrel{\text{def}}{=} [\![\exists v_i \varphi]\!]^{\mathcal{M}} = \left\{ \bar{a} \in M^\omega : \text{exists } \bar{a}' \in [\![\varphi]\!]^{\mathcal{M}} \text{ such that } a_j = a_j' \text{ if } j \neq i \right\}.$$

The meanings of formulas $v_i = v_j$ for arbitrary $i, j \in \omega$ are called the *diagonal elements*. Formally,

$$\mathsf{D}_{ij} \stackrel{\text{def}}{=} \{ \bar{a} \in M^\omega : a_i = a_j \}.$$

There are two other important constants in a concept algebra, namely the ones corresponding to the meanings of tautologies and self-contradictions, which can be introduced in meaning algebras, for example, as follows:

$$1 \stackrel{\text{def}}{=} [\![\exists v_0 \, v_0 = v_0]\!]^{\mathcal{M}} \quad \text{and} \quad 0 \stackrel{\text{def}}{=} [\![\forall v_0 \, v_0 \neq v_0]\!]^{\mathcal{M}}.$$

The *meaning algebra* $\mathfrak{Cs}(\mathcal{M})$ *of model* $\mathcal{M}$ is this natural algebra of meanings of formulas in $\mathcal{M}$:

$$\mathfrak{Cs}(\mathcal{M}) \stackrel{\text{def}}{=} \langle Cs(\mathcal{M}), +, \cdot, -, 0, 1, \mathsf{C}_i, \mathsf{D}_{ij} \rangle_{i,j<\omega},$$

where $Cs(\mathcal{M})$ denotes the set of all concepts definable in model $\mathcal{M}$. In this paper, we are going to use the following notation for the smallest class which is closed under isomorphism and contains all meaning algebras:

$$\mathsf{MA} \stackrel{\text{def}}{=} \{ \mathfrak{A} : \mathfrak{A} \cong \mathfrak{Cs}(\mathcal{M}) \text{ for some first-order model } \mathcal{M} \}.$$

Let us now see the other main type of concept algebras. Using the terminology of [4], by the *free algebra of formulas in language* $\Lambda$, we understand the natural algebra given on the set $\mathsf{Fm}_\Lambda$ of formulas when the logical connectives considered as operations:

$$\mathfrak{Fm}_\Lambda \stackrel{\text{def}}{=} \langle \mathsf{Fm}_\Lambda, \vee, \wedge, \neg, \top, \bot, \exists v_i, v_i = v_j \rangle_{i,j<\omega}.$$

We have that

$$\varphi \equiv_T \psi \stackrel{\text{def}}{\Longleftrightarrow} T \models \varphi \leftrightarrow \psi$$

is a congruence relation on $\mathfrak{Fm}_\Lambda$ for every theory $T$ of $\Lambda$ (see, e.g., [5, theorem 4.3.13, p. 156]). Let $T$ be a theory of language $\Lambda$. Then the *Lindenbaum–Tarski algebra of* $T$ *in* $\Lambda$ is the formula algebra of language $\Lambda$ factorized by relation $\equiv_T$, i.e.,

$$\mathfrak{Fm}_T^\Lambda \stackrel{\text{def}}{=} \langle \mathsf{Fm}_\Lambda /_{\equiv_T}, +, \cdot, -, 1, 0, \mathsf{C}_i, \mathsf{D}_{ij} \rangle_{i,j<\omega}.{}^{[1]}$$

Herein, we are going to use the following notation for the class of algebras isomorphic to some Lindenbaum–Tarski algebra:

$$\mathsf{LT} = \left\{ \mathfrak{A} : \mathfrak{A} \cong \mathfrak{Fm}_T^\Lambda \text{ for some Lindenbaum–Tarski algebra } \mathfrak{Fm}_T^\Lambda \right\}.$$

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[1] The operations here are the operations of $\mathfrak{Fm}_\Lambda$ after taking the quotient. They are denoted by the corresponding operations symbols of concept algebras.

There is a strong connection between meaning algebras and Lindenbaum–Tarski algebras. To see this connection, let $\mathsf{Th}(\mathcal{M})$ denote the complete theory of structure $\mathcal{M}$, i.e., the set of all statements (formulated in the language of $\mathcal{M}$) that are true in $\mathcal{M}$. Let $\Lambda$ be a language and $\mathcal{M}$ be a model for that language, then

$$\mathfrak{Cs}(\mathcal{M}) \cong \mathfrak{Fm}^{\Lambda}_{\mathsf{Th}(\mathcal{M})}. \tag{1}$$

To see this, let $[\varphi]$ denote the equivalence class of $\varphi$ in $\mathfrak{Fm}^{\Lambda}_{\mathsf{Th}(\mathcal{M})}$ and consider map $f : [\varphi] \mapsto \llbracket \varphi \rrbracket^{\mathcal{M}}$ for all $\varphi \in \mathsf{Fm}_{\Lambda}$. Map $f$ is an isomorphism because of the following equivalences

$$\mathsf{Th}(\mathcal{M}) \models \varphi \leftrightarrow \psi \iff \mathcal{M} \models \varphi \leftrightarrow \psi \iff \llbracket \varphi \rrbracket^{\mathcal{M}} = \llbracket \psi \rrbracket^{\mathcal{M}}$$

for all $\varphi, \psi \in \mathsf{Fm}_{\Lambda}$.

§3. Conceptual distance of theories and generator distance of algebras. In this section, we recall two notions of distance from the literature. One of them is between theories and called conceptual distance. The other one is between algebras of the same similarity type and called generator distance.

For convenience and simplicity, we use the same subscripts and superscripts for theories and their corresponding languages, etc.

Conceptual distance is a natural refinement of definitional equivalence. The basic idea behind definitional equivalence is that introducing new definitions does not change a theory. In this spirit, *definitional equivalence* can be introduced as the transitive and symmetric closure of definitional extension.[2] There are many equivalent reformulation of this notion (see, e.g., [11]). We are going to write $T_1 \rightleftarrows T_2$ for that theories $T_1$ and $T_2$ are definitionally equivalent. From [7], we recall that *theory $T^+$ is a one-concept-extension of theory $T$* iff

- the language of $T^+$ is that of $T$ plus one relation symbol, i.e., $\Lambda^+ = \Lambda \cup \{R\}$ for some relation symbol $R$, and
- $T^+$ is a conservative extension of $T$, i.e., $\mathsf{Fm}_{\Lambda} \subseteq \mathsf{Fm}_{\Lambda^+}$ and $T^+ \models \varphi$ iff $T \models \varphi$ for all $\varphi \in \mathsf{Fm}_{\Lambda}$.

In this case, we write $T \rightsquigarrow T^+$ to denote this relation, and we also say that *$T$ is a one-concept-removal of $T^+$*. By $\leftrightsquigarrow$, we denote the symmetric closure of relation $\rightsquigarrow$.

The *conceptual distance* $\mathsf{Cd}(T_1, T_2)$ *between theories $T_1$ and $T_2$* is defined to be the minimum number of one-concepts-extension and one-concept-removal steps needed to be taken to reach theory $T_2$ from theory $T_1$. Note that $\mathsf{Cd}(T_1, T_2) = \infty$ is possible. This notion of distance satisfies all the axioms of the metric spaces, up to definitional equivalence and allowing infinite distance:

- $\mathsf{Cd}(T_1, T_2) \geq 0$, and $\mathsf{Cd}(T_1, T_2) = 0 \iff T_1 \rightleftarrows T_2$.
- $\mathsf{Cd}(T_1, T_2) = \mathsf{Cd}(T_2, T_1)$.
- $\mathsf{Cd}(T_1, T_3) \leq \mathsf{Cd}(T_1, T_2) + \mathsf{Cd}(T_2, T_3)$.

In Section 5, we will see some simple examples such as that the complete theory of the cherry graph (path graph of length two) is of conceptual distance 2 from that of the

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2  A *definitional extension of a theory* means extending its language with some new relation symbols and extending the theory with the explicit definitions of these new relation symbols.

directed circle graph of length 3. This is clear intuitively because none of those relations can be defined from the other one, but both can be reached from the theory of the three-element set by adding a new relation. For further examples, see [7], where among others it is shown that conceptual distance between special relativity and classical kinematics is 1.

To recall the generator distance, we need some definitions. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two algebras of the same similarity type. We say that $\mathfrak{A}$ is a *large subalgebra in* $\mathfrak{B}$ iff $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, and there is an element $b \in B$ such that $A \cup \{b\}$ generates the whole big algebra $\mathfrak{B}$. Through out this paper, we use the notation $\langle X \rangle$ for the *the subalgebra generated by set X*. With this notation $\mathfrak{A}$ is a large subalgebra of $\mathfrak{B}$ iff $\mathfrak{B} = \langle A \cup \{b\} \rangle$ for some $b \in B$. By *large embedding*, we mean an injective homomorphism whose range is a large subalgebra, and we denote the fact that $\mathfrak{A}$ is largely embeddable into $\mathfrak{B}$ by $\mathfrak{A} \xrightarrow{\langle \bullet \rangle} \mathfrak{B}$.

Let K be a class of similar algebras. Consider the following possibly infinite network on K: the nodes are the elements of K, two nodes are connected by a blue edge if one of them is a large subalgebra of the other, and two nodes are connected by a red edge if they are isomorphic. Then the *generator distance* $d_K : K \times K \to \mathbb{N} \cup \{\infty\}$ on this network of K is defined as follows: $d_K(\mathfrak{A}, \mathfrak{B})$ is the minimum number of blue edges that we can have in a finite red-blue path connecting $\mathfrak{A}$ and $\mathfrak{B}$ in the network, and $d_K(\mathfrak{A}, \mathfrak{B}) = \infty$ if there is no finite path between $\mathfrak{A}$ and $\mathfrak{B}$.

A class K of similar algebras equipped with the generator distance $d_K : K \times K \to \mathbb{N} \cup \{\infty\}$ satisfies all the axioms of the metric spaces, up to isomorphism and allowing infinite distance, i.e., for all $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K$,

- $d_K(\mathfrak{A}, \mathfrak{B}) \geq 0$, and $d_K(\mathfrak{A}, \mathfrak{B}) = 0 \iff \mathfrak{A} \cong \mathfrak{B}$,
- $d_K(\mathfrak{A}, \mathfrak{B}) = d_K(\mathfrak{B}, \mathfrak{A})$, and
- $d_K(\mathfrak{A}, \mathfrak{C}) \leq d_K(\mathfrak{A}, \mathfrak{B}) + d_K(\mathfrak{B}, \mathfrak{C})$.

As an illustration for the generator distance, we mention the following example. In the class K of vector spaces over the same field,

$$d_K(V, W) = \begin{cases} 0, & \text{if } V \cong W, \\ |dim(V) - dim(W)|, & \text{if } dim(V), dim(W) < \infty, \\ \infty, & \text{otherwise.} \end{cases}$$

For further details and examples, see [1, 6].

§4. The conceptual distance between theories is equal to the generator distance of their Lindenbaum–Tarski algebras. In this section, we are going to prove that the conceptual distance between any two theories is equal to the generator distance of their Lindenbaum–Tarski algebras. To do so, in Proposition 4.1, first we show that one-concept-extension implies large embeddibility of the corresponding Lindenbaum–Tarski algebras. The converse is not necessarily true, but in Proposition 4.2, we give a characterization of large embeddibility of Lindenbaum–Tarski algebras in terms of one-concept-extensions between theories.

PROPOSITION 4.1. *Let $T$ and $T^+$ be two arbitrary theories on respective languages $\Lambda$ and $\Lambda^+$, and let $\mathfrak{Fm}_T^\Lambda$ and $\mathfrak{Fm}_{T^+}^{\Lambda^+}$ be their corresponding Lindenbaum–Tarski algebras. Then*

$$T \rightsquigarrow T^+ \implies \mathfrak{Fm}_T^\Lambda \xrightarrow{\langle \bullet \rangle} \mathfrak{Fm}_{T^+}^{\Lambda^+}.$$

*Proof.* Let $[\varphi]$ and $[\varphi]^+$ denote the $\equiv_T$ and $\equiv_{T^+}$ equivalence classes of formula $\varphi$, respectively.

Since $T \rightsquigarrow T^+$, we have that $\Lambda^+ = \Lambda \cup \{R\}$ for some relation symbol $R$ and $T^+$ is a conservative extension of $T$. Consider map $f : \mathfrak{Fm}_T^\Lambda \to \mathfrak{Fm}_{T^+}^{\Lambda^+}$ that maps $[\varphi]$ to $[\varphi]^+$. To show that $f$ is well-defined, let $\psi \in [\varphi]$ be another formula representing the equivalence class of $\varphi$ in $\mathfrak{Fm}_T^\Lambda$. Then $T \models \varphi \leftrightarrow \psi$, and from this, because $T^+$ is a conservative extension of $T$, we have $T^+ \models \varphi \leftrightarrow \psi$, i.e., $\psi \in [\varphi]^+$. Hence $f$ is well-defined.

From the definitions, it is straightforward to check that $f$ is a homomorphism. As an illustration, we show this in the case of operation $\wedge$. To do so, let $[\varphi]$ and $[\psi]$ be two arbitrary elements of $\mathfrak{Fm}_T^\Lambda$. By definitions, we have $[\varphi \wedge \psi] = [\varphi] \cdot [\psi]$ and $[\varphi \wedge \psi]^+ = [\varphi]^+ \cdot [\psi]^+$. Therefore,

$$f([\varphi] \cdot [\psi]) = f([\varphi \wedge \psi]) = [\varphi \wedge \psi]^+ = [\varphi]^+ \cdot [\psi]^+ = f([\varphi]) \cdot f([\psi]).$$

So $f$ preserves operation $\cdot$. Showing that $f$ preserves the other operations is completely analogous.

To show that $f$ is one-to-one, let $\varphi, \psi \in \mathsf{Fm}_\Lambda$ such that $[\varphi]^+ = [\psi]^+$, i.e., $T^+ \models \varphi \leftrightarrow \psi$. Since $T^+$ is a conservative extension of $T$ and $\varphi, \psi \in \mathsf{Fm}_\Lambda$, we have $T \models \varphi \leftrightarrow \psi$, and hence $[\varphi] = [\psi]$. Consequently, $f$ is an embedding of $\mathfrak{Fm}_T^\Lambda$ to $\mathfrak{Fm}_{T^+}^{\Lambda^+}$.

To show that $f$ is a large embedding, it is enough prove that the $f$-image of $\mathfrak{Fm}_T^\Lambda$ plus the element $[R]^+ \in \mathfrak{Fm}_{T^+}^{\Lambda^+}$ generates the whole Lindenbaum–Tarski algebra $\mathfrak{Fm}_{T^+}^{\Lambda^+}$. This is so because the formula algebra $\mathfrak{Fm}_{T^+}^{\Lambda^+}$ is generated by formulas $\mathsf{Fm}_\Lambda$ and formula $R$ since $\Lambda^+ = \Lambda \cup \{R\}$. This completes the proof.                    □

The converse of Proposition 4.1 is not necessarily true because it is possible that $\mathfrak{Fm}_T^\Lambda \xrightarrow{\langle \bullet \rangle} \mathfrak{Fm}_{T^+}^{\Lambda^+}$ holds but there are more than one relation symbols in $\Lambda^+ \setminus \Lambda$. Nevertheless, even in such cases, it is possible to replace the language $\Lambda^+$ to another language $\Lambda_*^+$ and find a theory $T_*^+$ in this language such that $T \rightsquigarrow T_*^+$ and $\mathfrak{Fm}_{T_*^+}^{\Lambda_*^+} \cong \mathfrak{Fm}_{T^+}^{\Lambda^+}$:

PROPOSITION 4.2. *Let $T$ and $T^+$ be two arbitrary theories on respective languages $\Lambda$ and $\Lambda^+$; and let $\mathfrak{Fm}_T^\Lambda$ and $\mathfrak{Fm}_{T^+}^{\Lambda^+}$ be their corresponding Lindenbaum–Tarski algebras. Then*

$$\mathfrak{Fm}_T^\Lambda \xrightarrow{\langle \bullet \rangle} \mathfrak{Fm}_{T^+}^{\Lambda^+} \iff T \rightsquigarrow T_*^+ \text{ for some theory } T_*^+ \text{ for which } \mathfrak{Fm}_{T_*^+}^{\Lambda_*^+} \cong \mathfrak{Fm}_{T^+}^{\Lambda^+}.$$

In the proof of Proposition 4.2, we need to introduce some terminology. The so-called *dimension set of $x$* is defined as $\Delta x := \{i \in \omega : \mathsf{C}_i x \neq x\}$. We denote *the rank function* that associate their arity with predicates (relation symbols) by $\rho$. A key step in the proof is based on the following fact of algebraic logic (see [5, theorem 4.3.15, p. 156, theorem 4.3.17, p. 156 and theorem 4.3.28, p. 161] and [4, p. 349]):
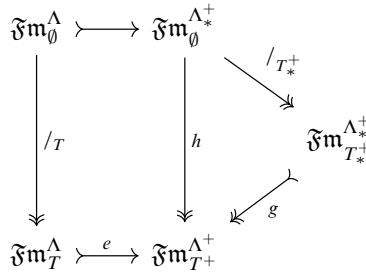
Fig. 1. The diagram illustrates the connection of algebras used in the proof of Proposition 4.2.

(CA1)   Let $\Lambda$ be an arbitrary language, then $\mathfrak{Fm}_\emptyset^\Lambda$ is a dimension-restricted free algebra. That is, for any $\mathfrak{A} \in \mathsf{LT}$, that is generated by $\langle x_R : R \in \Lambda \rangle$ such that $\Delta x_R \subseteq \rho R$, there is a (unique) homomorphism $h : \mathfrak{Fm}_\emptyset^\Lambda \to \mathfrak{A}$ satisfying $h(R) = x_R$ for all $R \in \Lambda$.

Note that, for the sake of notational simplicity, $R$ is also used as an abbreviation for atomic formula $R(v_0, \dots, v_{\rho R - 1})$ and not just for relation symbol $R \in \Lambda$.

*Proof.* Direction $\Longleftarrow$ follows immediately from Proposition 4.1. To show the converse direction, as before, let $[\varphi]$, $[\varphi]^+$ and $[\varphi]_*^+$ denote the $\equiv_T$, $\equiv_{T^+}$ and $\equiv_{T_*^+}$ equivalence classes of formula $\varphi$.

Since $\mathfrak{Fm}_T^\Lambda \xrightarrow{\langle \bullet \rangle} \mathfrak{Fm}_{T^+}^{\Lambda^+}$, there is an embedding $e : \mathfrak{Fm}_T^\Lambda \to \mathfrak{Fm}_{T^+}^{\Lambda^+}$ and element $b \in \mathfrak{Fm}_{T^+}^{\Lambda^+}$ such that $\mathfrak{Fm}_{T^+}^{\Lambda^+} = \langle \mathrm{Im}(e) \cup \{b\} \rangle$, where $\mathrm{Im}(e)$ denotes the image $\{e(x) : x \in \mathfrak{Fm}_T^\Lambda\}$ of $e$. Let $\Lambda_*^+ := \Lambda \cup \{R_b\}$ for some relation symbol $R_b \notin \Lambda$ and let $\rho_*^+$ be an extension of $\rho$ such that

$$\rho_*^+(R_b) := \bigcap \{\beta < \omega : \Delta b \subseteq \beta\}.$$

By its definition, we have that $\rho_*^+(R_b)$ is finite, and hence $\Lambda_*^+$ is a first-order language.

Since $e$ is a large embedding, $\mathfrak{Fm}_{T^+}^{\Lambda^+}$ can be generated by $R_b$ all the relations in $\Lambda$. Hence, by (CA1), there is a homomorphism $h$ from $\mathfrak{Fm}_\emptyset^{\Lambda_*^+}$ to $\mathfrak{Fm}_{T^+}^{\Lambda^+}$ such that $h(R_b) = b$ and $h(R) = [R]^+$ for all $R \in \Lambda$.[3] Let

$$T_*^+ := \left\{ \varphi \in \mathsf{Fm}_{\Lambda_*^+} : h(\varphi) = 1 \right\}.$$

Let us define $g : \mathfrak{Fm}_{T_*^+}^{\Lambda_*^+} \to \mathfrak{Fm}_{T^+}^{\Lambda^+}$ as $g : [\varphi]_*^+ \mapsto h(\varphi)$ (see Figure 1).

It is straightforward to check that $g$ is an isomorphism, and hence $\mathfrak{Fm}_{T^+}^{\Lambda^+} \cong \mathfrak{Fm}_{T_*^+}^{\Lambda_*^+}$: it is easy to check that $g$ is a homomorphism; $g$ is one-to-one because $g([\varphi]_*^+) = g([\psi]_*^+)$ iff $h(\varphi) = h(\psi)$ iff $h(\varphi \leftrightarrow \psi) = 1$ iff $\varphi \leftrightarrow \psi \in T_*^+$, which implies $[\varphi]_*^+ = [\psi]_*^+$; $g$ is surjective because $h$ is so.

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[3]   Technically, we should write the $\equiv_\emptyset^{\Lambda_*^+}$ equivalence classes of formulas in homomorphism $h$, but we drop this detail from the notation to make the ideas of the proof easier to be followed.

To show that $T \subseteq T_*^+$, let $\varphi \in \mathsf{Fm}_\Lambda \subset \mathsf{Fm}_{\Lambda_*^+}$. If $\varphi \in T$, we have $[\varphi] = 1$. Hence, $[\varphi]^+ = e([\varphi]) = 1$, and thus $h(\varphi) = 1$ because $[\varphi]^+ = h(\varphi)$ as $[R]^+ = h(R)$ for all $R \in \Lambda$. Consequently, $\varphi \in T_*^+$.

Theory $T_*^+$ is a conservative extension of $T$ because, for all $\varphi \in \mathsf{Fm}_\Lambda$, we have

$$T_*^+ \models \varphi \iff \varphi \in T_*^+ \iff h(\varphi) = 1 \iff [\varphi]^+ = e([\varphi]) = 1 \iff T \models \varphi.$$

Consequently, $T \rightsquigarrow T_*^+$ because $\Lambda_*^+ = \Lambda \cup \{R_b\}$. $\qquad\square$

Now we show that conceptual distance can be reduced to generator distance between certain algebras of logic. By Theorem 4.3 below, the conceptual distance between theories is the same as the generator distance between their Lindenbaum–Tarski algebras in the class LT.

THEOREM 4.3. *Let $T_1$ and $T_2$ be two arbitrary theories and let $\mathfrak{Fm}_{T_1}^{\Lambda_1}$ and $\mathfrak{Fm}_{T_1}^{\Lambda_1}$ be their Lindenbaum–Tarski algebras. Then*

$$\mathsf{Cd}(T_1, T_2) = d_{\mathsf{LT}}\left(\mathfrak{Fm}_{T_1}^{\Lambda_1}, \mathfrak{Fm}_{T_2}^{\Lambda_2}\right).$$

In the proof of Theorem 4.3, we are going to use the following fact of algebraic logic (see [5, theorems 4.3.15, 4.3.28(ii) and 4.3.43]):

(CA2)   ∘ For any two theories $T_1$ and $T_2$, $T_1 \rightleftarrows T_2$ iff $\mathfrak{Fm}_{T_1}^{\Lambda_1} \cong \mathfrak{Fm}_{T_2}^{\Lambda_2}$.
   ∘ For all $\mathfrak{A}, \mathfrak{B} \in \mathsf{LT}$, we have that $\mathfrak{A} \cong \mathfrak{B}$ iff there are definitionally equivalent theories $T_1$ and $T_2$ of respective languages $\Lambda_1$ and $\Lambda_2$ such that $\mathfrak{A} \cong \mathfrak{Fm}_{T_1}^{\Lambda_1}$ and $\mathfrak{B} \cong \mathfrak{Fm}_{T_2}^{\Lambda_2}$.

*Proof.* We have $\mathsf{Cd}(T_1, T_2) \le d_{\mathsf{LT}}\left(\mathfrak{Fm}_{T_1}^{\Lambda_1}, \mathfrak{Fm}_{T_2}^{\Lambda_2}\right)$ because, by (CA2) and Proposition 4.1, we can associate an $n$-long path of algebras in LT connecting $\mathfrak{Fm}_{T_1}^{\Lambda_1}$ and $\mathfrak{Fm}_{T_2}^{\Lambda_2}$ to every $n$-long path of theories connecting $T_1$ and $T_2$. We have $\mathsf{Cd}(T_1, T_2) \ge d_{\mathsf{LT}}\left(\mathfrak{Fm}_{T_1}^{\Lambda_1}, \mathfrak{Fm}_{T_2}^{\Lambda_2}\right)$ because, by (CA2) and Proposition 4.2, we can associate an $n$-long path of theories connecting $T_1$ and $T_2$ to every $n$-long path of algebras in LT connecting $\mathfrak{Fm}_{T_1}^{\Lambda_1}$ and $\mathfrak{Fm}_{T_2}^{\Lambda_2}$. $\qquad\square$

The following is a corollary of Theorem 4.3 and (1) on page 3:

COROLLARY 4.4. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two arbitrary mathematical structures. Then*

$$\mathsf{Cd}\big(\mathsf{Th}(\mathcal{M}_1), \mathsf{Th}(\mathcal{M}_2)\big) = d_{\mathsf{LT}}\big(\mathfrak{Cs}(\mathcal{M}_1), \mathfrak{Cs}(\mathcal{M}_2)\big).$$

In case of complete theories, one may wonder if it is possible to find a minimal path representing their conceptual distance that goes through complete theories only. To answer this natural question positively in Corollary 4.5, let us see some consequences of the things above. By Proposition 4.2, (CA2) and (1), we have

$$\mathfrak{Cs}(\mathcal{M}) \xrightarrow{\langle\bullet\rangle} \mathfrak{Cs}(\mathcal{M}^+) \iff \mathsf{Th}(\mathcal{M}) \rightsquigarrow \mathsf{Th}(\mathcal{M}^+) \tag{2}$$

for all models $\mathcal{M}$ and $\mathcal{M}^+$. By (1) and (CA2), we have

$$\mathfrak{Cs}(\mathcal{M}_1) \cong \mathfrak{Cs}(\mathcal{M}_2) \iff \mathsf{Th}(\mathcal{M}_1) \rightleftarrows \mathsf{Th}(\mathcal{M}_2) \tag{3}$$

for all $\mathcal{M}_1$ and $\mathcal{M}_2$. From (2) and (3), it follows straightforwardly that, for all $\mathcal{M}_1$ and $\mathcal{M}_2$,

$$d_{\mathsf{MA}}\big(\mathfrak{Cs}(\mathcal{M}_1), \mathfrak{Cs}(\mathcal{M}_2)\big) = \mathsf{Cd}\big(\mathsf{Th}(\mathcal{M}_1), \mathsf{Th}(\mathcal{M}_2)\big).$$

From this, by Theorem 4.3 and (1), we get the following:

COROLLARY 4.5. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two arbitrary mathematical structures. Then*

$$d_{\mathsf{LT}}\big(\mathfrak{Cs}(\mathcal{M}_1), \mathfrak{Cs}(\mathcal{M}_2)\big) = d_{\mathsf{MA}}\big(\mathfrak{Cs}(\mathcal{M}_1), \mathfrak{Cs}(\mathcal{M}_2)\big);$$

*in other words, it does not matter if one calculates the distance of $\mathfrak{Cs}(\mathcal{M}_1)$ and $\mathfrak{Cs}(\mathcal{M}_2)$ in the class* LT *of all Lindenbaum–Tarski algebras or in the class* MA *of all meaning algebras.*

So, by Corollary 4.5, if there is a minimal path

$$\mathsf{Th}(\mathcal{M}_1) \longleftrightarrow T_1 \longleftrightarrow T_2 \longleftrightarrow \cdots \longleftrightarrow T_n \longleftrightarrow \mathsf{Th}(\mathcal{M}_2)$$

representing the distance $\mathsf{Cd}\big(\mathsf{Th}(\mathcal{M}_1), \mathsf{Th}(\mathcal{M}_2)\big)$ with some not necessarily complete theories $T_1, T_2, ..., T_n$, then there are models $\mathcal{N}_1, \mathcal{N}_2, ..., \mathcal{N}_n$ such that

$$\mathsf{Th}(\mathcal{M}_1) \longleftrightarrow Th(\mathcal{N}_1) \longleftrightarrow Th(\mathcal{N}_2) \longleftrightarrow \cdots \longleftrightarrow Th(\mathcal{N}_n) \longleftrightarrow \mathsf{Th}(\mathcal{M}_2).$$

**§5. Examples and applications.** In this section, we give some illustrative examples and applications of the previous results. We are going to call two models *conceptually equivalent* iff their theories are definitionally equivalent.[4]

For every nonzero cardinality $\kappa$, let $\mathfrak{Min}_\kappa$ denote *the minimal algebra of concepts* definable in a set of cardinality $\kappa$ only by equality. For different countable cardinalities, these algebras are mutually non-isomorphic, and each meaning algebra contains a unique isomorphic copy of $\mathfrak{Min}_\kappa$ for some $\kappa \leq \omega$ as a minimal subalgebra. If $\kappa \geq \omega$, then $\mathfrak{Min}_\kappa \cong \mathfrak{Min}_\omega$. Let $\mathfrak{Full}_\kappa$ denote *the full algebra of concepts* containing all relations having finite arity over a nonempty set of cardinality $\kappa$. These algebras are mutually non-isomorphic for all different cardinalities, and each meaning algebra can be embedded to one of them. For all nonzero finite $n$, we have $\mathfrak{Min}_n \xrightarrow{\langle \bullet \rangle} \mathfrak{Full}_n$ since $\mathfrak{Full}_n \cong \mathfrak{Cs}(\langle n, \leq \rangle)$ as every element of $n$ can be defined from $\leq$ and this is enough to define every finitary relation over $n$. Note that $\mathfrak{Full}_\omega \not\cong \mathfrak{Cs}(\langle \omega, \leq \rangle)$.

We say that concept $x$ is an *$n$-ary concept* if $\mathsf{C}_i x = x$ for all $i \geq n$. Such concepts are essentially the definable relations of arity $n$. Note that $n$-ary concepts are also $(n+1)$-ary ones, and thus 0 and 1 are $n$-ary concepts for all natural number $n$. Hence, we are going to call these two *trivial $n$-ary concepts*.

Let $|M| = n$ for some natural number $n \geq 1$ and let $[\![R]\!]^{\mathcal{M}}$ be an *$m$-ary* concept of $\mathcal{M}$ for some $m \geq n + 1$. Equation $\sum_{i<j\leq n} \mathsf{D}_{ij} = 1$ holds in $\mathfrak{Cs}(\mathcal{M})$ because it is not possible to take more than $n$ distinct elements of $\mathcal{M}$. From this, it follows that

$$[\![R]\!]^{\mathcal{M}} = \sum_{i<j\leq n} [\![R]\!]^{\mathcal{M}} \cdot \mathsf{D}_{ij}. \tag{4}$$

---

[4] Two models are conceptually equivalent iff one of them is definitionally equivalent to a model which is elementarily equivalent to the other one (see [13, theorem 3.3]).

So $[\![R]\!]^{\mathcal{M}}$ can be reconstructed from the concepts $[\![R]\!]^{\mathcal{M}} \cdot \mathsf{D}_{ij} = [\![R \wedge v_i = v_j]\!]^{\mathcal{M}}$, which are $(m-1)$-ary ones up to substitution of variables.[5] Repeating this construction enough times, one can reconstruct any relation from (at most) $n$-ary ones. Hence,

If $|M| = n$, every concept of $\mathfrak{Cs}(\mathcal{M})$ can be generated by (at most) $n$-ary ones.  (5)

Actually, statement (5) can be strengthened: it is enough to consider (at most) $(n-1)$-ary relations. For a precise proof of this claim, see Lemma 6.2 herein.

If $|M| = n$, every concept of $\mathfrak{Cs}(\mathcal{M})$ can be generated by (at most) $(n-1)$-ary ones.
(6)

Now we will list the meaning algebras of all finite models of size at most 3, up to isomorphism. Note that by (6), it is enough to search for meaning algebras of models whose relation symbols are of arity at most $n-1$, where $n = |M|$. Let $\mathcal{M}$ be an arbitrary model such that $|M| \leq 3$.

Suppose that $|M| = 1$. The are only two 1-ary concepts that can be defined on a one-element set: the empty concept, and its complement. Each of these relations is defined by equality. Thus

$$\mathfrak{Cs}(\mathcal{M}) \cong \mathfrak{Min}_1 \cong \mathfrak{Full}_1.$$

Suppose that $|M| = 2$. Note that $\mathfrak{Min}_2 \not\cong \mathfrak{Full}_2$, but still we have

$$\mathfrak{Cs}(\mathcal{M}) \cong \mathfrak{Min}_2 \text{ or } \mathfrak{Cs}(\mathcal{M}) \cong \mathfrak{Full}_2.$$

This is so because a 1-ary relation on $\mathcal{M}$ is either $\emptyset$, $\mathcal{M}$ itself or a one-element subset of $\mathcal{M}$.

Suppose that $|M| = 3$. There are exactly four meaning algebras up to isomorphism:

$$\mathfrak{Min}_3, \ \mathfrak{Full}_3, \ \mathfrak{Cs}(\text{⬚}), \text{ and } \mathfrak{Cs}(\wedge),$$

where $\mathfrak{Cs}(\text{⬚})$ is the meaning algebra of the directed cycle graph of length 3, and $\mathfrak{Cs}(\wedge)$ is the meaning algebra of the cherry graph (i.e., path graph of length two). In other words, up to conceptual equivalence there are only four structures that one can give on a three-element set. Let us see why this is so. By (6), it is enough to consider structures given by binary relations. If none of the three elements is distinguishable, i.e., any one of them can be mapped to any other one by an automorphism, then these binary relations can only be the unions of the following ones: the one giving a directed cycle, its converse or the equality. These cases give exactly $\mathfrak{Cs}(\text{⬚})$ and $\mathfrak{Min}_3$ up to conceptual equivalence. If exactly two of the three elements are indistinguishable but the third is not, then we get $\mathfrak{Cs}(\wedge)$. Finally, if all the three elements are distinguishable, then we get $\mathfrak{Full}_3$. Let us also note that $\mathfrak{Cs}(\text{⬚})$ is the only one of these four algebras that cannot be generated by unary relations.

---

[5]  For technical reasons, we defined arity such that it does not care about a lower dimension where cylindrification does nothing if there is a higher one where it does. So, for example, $[\![v_0 < v_9]\!]^{\mathcal{M}}$ is not a binary but a 10-ary concept; nevertheless, by substitution of variables it can be easily reduced to the binary $[\![v_0 < v_1]\!]^{\mathcal{M}}$. Since substitution of variables can be defined from basic operations of concept algebras, for example, $[\![v_0 < v_9]\!]^{\mathcal{M}} = [\![\exists v_9(v_0 < v_1 \wedge v_1 = v_9)]\!]^{\mathcal{M}} = \mathsf{C}_9([\![v_0 < v_1]\!]^{\mathcal{M}} \cdot D_{09})$, it is enough if we reduce the arity up to substitution. For a general and precise proof, see Lemma A.1 herein.
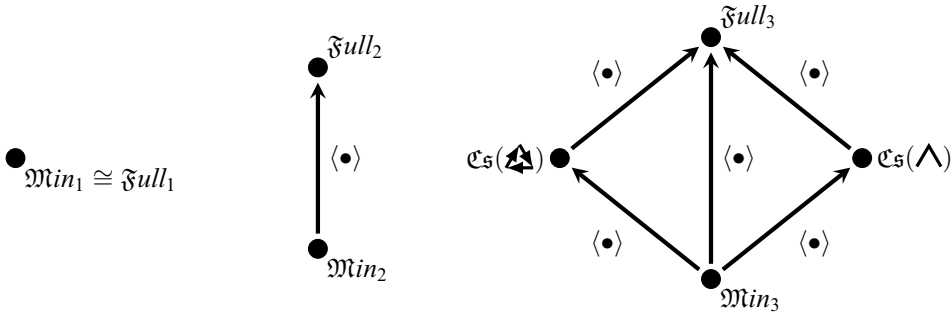
Fig. 2. The figure shows a complete description for the large embedding network of meaning algebras if $|M| \leq 3$.

Clearly, we have $\mathfrak{Min}_3 \xrightarrow{\langle \bullet \rangle} \mathfrak{Cs}(\mathbf{\&})$ and $\mathfrak{Min}_3 \xrightarrow{\langle \bullet \rangle} \mathfrak{Cs}(\bigwedge)$ as $\mathfrak{Cs}(\mathbf{\&})$ and $\mathfrak{Cs}(\bigwedge)$ are both given by only one relation. Neither $\mathfrak{Cs}(\mathbf{\&})$ nor $\mathfrak{Cs}(\bigwedge)$ can be largely embedded to the other. $\mathfrak{Cs}(\bigwedge)$ cannot be embedded to $\mathfrak{Cs}(\mathbf{\&})$ because there are two nontrivial unary concepts in $\mathfrak{Cs}(\bigwedge)$ and there are no such unary concepts in $\mathfrak{Cs}(\mathbf{\&})$ at all. $\mathfrak{Cs}(\mathbf{\&})$ cannot be embedded to $\mathfrak{Cs}(\bigwedge)$ because the concept $\mathbf{\&}$ has to be mapped to a nontrivial binary concept under $- D_{01}$ in $\mathfrak{Cs}(\bigwedge)$. Every such concept is the union of the following three ___, $\mathbf{\Lambda\hspace{-0.3em}\nwarrow}$ and $\mathbf{\nwarrow\hspace{-0.3em}\Lambda}$. It is straightforward to check that any of these nontrivial binary concepts generate the whole algebra $\mathfrak{Cs}(\bigwedge)$, and hence it cannot be the image of $\mathbf{\&}$.

Based on the observations above, we can give a complete description for the large embedding network of meaning algebras if $|M| \leq 3$ (see Figure 2). In this small network, the distances are either 0, 1, 2 or $\infty$. We are going to show that only these four distances appear even in the network of all meaning algebras (cf. also Theorem 5.1 below).

By *the **size** of a model* $\mathcal{M}$, we mean its cardinality if it is finite and $\infty$ otherwise, i.e.,

$$\mathsf{size}(\mathcal{M}) = \begin{cases} |M|, & \text{if } |M| < \infty, \\ \infty, & \text{otherwise.} \end{cases}$$

Let us first note that if two models $\mathcal{M}_1$ and $\mathcal{M}_2$ have different sizes, then their minimal subalgebras are non-isomorphic, and hence we have that

$$\mathsf{size}(\mathcal{M}_1) \neq \mathsf{size}(\mathcal{M}_2) \implies d_{\mathsf{LT}}(\mathfrak{Cs}(\mathcal{M}_1), \mathfrak{Cs}(\mathcal{M}_2)) = \infty. \tag{7}$$

Now, borrowing some ideas from the proof of [4, theorem 2.3.22], which shows that certain finitely generated algebras can be generated by a single element, we show that

$$\mathfrak{Cs}(\mathcal{M}_1) \xrightarrow{\langle \bullet \rangle} \mathfrak{Cs}(\mathcal{M}_2) \xrightarrow{\langle \bullet \rangle} \cdots \xrightarrow{\langle \bullet \rangle} \mathfrak{Cs}(\mathcal{M}_n) \implies \mathfrak{Cs}(\mathcal{M}_1) \xrightarrow{\langle \bullet \rangle} \mathfrak{Cs}(\mathcal{M}_n). \tag{8}$$

By induction, it is enough to show that (8) holds for $n = 3$. If $\mathfrak{Cs}(\mathcal{M}_1) \cong \mathfrak{Min}_1$, then $|M_1| = 1$, and hence $|M_2| = |M_3| = 1$ and $\mathfrak{Cs}(\mathcal{M}_2) \cong \mathfrak{Cs}(\mathcal{M}_3) \cong \mathfrak{Min}_1$, and thus $\mathfrak{Cs}(\mathcal{M}_1) \xrightarrow{\langle \bullet \rangle} \mathfrak{Cs}(\mathcal{M}_3)$ holds trivially. Otherwise, we have $|M_1| = |M_2| = |M_3| \geq 2$. Let $\rho$ and $\sigma$ be the formulas representing the concepts that we have added to $\mathfrak{Cs}(\mathcal{M}_1)$ and $\mathfrak{Cs}(\mathcal{M}_2)$ to reach $\mathfrak{Cs}(\mathcal{M}_3)$, and let $\tau \overset{\text{def}}{=} (\rho \wedge v_i = v_j) \vee (\sigma \wedge v_i \neq v_j)$ for some variables $v_i$ and $v_j$ none of which is free in $\rho$ and $\sigma$. Then because

$|M_3| \geq 2$, we have $[\![\rho]\!]^{\mathcal{M}_3} = [\![\exists v_i(\tau \wedge v_i = v_j)]\!]^{\mathcal{M}_3} = \mathsf{C}_i([\![\tau]\!]^{\mathcal{M}_3} \cdot \mathsf{D}_{ij})$ and $[\![\sigma]\!]^{\mathcal{M}_3} = [\![\exists v_i(\tau \wedge v_i \neq v_j)]\!]^{\mathcal{M}_3} = \mathsf{C}_i([\![\tau]\!]^{\mathcal{M}_3} \cdot - \mathsf{D}_{ij})$. This means that instead of the two concepts represented by $\rho$ and $\sigma$ we could have added the one represented by $\tau$ to reach $\mathfrak{Cs}(\mathcal{M}_3)$ from $\mathfrak{Cs}(\mathcal{M}_1)$, and this completes the proof of (8).

By [1, proposition 2.12] and the fact that the class of LT has the amalgamation property (see, e.g., [12, table 1] or [8]), we have that there is any path between two algebras $\mathfrak{A}, \mathfrak{B} \in \mathsf{LT}$, then there is an algebra $\mathfrak{D} \in \mathsf{LT}$ such that $\mathfrak{D}$ is reachable from both $\mathfrak{A}$ and $\mathfrak{B}$ by finitely many large embedding steps. Since up to isomorphism every algebra in LT is a meaning algebra, we get from this that, for any two models $\mathcal{M}_1$ and $\mathcal{M}_2$,

$$d_{\mathsf{LT}}(\mathfrak{Cs}(\mathcal{M}_1), \mathfrak{Cs}(\mathcal{M}_2)) \leq 2 \quad \text{if} \quad d_{\mathsf{LT}}(\mathfrak{Cs}(\mathcal{M}_1), \mathfrak{Cs}(\mathcal{M}_2)) < \infty. \tag{9}$$

From the above, it also follows that, if the languages of $\mathcal{M}_1$ and $\mathcal{M}_2$ are finite and $\mathsf{size}(\mathcal{M}_1) = \mathsf{size}(\mathcal{M}_2)$, then $d_{\mathsf{LT}}(\mathfrak{Cs}(\mathcal{M}_1), \mathfrak{Cs}(\mathcal{M}_2)) \leq 2$. This is so because, in this case, $\mathfrak{Cs}(\mathcal{M}_1)$ and $\mathfrak{Cs}(\mathcal{M}_2)$ have isomorphic minimal subalgebras from which they can be reached in one large embedding step by (8).

So if $\mathsf{size}(\mathcal{M}_1), \mathsf{size}(\mathcal{M}_2) < \infty$, then by (6), we have two cases:

either $\mathsf{size}(\mathcal{M}_1) = \mathsf{size}(\mathcal{M}_2)$, and then $d_{\mathsf{LT}}(\mathfrak{Cs}(\mathcal{M}_1), \mathfrak{Cs}(\mathcal{M}_2)) \leq 2$;
or $\mathsf{size}(\mathcal{M}_1) \neq \mathsf{size}(\mathcal{M}_2)$, and then $d_{\mathsf{LT}}(\mathfrak{Cs}(\mathcal{M}_1), \mathfrak{Cs}(\mathcal{M}_2)) = \infty$ by (7).

Consequently, determining the distance of meaning algebras corresponding to finite models boils down to three questions: Do they have the same size? If yes, are they definitionally equivalent? If not, can one of them largely embedded to the other?

What can we say in the case $\mathsf{size}(\mathcal{M}_1) = \mathsf{size}(\mathcal{M}_2) = \infty$?

Assume that $\mathsf{size}(\mathcal{M}) = \infty$, the language of $\mathcal{M}$ is countable and there is some natural number $n$ such that $\rho(R) \leq n$ for every relation symbol $R$ in the language of $\mathcal{M}$. Without loosing generality, we can assume that $\{R_i : i < \omega\}$ is a listing of these relation symbols and $\rho(R_i) = n$ for all natural number $i$. Now let us try encode the whole language of $\mathcal{M}$ with one extra relation symbol. By (8), it is enough if we can encode it with finitely many relations because adding finitely many extra relations can be replaced by adding only one extra relation if $|M| \geq 2$. Let $<$ be an ordering on a countable infinite subset of $M$ isomorphic to the ordering of natural numbers, there is such ordering because $\mathsf{size}(\mathcal{M}) = \infty$. From this ordering, all natural numbers $0, 1, 2$, etc. are definable individually as unary relations $0(x), 1(x), 2(x)$, etc. Now let $S$ be an $(n+1)$-ary relation such that $\mathcal{M} \models S(\bar{x}, y) \leftrightarrow R_i(\bar{x}) \wedge i(y)$ holds for every natural number $i$. From $<$ and $S$, each $R_i$ can be defined back because $\mathcal{M} \models R_i(\bar{x}) \leftrightarrow \exists y(S(\bar{x}, y) \wedge i(y))$. Consequently, there is a structure $\mathcal{M}^+$ whose language contains only one relation symbol, but all the countably infinite relations of $\mathcal{M}$ can be defined from that relation. In algebraic terms, this means that $\mathfrak{Cs}(\mathcal{M})$ can be largely embedded to a one-element generated meaning algebra.

From the above, it follows that, if language of models $\mathcal{M}_1$ and $\mathcal{M}_2$ are countable such that there is a finite bound for the rank of the relation symbols of these languages and $\mathsf{size}(\mathcal{M}_1) = \mathsf{size}(\mathcal{M}_2)$, then $d_{\mathsf{LT}}(\mathfrak{Cs}(\mathcal{M}_1), \mathfrak{Cs}(\mathcal{M}_2)) \leq 2$.

By Corollary 4.4, the above considerations can be summarized in terms of conceptual distance as follows.

THEOREM 5.1. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be models of arbitrary languages. Then*

$$\mathsf{Cd}(\mathsf{Th}(\mathcal{M}_1), \mathsf{Th}(\mathcal{M}_2)) \leq 2 \text{ or } \mathsf{Cd}(\mathsf{Th}(\mathcal{M}_1), \mathsf{Th}(\mathcal{M}_2)) = \infty \tag{10}$$

*and*

$$\text{size}(\mathcal{M}_1) \neq \text{size}(\mathcal{M}_2) \implies \text{Cd}(\text{Th}(\mathcal{M}_1), \text{Th}(\mathcal{M}_2)) = \infty. \qquad (11)$$

*If the languages of $\mathcal{M}_1$ and $\mathcal{M}_2$ are countable and there is some natural number* n *such that if the rank $\rho(R) \leq n$ for every relation R in the two languages, then*

$$\text{size}(\mathcal{M}_1) = \text{size}(\mathcal{M}_2) \implies \text{Cd}(\text{Th}(\mathcal{M}_1), \text{Th}(\mathcal{M}_2)) \leq 2. \qquad (12)$$

*Proof.* See the discussion above the statement starting with observation (7). □

Now we are going to show that the arity-bound condition of (12) in Theorem 5.1 cannot be omitted:

PROPOSITION 5.2. *There is a model $\mathcal{M}$ on a countable language such that*

$$\text{size}(\mathcal{M}) = \infty \text{ and } d_{\text{LT}}(\mathfrak{Min}_\omega, \mathfrak{Cs}(\mathcal{M})) = \infty.$$

*Proof.* Let $\mathcal{M} = \langle \omega, R_0 = \{0\}, R_1 = \{(1,2)\}, R_2 = \{(3,4,5)\}, \dots \rangle$, i.e., for each natural number $n$, $R_n \subset \omega^n$ is an $n$-ary relation containing exactly one sequence of elements which do not appear in the range or domain of relations $R_0, \dots, R_{n-1}$. Then clearly $R_{n+1}$ is not definable form relations $R_0, \dots, R_n$ because the infinitely many elements that appear neither in the range nor in the domain of these relations can be freely permuted by an automorphism of structure $\mathcal{M}_n = \langle \omega, R_0, R_1, \dots, R_n \rangle$. Hence it is not enough to add only finitely many relations to $\omega$ to reach $\mathcal{M}$. Consequently, $d_{\text{LT}}(\mathfrak{Min}_\omega, \mathfrak{Cs}(\mathcal{M})) = \infty$ as desired. □

**§A. Appendix.** For completeness, here we give precise proofs for equations (5) and (6) referring to general theorems of algebraic logic and prove a general statement true for the meaning algebras of all finite structures. The lemmas of this appendix are quite interesting in their own. Even though Lemma A.1 is not a surprise for us, we could not find any reference for it in the literature. So, we include a detailed proof herein.

LEMMA A.1. *Suppose that $|M| = n$, then the meaning algebra $\mathfrak{Cs}(\mathcal{M})$ is generated by*

$$\{x \in \mathfrak{Cs}(\mathcal{M}) : x \text{ is an n-ary concept}\}.$$

*Proof.* First, we need to recall the substitution operation on $\mathfrak{Cs}(\mathcal{M})$ (cf. [4, sec. 1.5]): For every $i, j \in \omega$, and every $x \in \mathfrak{Cs}(\mathcal{M})$,

$$S_j^i x = \begin{cases} x, & \text{if } i = j, \\ \mathsf{C}_i(x \cdot \mathsf{D}_{ij}), & \text{if } i \neq j. \end{cases}$$

Let $x \in \mathfrak{Cs}(\mathcal{M})$ be an $m$-ary concept of $\mathcal{M}$ for some $m \geq n + 1$. We will show that $x$ can be constructed from some $(m-1)$-ary concepts. Note that the equation

$$\sum_{i < j \leq n} \mathsf{D}_{ij} = 1$$

holds in $\mathfrak{Cs}(\mathcal{M})$ because it is not possible to take more than $n$ distinct elements of $\mathcal{M}$. Then, it follows that

$$x = \sum_{i < j \leq n} x \cdot \mathsf{D}_{ij}.$$

Let $i < j \leq n$. It is enough now to show that each $x_{ij} := x \cdot \mathsf{D}_{ij}$ can be constructed from an $(m-1)$-ary concepts.

– Case 1: Suppose that $i < j < m-1$. In this case, let $y_{ij} := S_i^{m-1}\mathsf{C}_i x_{ij}$. By [4, theorems 1.6.8 and 1.6.13], we know that $y_{ij}$ is an $(m-1)$-ary concept. Moreover,

$$
\begin{aligned}
S_{m-1}^i y_{ij} \cdot \mathsf{D}_{ij} &= S_{m-1}^i S_i^{m-1}\mathsf{C}_i x_{ij} \cdot \mathsf{D}_{ij} \\
&= S_{m-1}^i S_{m-1}^{m-1}\mathsf{C}_i x_{ij} \cdot \mathsf{D}_{ij} && \text{by [4, Thm 1.5.10 (ii)]} \\
&= S_{m-1}^i \mathsf{C}_i x_{ij} \cdot \mathsf{D}_{ij} \\
&= \mathsf{C}_i x_{ij} \cdot \mathsf{D}_{ij} && \text{by [4, Thm 1.5.8 (i)]} \\
&= x_{ij} && \text{by [4, Theorem 1.3.9].}
\end{aligned}
$$

– Case 2: Suppose that $j = m-1$. Thus, we must have $m = n+1$. Now, we let $y_{in} := \mathsf{C}_n x_{in}$. By [4, theorem 1.6.8] and [4, theorem 1.3.9], we have $y_{in}$ is an $n$-ary concept, and

$$
x_{in} = \mathsf{C}_n x_{in} \cdot \mathsf{D}_{in} = y_{in} \cdot \mathsf{D}_{in}.
$$

Thus, $x$ can be constructed from $(m-1)$-ary concepts. By repeating this process enough many times, the desired follows. □

Surprisingly, Lemma A.1 can be even strengthened more (see Lemma A.2). The logical interpretation of Lemma A.1 indicates that a model of size $n$ is definitionally equivalent to a model of size $n$ whose language consists of relation symbols of arity at most $n-1$.
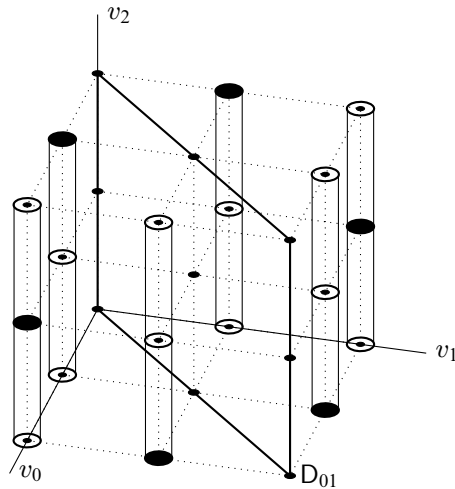


● – elements of $\bar{\mathsf{D}}_2$          ◉ – elements of $\mathsf{C}_2\bar{\mathsf{D}}_2 - \bar{\mathsf{D}}_2$

Fig. 3. If $|M| = n+1$, then, for every $(a_0, \ldots, a_{n-1}) \in M^n$ of pairwise distinct elements, there is a unique $a_n \in M$ such that sequence $(a_0, \ldots, a_n)$ can be extended to an element of $\bar{\mathsf{D}}_n$. Hence, $\bar{\mathsf{D}}_n \cap S = \bar{\mathsf{D}}_n \cap \mathsf{C}_n S$ for every $S \subseteq M^\omega$.

LEMMA A.2. *Suppose that* $|M| = n$, *then the meaning algebra* $\mathfrak{Cs}(\mathcal{M})$ *is generated by*

$$\{x \in \mathfrak{Cs}(\mathcal{M}) : x \text{ is an } (n-1)\text{-ary concept}\}.$$

*Proof.* By Lemma A.1, it is enough to show that an $n$-ary concept can be constructed from $(n-1)$-ary concepts. Let $x \in \mathfrak{Cs}(\mathcal{M})$, then

$$x = x \cdot \bar{\mathsf{D}}_{n-1} + \sum_{i < j \leq n-1} x \cdot \mathsf{D}_{ij},$$

where $\bar{\mathsf{D}}_{n-1} := \prod_{i < j \leq n-1} - \mathsf{D}_{ij}$. Similarly to the proof of Lemma A.1, one can see that the element $x \cdot \mathsf{D}_{ij}$, for each $i < j \leq n-1$, is generated by the diagonal $\mathsf{D}_{ij}$ together with an $(n-1)$-ary concept. So, it is enough to consider the element $x \cdot \bar{\mathsf{D}}_{n-1}$. We claim that

$$x \cdot \bar{\mathsf{D}}_{n-1} = \bar{\mathsf{D}}_{n-1} \cdot \mathsf{C}_{n-1}(x \cdot \mathsf{D}_{n-1}).$$

This is true because the $(n-1)$-th coordinate of a sequence in $x \cdot \mathsf{D}_{n-1}$ is uniquely determined from the previous coordinates as the only remaining element of $M$ which is not yet listed in the first $(n-1)$ coordinates. Thus, intersecting with $\mathsf{D}_{n-1}$ after taking its cylindrification $\mathsf{C}_{n-1}$ gives back $x \cdot \mathsf{D}_{n-1}$ (cf. Figure 3). Obviously, by the facts in [4, theorems 1.6.4 and 1.6.6–1.6.8], the element $\mathsf{C}_{n-1} x \cdot \bar{\mathsf{D}}_{n-1}$ is an $(n-1)$-ary concept.    □

BIBLIOGRAPHY

[1] Aslan, T., Khaled, M., & Székely, G. (2021). On the networks of large embeddings, preprint, arXiv:2112.12587.

[2] Barrett, T. W. (2017). On the Structure and Equivalence of Theories. Ph.D. Thesis, Princeton University.

[3] Coffey, K. (2014). Theoretical equivalence as interpretative equivalence. *The British Journal for the Philosophy of Science*, **65**(4), 821–844.

[4] Henkin, L., Monk, J. D., & Tarski, A. (1971). *Cylindric Algebras. Part I*. Amsterdam: North-Holland.

[5] ———. (1985). *Cylindric Algebras. Part II*. Amsterdam: North-Holland.

[6] Khaled, M., & Székely, G. (2021). Algebras of concepts and their networks. In Allahviranloo, T., Salahshour, S., & Arica, N., editors. *Progress in Intelligent Decision Science. Proceeding of IDS 2020*. Cham: Springer, pp. 611–622.

[7] Khaled, M., Székely, G., Lefever, K., & Friend, M. (2020). Distances between formal theories. *Review of Symbolic Logic*, **13**(3), 633–654.

[8] Kiss, E. W., Márki, L., Pröhle, P., & Tholen, W. (1983). Categorical algebraic properties. A compendium on amalgamation, congruence extension, epimorphisms, residual smallness, and injectivity. *Studia Scientiarum Mathematicarum Hungarica*, **18**, 79–141.

[9] Lefever, K. (2017). Using Logical Interpretation and Definitional Equivalence to Compare Classical Kinematics and Special Relativity Theory. Ph.D. Thesis, Vrije Universiteit Brussel.

[10] Lefever, K., & Székely, G. (2018). Comparing classical and relativistic kinematics in first-order logic. *Logique et Analyse*, **61**(241), 57–117.

[11] ———. (2019). On generalization of definitional equivalence to non-disjoint languages. *Journal of Philosophical Logic*, **48**(4), 709–729.

[12] Madárasz, J., & Sayed-Ahmed, T. (2007). Amalgamation, interpolation and epimorphisms in algebraic logic. *Algebra Universalis*, **56**(2), 179–210.

[13] Monk, J. D. (2000). An introduction to Cylindric set algebras. *Logic Journal of the IGPL*, **8**(4), 451–492.

[14] Rosenstock, S., Barrett, T. W., & Weatherall, J. O. (2015). On Einstein algebras and relativistic spacetimes. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, **5**(B), 309–316.

[15] Weatherall, J. O. (2016). Are Newtonian gravitation and geometrized Newtonian gravitation theoretically equivalent? *Erkenn*, **81**, 1073–1091.

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