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On Mertens' Theorem for Beurling Primes

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Abstract. Let $1 < p_1 \leq p_2 \leq p_3 \leq \dots$ be an infinite sequence $\mathscr{P}$ of real numbers for which $p_i \to \infty$, and associate with this sequence the *Beurling zeta function* $\zeta_{\mathscr{P}}(s) := \prod_{i=1}^{\infty}(1 - p_i^{-s})^{-1}$. Suppose that for some constant $A > 0$, we have $\zeta_{\mathscr{P}}(s) \sim A/(s-1)$, as $s \downarrow 1$. We prove that $\mathscr{P}$ satisfies an analogue of a classical theorem of Mertens: $\prod_{p_i \leq x}(1 - 1/p_i)^{-1} \sim Ae^{\gamma} \log x$, as $x \to \infty$. Here $e = 2.71828\dots$ is the base of the natural logarithm and $\gamma = 0.57721\dots$ is the usual Euler–Mascheroni constant. This strengthens a recent theorem of Olofsson.

1 Introduction

Let $\mathscr{M}$ be the free commutative monoid on the symbols $p_1, p_2, p_3, \dots$, so that the elements of $\mathscr{M}$ correspond precisely to the products $\prod_{i=1}^{\infty} p_i^{e_i}$, where each $e_i \geq 0$ and all but finitely many of the e_i vanish. By a *Beurling system*, we mean a map $|\cdot|\colon \mathscr{M} \to \mathbf{R}_{>0}$ with the following properties:

(i) $|p_i| > 1$ for all i,
(ii) $|ab| = |a||b|$ for all $a, b \in \mathscr{M}$,
(iii) for all x, there are only finitely many $m \in \mathscr{M}$ with $|m| \leq x$.

The elements of $\mathscr{M}$ are called *Beurling integers*, and the p_i are called *Beurling primes.* Throughout this article, we use the letter $\mathscr{B}$ to denote a Beurling system (*i.e.,* a choice of norm $|\cdot|$).

In the theory of Beurling primes, the key objects of study are the counting functions

$$\pi_{\mathscr{B}}(x) := \sum_{|p| \leq x} 1 \qquad \text{and} \qquad N_{\mathscr{B}}(x) := \sum_{|n| \leq x} 1,$$

where p runs over all Beurling primes of norm $\leq x$ in the former sum and n runs over all Beurling integers of norm $\leq x$ in the latter. One views $\pi_{\mathscr{B}}(x)$ as an analogue of the classical prime counting function $\pi(x)$ and $N_{\mathscr{B}}(x)$ as an analogue of the integer counting function $\lfloor x \rfloor$. The fundamental goal in *Beurling prime number theory* is to show that if one of $\pi_{\mathscr{B}}(x)$ or $N_{\mathscr{B}}(x)$ behaves like its classical counterpart, then so does the other.

Beurling's initial theorem [1] was the following generalization of the classical prime number theorem. Suppose that for some fixed $A > 0$, we have the estimate

$$N_{\mathscr{B}}(x) = Ax + O\big(x/(\log x)^a\big) \tag{1.1}$$

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with an exponent $a > 3/2$. Under these conditions, $\pi_{\mathscr{B}}(x) \sim x/\log x$ (as $x \to \infty$). In other words, the analogue of the prime number theorem holds for the Beurling system. It is known ([2]) that Beurling's result is sharp. The asymptotic relation may fail if we only assume (1.1) with $a = 3/2$. Moreover, for any $a < 1$, there are Beurling systems for which (1.1) holds and for which $\limsup_{x\to\infty} \pi_{\mathscr{B}}(x)/(x/\log x) = \infty$ and $\liminf_{x\to\infty} \pi_{\mathscr{B}}(x)/(x/\log x) = 0$ (see [5], and *cf.* [3]). So the following theorem of Olofsson [8, Theorem 1.1], which assumes comparatively little about $N_{\mathscr{B}}(x)$, is perhaps a bit surprising.

Theorem A *Assume that there is a constant $A > 0$ with $N_{\mathscr{B}}(x) \sim Ax$, as $x \to \infty$. Then one has an analogue of Mertens' theorem: As $x \to \infty$,*

$$\prod_{|p|\le x} (1 - 1/|p|)^{-1} \sim Ae^{\gamma} \log x. \tag{1.2}$$

In this note, we investigate weakening the hypothesis in Olofsson's result. Define the *Beurling zeta-function* by the Euler-product formula

$$\zeta_{\mathscr{B}}(s) := \prod_p \frac{1}{1 - \frac{1}{|p|^s}}.$$

Since $\zeta_{\mathscr{B}}(s) = \int_{1^-}^{\infty} t^{-s}\, dN_{\mathscr{B}}(t)$, integration by parts shows that if $N_{\mathscr{B}}(x) \sim Ax$ (with $A > 0$), then

$$\zeta_{\mathscr{B}}(s) \sim \frac{A}{s-1}, \qquad \text{as } s \downarrow 1. \tag{1.3}$$

Our first theorem shows that one can replace Olofsson's hypothesis on $N_{\mathscr{B}}(x)$ with (1.3). In fact, (1.2) and (1.3) turn out to be equivalent.

Theorem 1.1 *Let $A > 0$. If (1.3) holds for a Beurling system $\mathscr{B}$, then so does (1.2). Conversely, the Mertens-type formula (1.2) implies (1.3).*

Remark 1.2 Theorem 1.1 could also have been formulated with the hypothesis (1.3) on $\zeta_{\mathscr{B}}(s)$ replaced by the assumption that $\sum_{|n|\le x} \frac{1}{|n|} \sim A\log x$, as $x \to \infty$. Indeed, these two conditions on a Beurling system turn out to be equivalent. The forward direction of this equivalence is proved below as Lemma 2.1; the other direction follows by partial summation, using the fact that $\zeta_{\mathscr{B}}(s) = \sum_n \frac{1}{|n|^s}$.

If K is a number field, one can put the p_i in one-to-one correspondence with the prime ideals of the ring of integers of K, setting $|p_i|$ equal to the norm of the corresponding prime ideal. Then there is a norm-preserving isomorphism between the Beurling system $\mathscr{B}$ so obtained and the monoid of integral ideals of K. After Dirichlet and Dedekind, one knows that $N_{\mathscr{B}}(x) \sim Ax$, where A depends on certain arithmetic invariants of K. Thus, Olofsson's Theorem A gives another proof of Mertens' theorem for number fields. (For a precise statement, see [12, Theorem 2].) Unfortunately, Theorem A does not apply in the global function field case, since the number of divisors of norm $\le x$ is not asymptotic to a constant multiple of x. However, it is still true

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that the associated zeta function has a simple pole at $s = 1$ with an easily described residue, and so Theorem 1.1 permits one to to recover the function field analogue of Mertens' theorem [12, Theorem 3], although without an explicit error term. Thus, we achieve a unified proof of Mertens' theorem for global fields. For further work on these generalizations; see [7].

Using the known results on the analytic behavior of Ruelle zeta functions, Theorem 1.1 immediately yields Sharp's analogue of Mertens' theorem for hyperbolic flows [13]; *cf.* [10]. (The needed properties of these zeta functions are established in [9, Theorem 5.6, p. 84], the example on pp. 85–86, and the calculation at the top of p. 96. See also the survey in [11, §6].)

One may try weakening Olofsson's hypotheses even further. As remarked above, the hypothesis (1.3) is equivalent to the assumption that $\sum_{|n|\le x} 1/|n| \sim A\log x$. In the next theorem, we consider the situation where one supposes only that $\sum_{|n|\le x} 1/|n| \asymp \log x$.

Theorem 1.3 *Suppose $\mathscr{B}$ is a Beurling system for which*

$$\sum_{|n|\le x} \frac{1}{|n|} \asymp \log x \tag{1.4}$$

for all $x \ge 2$. Then for $x \ge 2$,

$$\prod_{|p|\le x} (1 - 1/|p|)^{-1} \asymp \log x. \tag{1.5}$$

This conclusion is best possible, in the sense that (1.4) *may hold without the quotient of the left and right-hand sides of* (1.5) *tending to a limit (as $x \to \infty$).*

Notation

We continue to use $\mathscr{B}$ to denote a fixed Beurling system. In what follows, the letters m, n, and d are reserved for Beurling integers, and the letter p for Beurling primes. We always assume (relabelling the p_i if necessary) that $|p_1| = \min_{i\ge 1} |p_i|$. We use Ei for the exponential integral, so that $\mathrm{Ei}(x) := \int_{-\infty}^{x} \frac{e^t}{t}\, dt$.

We remind the reader that "$f = O(g)$", "$f \ll g$", and "$g \gg f$" all mean that for some positive constant C, we have $|f| \le Cg$ for all values of the variables under consideration. The notation $f \asymp g$ means that $f \ll g$ and $g \ll f$. All implied constants may depend on $\mathscr{B}$.

2 Proof of Theorem 1.1

We begin by quoting two Tauberian theorems from the literature (see [6, Theorems 15.1 and 15.3, p. 30]).

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Theorem B *Let $s(v)$ vanish for $v < 0$, be nondecreasing, continuous from the right, and suppose that for some fixed $\alpha > 0$ and some constant $L > 0$, we have*

$$\int_{0-}^{\infty} \mathrm{e}^{-rv}\, ds(v) \sim L/r^{\alpha}, \qquad \textit{as } r \downarrow 0.$$

Then as $u \to \infty$, we have $s(u) \sim \frac{L}{\Gamma(\alpha+1)} u^{\alpha}$.

Theorem C *Suppose $a(v)$ is locally Riemann integrable and that the improper integral $F(r) := \int_0^\infty a(v)\mathrm{e}^{-rv}\,dv$ exists for all $r > 0$. Suppose that $F(r) \to L$ as $r \downarrow 0$ and that for all sufficiently large values of v, one has $a(v) \geq -C/v$ for some constant $C > 0$. Then (as an improper Riemann integral) $\int_0^\infty a(v)\,dv = L$.*

We suppose for the remainder of this section that we have fixed a Beurling system $\mathscr{B}$ satisfying (1.3). The next sequence of lemmas develops the basic analytic number theory of the corresponding Beurling integers.

Lemma 2.1 *As $x \to \infty$, we have $\sum_{|n| \leq x} \frac{1}{|n|} \sim A \log x$.*

Proof We apply Theorem B with $s(v) := \sum_{|n| \leq e^{v}} |n|^{-1}$. Then

$$\int_{0-}^{\infty} \mathrm{e}^{-rv}\, ds(v) = \sum_{n} |n|^{-1-r} = \zeta_{\mathscr{B}}(1+r) \sim A/r, \qquad \text{as } r \downarrow 0.$$

So by Theorem B (with $L = A$ and $\alpha = 1$), we have $s(u) \sim Au$ as $u \to \infty$. Now put $u = \log x$ to obtain the lemma. ∎

For each Beurling integer n, set $\Lambda(n) = \log |p|$ if n is a power of the Beurling prime p, and put $\Lambda(n) = 0$ otherwise.

Lemma 2.2 *As $x \to \infty$, we have*

$$\sum_{|d| \leq x} \frac{\Lambda(d)}{|d|} \sum_{|m| \leq x/|d|} \frac{1}{|m|} \sim \frac{A}{2} (\log x)^2.$$

Proof Since $\log |n| = \sum_{d|n} \Lambda(d)$, we have

$$\sum_{|n| \leq x} \frac{\log |n|}{|n|} = \sum_{|n| \leq x} \frac{1}{|n|} \sum_{d|n} \Lambda(d) = \sum_{|d| \leq x} \Lambda(d) \sum_{\substack{|n| \leq x \\ d|n}} \frac{1}{|n|} = \sum_{|d| \leq x} \frac{\Lambda(d)}{|d|} \sum_{|m| \leq x/|d|} \frac{1}{|m|}.$$

On the other hand,

$$\sum_{|n| \leq x} \frac{\log |n|}{|n|} = \int_{1^-}^{x} \log t \, d\Big(\sum_{|n| \leq t} \frac{1}{|n|} \Big) \sim \frac{A}{2} (\log x)^2$$

by Lemma 2.1 and an easy integration by parts. Comparing these two expressions gives the lemma. ∎

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Lemma 2.3 *As $x \to \infty$, we have*

$$\Big(\frac{1}{2}+o(1)\Big)\,\log x \le \sum_{|d|\le x}\frac{\Lambda(d)}{|d|} \le \big(2+o(1)\big)\,\log x.$$

Proof First, observe that from Lemma 2.2,

$$\sum_{|d|\le x}\frac{\Lambda(d)}{|d|}\sum_{|m|\le x}\frac{1}{|m|} \ge \big(1+o(1)\big)\,\frac{A}{2}(\log x)^2;$$

using that $\sum_{|m|\le x} 1/|m| \le (1+o(1))A\log x$, we obtain the lower estimate of the lemma. Next, note that from Lemma 2.2,

$$\sum_{|d|\le\sqrt{x}}\frac{\Lambda(d)}{|d|}\sum_{|m|\le\sqrt{x}}\frac{1}{|m|} \le \sum_{|d|\le x}\frac{\Lambda(d)}{|d|}\sum_{|m|\le x/|d|}\frac{1}{|m|} \le \big(1+o(1)\big)\,\frac{A}{2}(\log x)^2.$$

Since $\sum_{|m|\le\sqrt{x}}\frac{1}{|m|} \ge (\frac{1}{2}+o(1))A\log x$, we find that

$$\sum_{|d|\le\sqrt{x}}\frac{\Lambda(d)}{|d|} \le \big(1+o(1)\big)\,\log x.$$

Now replace x with x^2 to obtain the upper estimate. ∎

Lemma 2.4 *As $x \to \infty$,*

$$\sum_{|d|\le x}\frac{\Lambda(d)}{|d|}\log\frac{x}{|d|} \sim \frac{1}{2}(\log x)^2. \tag{2.1}$$

Proof Write $\sum_{|m|\le t}|m|^{-1} = A\log t + E(t)$, so that $E(t) = o(\log t)$ as $t\to\infty$. By Lemma 2.2, it suffices to show that as $x\to\infty$,

$$\sum_{|d|\le x}\frac{\Lambda(d)}{|d|}E\big(x/|d|\big) = o\big((\log x)^2\big).$$

Fix $\epsilon > 0$. Choose t_0 so that $|E(t)| < \epsilon\log t$ once $t \ge t_0$. Then with B denoting an upper bound on $|E(t)|$ for $t \le t_0$, we find that

$$\begin{aligned}\Big|\sum_{|d|\le x}\frac{\Lambda(d)}{|d|}E\big(x/|d|\big)\Big| &\le \epsilon\sum_{|d|\le x/t_0}\frac{\Lambda(d)}{|d|}\log\frac{x}{|d|} + \sum_{x/t_0<|d|\le x}\frac{\Lambda(d)}{|d|}\big|E\big(x/|d|\big)\big| \\ &\le \epsilon\log x\sum_{|d|\le x}\frac{\Lambda(d)}{|d|} + B\sum_{|d|\le x}\frac{\Lambda(d)}{|d|}.\end{aligned}$$

Using the upper bound of Lemma 2.3, we see that

$$\limsup_{x\to\infty}\frac{\Big|\sum_{|d|\le x}\frac{\Lambda(d)}{|d|}E(x/|d|)\Big|}{(\log x)^2} \le 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary, the lemma follows. ∎

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We now improve Lemma 2.3 to an asymptotic result.

Lemma 2.5 *As $x \to \infty$, we have*

$$\sum_{|d| \le x} \frac{\Lambda(d)}{|d|} \sim \log x.$$

Proof Write $G(x)$ for the sum appearing on the left-hand side of (2.1). For each fixed $\epsilon > 0$, we have

$$G(x^{1+\epsilon}) - G(x) \ge \sum_{|d| \le x} \frac{\Lambda(d)}{|d|} \log(x^{\epsilon}) = \epsilon \log x \sum_{|d| \le x} \frac{\Lambda(d)}{|d|}.$$

On the other hand, using the asymptotic formula provided by Lemma 2.4,

$$G(x^{1+\epsilon}) - G(x) \sim \frac{1}{2}\big((1+\epsilon)^2 - 1\big)(\log x)^2.$$

Comparing these two expressions, we find that

$$\limsup_{x \to \infty} \frac{\sum_{|d| \le x} \frac{\Lambda(d)}{|d|}}{\log x} \le \frac{1}{2} \frac{(1+\epsilon)^2 - 1}{\epsilon}.$$

Letting $\epsilon \downarrow 0$, we get the upper-bound estimate implicit in the statement of Lemma 2.5. The lower bound can be handled similarly, upon noting that

$$\begin{aligned} G(x) - G(x^{1-\epsilon}) &= \sum_{|d| \le x^{1-\epsilon}} \frac{\Lambda(d)}{|d|} \log(x^{\epsilon}) + \sum_{x^{1-\epsilon} < |d| \le x} \frac{\Lambda(d)}{|d|} \log \frac{x}{|d|} \\ &\le \sum_{|d| \le x} \frac{\Lambda(d)}{|d|} \log(x^{\epsilon}) = \epsilon \log x \sum_{|d| \le x} \frac{\Lambda(d)}{|d|}. \end{aligned}$$

■

We next prove two crude bounds on $N_{\mathscr{B}}(x)$ and $\pi_{\mathscr{B}}(x)$.

Lemma 2.6 *For $x \ge 3$, we have $N_{\mathscr{B}}(x) \ll x \log x$.*

Proof We have only to observe that

$$N_{\mathscr{B}}(x) = \sum_{|n| \le x} 1 \le x \sum_{|n| \le x} \frac{1}{|n|} \ll x \log x,$$

using Lemma 2.1 in the last step. ■

To establish the needed estimate on $\pi_{\mathscr{B}}(x)$, we first isolate a simple consequence of Lemmas 2.5 and 2.6.

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Lemma 2.7 *As $x \to \infty$, we have*

$$\sum_{|p| \le x} \frac{\log |p|}{|p|} \sim \log x.$$

Proof After Lemma 2.5, it suffices to observe that the contribution to $\sum_{|d| \le x} \frac{\Lambda(d)}{|d|}$ from those d which are not prime is

$$\le \sum_{|p| \le x} \log |p| \Big(\frac{1}{|p|^2} + \frac{1}{|p|^3} + \cdots \Big) \ll \sum_{|p| \le x} \frac{\log |p|}{|p|^2} \le \int_1^\infty \frac{\log t}{t^2} \, dN_{\mathscr{B}}(t) \ll 1,$$

using Lemma 2.6 to estimate the integral. ∎

Lemma 2.8 $\pi_{\mathscr{B}}(x)/x \to 0$ *as $x \to \infty$.*

Proof As $x \to \infty$,

$$\#\big\{ p : x/(\log x)^2 < |p| \le x \big\} \cdot \frac{\log x}{x} \le \sum_{x/(\log x)^2 < |p| \le x} \frac{\log |p|}{|p|} = o(\log x),$$

by Lemma 2.7. Hence, the number of primes p with $x/(\log x)^2 < |p| \le x$ is $o(x)$. But the number of p with $|p| \le x/(\log x)^2$ is trivially at most $N_{\mathscr{B}}(x/(\log x)^2) \ll x/\log x$ (using Lemma 2.6), and so is also $o(x)$. ∎

We can now complete the proof of Theorem 1.1. We begin with the forward direction.

Proof that (1.3) $\Rightarrow$ (1.2) The proof of (1.2) is completed in the same manner as Olofsson's Theorem A. To keep the paper self contained, we give the details, closely following [8]. For real $s > 1$,

(2.2)
$$\begin{aligned}
&\log \zeta_{\mathscr{B}}(s) \\
&\quad = -\int_{|p_1|^-}^\infty \log(1 - t^{-s}) \, d\pi_{\mathscr{B}}(t) = s \int_{|p_1|}^\infty \frac{\pi_{\mathscr{B}}(t)}{t(t^s - 1)} \, dt \\
&\quad = s \int_{|p_1|}^\infty \Big(\pi_{\mathscr{B}}(t) - \frac{t}{\log t} \Big) t^{-(s+1)} \, dt + s \int_{|p_1|}^\infty \frac{\pi_{\mathscr{B}}(t)}{t^{s+1}(t^s - 1)} \, dt + s \int_{|p_1|}^\infty \frac{t^{-s}}{\log t} \, dt.
\end{aligned}$$

Thus, as $s \downarrow 1$,

$$\begin{aligned}
\log \zeta_{\mathscr{B}}(s) = s \int_{|p_1|}^\infty \Big(\frac{\pi_{\mathscr{B}}(t)}{t^2} - \frac{1}{t \log t} \Big) t^{-(s-1)} \, dt + \int_{|p_1|}^\infty \frac{\pi_{\mathscr{B}}(t)}{t^2(t-1)} \, dt \\
+ \log \frac{1}{s-1} - \log\log |p_1| - \gamma + o(1).
\end{aligned}$$

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(The final integral appearing in (2.2) can be seen to equal $-\mathrm{Ei}((1-s)\log|p_1|)$, by a change of variables; the estimate given above then follows from the relation $\mathrm{Ei}(t) = \log(-t) + \gamma + o(1)$, as $t \uparrow 0$. See, *e.g.*, [4, p. 884].) Put

$$I(s-1) := \int_{|p_1|}^{\infty} \Big(\frac{\pi_{\mathscr{B}}(t)}{t^2} - \frac{1}{t\log t} \Big) t^{-(s-1)}\, dt. \tag{2.3}$$

From the above and our hypothesis (1.3) that $(s-1)\zeta_{\mathscr{B}}(s) \to A$, we have that as $s \downarrow 1$,

$$\begin{aligned} s \cdot I(s-1) &= \log\big((s-1)\zeta_{\mathscr{B}}(s)\big) + \log\log|p_1| + \gamma - \int_{|p_1|}^{\infty} \frac{\pi_{\mathscr{B}}(t)}{t^2(t-1)}\, dt + o(1) \\ &= L + o(1), \end{aligned} \tag{2.4}$$

where

$$L := \log(Ae^{\gamma}) + \log\log|p_1| - \int_{|p_1|}^{\infty} \frac{\pi_{\mathscr{B}}(t)}{t^2(t-1)}\, dt.$$

If we make the change of variables $t := e^v$, we find that

$$I(s-1) = \int_{\log|p_1|}^{\infty} \Big(\frac{\pi_{\mathscr{B}}(e^v)}{e^v} - \frac{1}{v} \Big) e^{-v(s-1)}\, dv.$$

As $s \downarrow 1$, we have seen that $I(s-1) \to L$; applying Theorem C with $a(v) := \pi_{\mathscr{B}}(e^v)e^{-v} - v^{-1}$ shows that

$$I(0) = L. \tag{2.5}$$

Applying partial summation once again, we find that

$$\begin{aligned} \log \prod_{|p|\le x} \big(1 - 1/|p|\big)^{-1} &= -\sum_{|p|\le x} \log\big(1-1/|p|\big) = -\int_{|p_1|^-}^{x} \log(1-1/t)\, d\pi_{\mathscr{B}}(t) \\ &= \int_{|p_1|}^{x} \frac{\pi_{\mathscr{B}}(t)}{t^2 - t}\, dt + O\big(\pi_{\mathscr{B}}(x)/x\big). \end{aligned}$$

The error term is $o(1)$ by Lemma 2.8. Keeping (2.5) in mind, we see that as $x \to \infty$,

$$\begin{aligned} \int_{|p_1|}^{x} \frac{\pi_{\mathscr{B}}(t)}{t^2 - t} &= \int_{|p_1|}^{x} \Big(\frac{\pi_{\mathscr{B}}(t)}{t^2} - \frac{1}{t\log t} \Big)\, dt + \int_{|p_1|}^{x} \frac{dt}{t\log t} + \int_{|p_1|}^{x} \frac{\pi_{\mathscr{B}}(t)}{t^2(t-1)}\, dt \\ &= I(0) + \log\log x - \log\log|p_1| + \int_{|p_1|}^{\infty} \frac{\pi_{\mathscr{B}}(t)}{t^2(t-1)}\, dt + o(1) \\ &= \log\left(Ae^{\gamma}\log x\right) + o(1). \end{aligned}$$

Collecting our estimates and exponentiating, we arrive at the estimate

$$\prod_{|p|\le x} \big(1 - 1/|p|\big)^{-1} \sim Ae^{\gamma} \log x,$$

which completes the proof. ∎

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The reverse direction is easier.

Proof that (1.2) $\Rightarrow$ (1.3) We have to show that if $\mathscr{B}$ is a Beurling system for which the Mertens-type theorem (1.2) holds with the constant $A > 0$, then $\zeta_{\mathscr{B}}(s) \sim A/(s-1)$ as $s \downarrow 1$. We first show that the crude estimates of Lemmas 2.6 and 2.8 are valid under the hypothesis (1.2). First, observe that

$$N_{\mathscr{B}}(x) \leq x \sum_{|n| \leq x} \frac{1}{|n|} \leq x \prod_{|p| \leq x} \bigl(1 - 1/|p|\bigr)^{-1} \ll x \log x.$$

Also, (1.2) gives that

$$\log \prod_{x/(\log x)^2 < |p| \leq x} \bigl(1 - 1/|p|\bigr)^{-1} \to 0, \qquad \text{as } x \to \infty.$$

But

$$\begin{aligned} \log \prod_{x/(\log x)^2 < |p| \leq x} \bigl(1 - 1/|p|\bigr)^{-1} &\geq \sum_{x/(\log x)^2 < |p| \leq x} \frac{1}{|p|} \\ &\geq \frac{1}{x} \# \bigl\{ p : x/(\log x)^2 < |p| \leq x \bigr\}. \end{aligned}$$

Hence, there are only $o(x)$ primes p with $x/(\log x)^2 < |p| \leq x$, as $x \to \infty$. As in the proof of Lemma 2.8, there are $\ll x/\log x$ primes p with $|p| \leq x/(\log x)^2$. So the number of p with $|p| \leq x$ is $o(x)$.

With this preparation out of the way, we can reason as in the proof of the forward direction to find that

$$\log \prod_{|p| \leq x} (1 - 1/|p|)^{-1} =$$
$$\int_{|p_1|}^{x} \Bigl(\frac{\pi_{\mathscr{B}}(t)}{t^2} - \frac{1}{t \log t} \Bigr)\, dt + \log\log x - \log\log|p_1| + \int_{|p_1|}^{\infty} \frac{\pi_{\mathscr{B}}(t)}{t^2(t-1)}\, dt + o(1).$$

We are assuming the Mertens-type formula (1.2) holds, so that

$$\log \prod_{|p| \leq x} \bigl(1 - 1/|p|\bigr)^{-1} = \log(Ae^{\gamma}) + \log\log x + o(1);$$

comparing with the previous expression, we deduce that if we define

$$L(x) := \int_{|p_1|}^{x} \Bigl(\frac{\pi_{\mathscr{B}}(t)}{t^2} - \frac{1}{t \log t} \Bigr)\, dt,$$

then $L(x) \to L$ as $x \to \infty$, where L is as in (2.4). In other words, defining $I(s-1)$ as in (2.3), we have $I(0) = L$.

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Returning to $\zeta_{\mathscr{B}}(s)$, we have (by the same arguments appearing in the forward direction) that as $s \downarrow 1$,

$$\log \zeta_{\mathscr{B}}(s) = sI(s-1) + \int_{|p_1|}^{\infty} \frac{\pi_{\mathscr{B}}(t)}{t^2(t-1)}\,dt + \log\frac{1}{s-1} - \log\log|p_1| - \gamma + o(1).$$

If we show that $I(s-1) \to I(0) = L$ as $s \downarrow 1$, substituting the value (2.4) of L into this expression will give that as $s \downarrow 1$,

$$\log \zeta_{\mathscr{B}}(s) = \log\frac{A}{s-1} + o(1),$$

and the desired result (1.3) will follow upon exponentiating. To see that $I(s-1) \to L$ as $s \downarrow 0$, we note that for $s > 1$,

$$I(s-1) = \int_{|p_1|}^{\infty} t^{-(s-1)}\,dL(t) = (s-1)\int_{|p_1|}^{\infty} L(t)t^{-s}\,dt.$$

Since $L(t) \to L$ as $t \to \infty$, it is straightforward to show (by writing $L(t) = L + E(t)$, where $E(t) = o(1)$ as $t \to \infty$) that as $s \downarrow 1$,

$$I(s-1) = (s-1)\int_{|p_1|}^{\infty} L\cdot t^{-s}\,dt + o(1) = L + o(1).$$

This completes the proof. ∎

3 Proof of Theorem 1.3

We now suppose that $\mathscr{B}$ is a fixed Beurling system for which $\sum_{|n|\le x}\frac{1}{|n|} \asymp \log x$ for $x \ge 2$. In this situation, it is easy to establish the lower estimate implicit in (1.5). We have

$$\prod_{|p|\le x}\bigl(1-1/|p|\bigr)^{-1} = \sum_{n:\ p|n \Rightarrow |p|\le x}\frac{1}{|n|} \ge \sum_{|n|\le x}\frac{1}{|n|} \gg \log x.$$

So we may focus our attention on the upper estimate. Now

(3.1)
$$\sum_{|p|\le x}\Bigl(\log\frac{1}{1-1/p} - \frac{1}{p}\Bigr) = \sum_{|p|\le x}\Bigl(\frac{1}{2|p|^2} + \frac{1}{3|p|^3} + \cdots\Bigr)$$
$$\ll \sum_{p}\frac{1}{|p|^2} \le \sum_{n}\frac{1}{|n|^2} = \int_{1^-}^{\infty}\frac{1}{t}\,d\Bigl(\sum_{|n|\le t}\frac{1}{|n|}\Bigr) \ll 1.$$

Thus, it is enough to show that for $x \ge 3$,

$$\sum_{|p|\le x}\frac{1}{|p|} \le \log\log x + O(1). \tag{3.2}$$

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To establish (3.2), we need the following crude upper bound for the sum considered in Lemmas 2.3 and 2.5:

$$\sum_{|d|\leq x} \frac{\Lambda(d)}{|d|} \ll \log x. \tag{3.3}$$

This is simple to prove. On one hand, $\sum_{|n|\leq x} \frac{\log|n|}{|n|} = \int_{1-}^{x} \log t \, d\big(\sum_{|n|\leq t} \frac{1}{|n|}\big) \ll (\log x)^2$. On the other hand,

$$\sum_{|n|\leq x} \frac{\log|n|}{|n|} = \sum_{|n|\leq x} \frac{1}{|n|} \sum_{d|n} \Lambda(d) = \sum_{|d|\leq x} \frac{\Lambda(d)}{|d|} \sum_{|m|\leq x/|d|} \frac{1}{|m|} \gg \log x \sum_{|d|\leq\sqrt{x}} \frac{\Lambda(d)}{|d|}.$$

Comparing these two estimates shows that $\sum_{|d|\leq\sqrt{x}} \frac{\Lambda(d)}{|d|} \ll \log x$. Replacing x by x^2 gives (3.3).

Next, observe that for $s > 1$, we have

$$\log \zeta_{\mathscr{B}}(s) = \log \prod_{p} (1 - |p|^{-s})^{-1} \geq \sum_{p} |p|^{-s}.$$

Our hypothesis (1.4) on the partial sums of $\frac{1}{|n|}$ shows that

$$\begin{aligned}\zeta_{\mathscr{B}}(s) &= \int_{1-}^{\infty} t^{-(s-1)} \, d\Big(\sum_{|n|\leq t} \frac{1}{|n|}\Big) = (s-1) \int_{1}^{\infty} \Big(\sum_{|n|\leq t} \frac{1}{|n|}\Big) t^{-s} \, dt \\ &\ll 1 + (s-1) \int_{1}^{\infty} (\log t) \cdot t^{-s} \, dt = 1 + (s-1) \cdot \frac{1}{(s-1)^2} = \frac{s}{s-1}.\end{aligned}$$

Thus, for $1 < s < 2$ (say),

$$\sum_{p} |p|^{-s} \leq \log \frac{1}{s-1} + O(1).$$

Taking $s = 1 + \frac{1}{\log x}$ shows that for $x \geq 3$,

$$\sum_{|p|\leq x} |p|^{-1-1/\log x} \leq \sum_{p} |p|^{-1-1/\log x} \leq \log\log x + O(1).$$

But

$$\begin{aligned}\sum_{|p|\leq x} |p|^{-1} - \sum_{|p|\leq x} |p|^{-1-1/\log x} &= \sum_{|p|\leq x} \frac{|p|^{1/\log x} - 1}{|p|^{1+1/\log x}} \\ &\ll \frac{1}{\log x} \sum_{|p|\leq x} \frac{\log|p|}{|p|} \ll 1,\end{aligned}$$

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by (3.3). This proves (3.2) and completes the proof of (1.5).

It remains to show that condition (1.4) does not imply that the ratio of the left and right-hand sides of (1.5) tends to a limit. By (3.1), this is equivalent to showing that there is a Beurling system satisfying (1.4) for which the differences

$$\sum_{|p|\le x} \frac{1}{|p|} - \log\log x \tag{3.4}$$

do not tend to a limit as $x \to \infty$. We will show more than this; we demonstrate how to construct a Beurling system where (3.4) fails to tend to a limit, but which has the property (stronger than (1.4)) that

$$N_{\mathscr{B}}(x) \asymp x \qquad (\text{for } x \ge 1).$$

The existence of such a system will be deduced from the following theorem of Zhang (see [14, Theorem 4.1]):.

Theorem D *Let $\mathscr{B}$ be a Beurling system, and suppose that with*

$$\Pi_{\mathscr{B}}(x) := \pi_{\mathscr{B}}(x) + \frac{1}{2}\pi_{\mathscr{B}}(x^{1/2}) + \frac{1}{3}\pi_{\mathscr{B}}(x^{1/3}) + \cdots, \tag{3.5}$$

we have both $\Pi_{\mathscr{B}}(x) \asymp x/\log x$ for large x and, as $s \downarrow 1$,

$$\int_1^\infty t^{-s}\, d\Pi_{\mathscr{B}}(t) - \log\frac{1}{s-1} \ll 1. \tag{3.6}$$

Then $N_{\mathscr{B}}(x) \asymp x$.

Our strategy is to first choose the norms $|p_i|$ to guarantee that (3.4) is bounded but not convergent and then to use Theorem D to show that the Beurling system determined by our choice satisfies $N_{\mathscr{B}}(x) \asymp x$.

We start by constructing a set $\mathscr{R}$ of natural numbers with counting function $R(x) \asymp x/\log x$ and with the property that

$$\sum_{r\in\mathscr{R}\cap[1,x]} \frac{1}{r} - \log\log x$$

is bounded for $x \ge 3$ but does not converge as $x \to \infty$. Using the letter q to denote a generic rational prime, choose the natural number x_1 minimally, so that if $\mathscr{R}_1$ consists of all the numbers of the form q or $q+1$ not exceeding x_1, then

$$\sum_{r\in\mathscr{R}_1} \frac{1}{r} - \sum_{q\le x_1} \frac{1}{q} > 1. \tag{3.7}$$

Next, choose $x_2 > x_1$ minimally, so that if $\mathscr{R}_2$ consists of the primes $\equiv 1 \pmod 4$ in $(x_1, x_2]$, then

$$\sum_{r\in\mathscr{R}_1\cup\mathscr{R}_2} \frac{1}{r} - \sum_{q\le x_2} \frac{1}{q} < -1. \tag{3.8}$$

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Then choose $x_3 > x_2$ minimally, so that if $\mathscr{R}_3$ consists of all the numbers of the form q or $q+1$ in $(x_2, x_3]$, then

$$\sum_{r \in \mathscr{R}_1 \cup \mathscr{R}_2 \cup \mathscr{R}_3} \frac{1}{r} - \sum_{q \le x_3} \frac{1}{q} > 1,$$

and continue defining $\mathscr{R}_4, \mathscr{R}_5, \ldots$, alternating as above. Finally, set $\mathscr{R} := \cup_{i=1}^{\infty} \mathscr{R}_i$.

Let us check that $\mathscr{R}$ has the desired properties. Clearly $R(x) \le 2\pi(x) \ll x/\log x$ (for large x), where $\pi(x)$ is the usual (rational) prime counting function. It is slightly more involved to obtain the corresponding lower estimate $R(x) \gg x/\log x$. We first show that for large i, we have $R(x_i) \gg \pi(x_i)$. For this, it is enough to show that for every large j, a proportion $\gg 1$ of the rational primes belonging to $(x_j, x_{j+1}]$ are included in $\mathscr{R}$. If j is even, then $\mathscr{R}$ includes every rational prime from that interval, so suppose that j is odd. In that case, subtracting the analogue of (3.8) for x_{j+1} from the analogue of (3.7) for x_j, we find that

$$\sum_{\substack{x_j < q \le x_{j+1} \\ q \equiv 3 \pmod 4}} \frac{1}{q} > 2,$$

which implies that $x_{j+1} > x_j^2$ (say) for large values of j. So by the prime number theorem for progressions, the number of primes $\equiv 1 \pmod 4$ in $(x_j, x_{j+1}]$ is $> \frac{1}{3}\pi(x_{j+1})$. This completes the proof that $R(x_i) \gg \pi(x_i)$.

Now we prove that $R(x) \gg x/\log x$ for large x. From our work in the last paragraph, we can assume that $x_i < x < x_{i+1}$ for some index i. We know that

$$R(x) \gg \pi(x_i) + \#\{r \in \mathscr{R} : x_i < r \le x\}.$$

If i is even, then every prime from $(x_i, x]$ is counted in the second summand, and so $R(x) \gg \pi(x) \gg x/\log x$ in this case. Suppose that i is odd. Then if $x \ge 2x_i$ (and large, as we are assuming), the number of primes included in $\mathscr{R}$ from $(x_i, x]$ is at least $\frac{1}{3}$ of the total number of such primes, and again we may conclude that $R(x) \gg \pi(x) \gg x/\log x$. But if $x \le 2x_i$, then

$$R(x) \gg \pi(x_i) \gg x_i/\log x_i \gg x/\log x,$$

and so the desired lower bound still holds. Finally, since $\sum_{q \le x} \frac{1}{q} - \log\log x$ tends to a limit, it is clear from our construction that

$$\sum_{r \in \mathscr{R} \cap [1,x]} \frac{1}{r} - \log\log x$$

is $O(1)$ but does not converge as $x \to \infty$.

Now we define a Beurling system $\mathscr{B}$ by setting $|p_i|$ to be the ith smallest element of $\mathscr{R}$. As shown above, $\pi_{\mathscr{B}}(t) \asymp t/\log t$ for large t. (In fact, $2 \in \mathscr{R}$, so that this estimate holds for $t \ge 2$.) So with $\Pi_{\mathscr{B}}(t)$ defined by (3.5), we have $\Pi_{\mathscr{B}}(t) = \pi_{\mathscr{B}}(t) +$

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$O(t^{1/2})$. Consequently, $\Pi_{\mathscr{B}}(t) \asymp t/\log t$, and to verify (3.6), it is enough to verify the corresponding condition with $\Pi_{\mathscr{B}}(t)$ replaced by $\pi_{\mathscr{B}}(t)$. But this variant follows from the estimate $\sum_{|p|\le x} \frac{1}{|p|} - \log\log x \ll 1$. Indeed, for $s > 1$, that estimate shows

$$\begin{aligned}\int_1^\infty t^{-s}\, d\pi_{\mathscr{B}}(t) &= \int_1^\infty t^{-(s-1)}\, d\Big(\sum_{|p|\le t} |p|^{-1}\Big)\\ &= (s-1)\int_{|p_1|}^\infty t^{-s}\Big(\sum_{|p|\le t}|p|^{-1}\Big)\, dt\\ &= (s-1)\int_{|p_1|}^\infty t^{-s}\log\log t\, dt + O(1),\end{aligned}$$

while as $s \downarrow 1$,

$$\begin{aligned}(s-1)\int_{|p_1|}^\infty t^{-s}\log\log t\, dt &= \log\log|p_1| + o(1) + \int_{|p_1|}^\infty \frac{t^{-s}}{\log t}\, dt\\ &= \log\frac{1}{s-1} - \gamma + o(1).\end{aligned}$$

(In the first of the two lines above, we have integrated by parts, and in the second line we have again used the asymptotic expansion of the exponential integral.) By Theorem D, our Beurling system has $N_{\mathscr{B}}(x) \asymp x$, as desired.

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