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A CHARACTERIZATION OF VARIETIES WITH A DIFFERENCE TERM, II: NEUTRAL = MEET SEMI-DISTRIBUTIVE

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ABSTRACT. We provide more characterizations of varieties with a weak difference term and of neutral varieties. We prove that a variety has a (weak) difference term (is neutral) with respect to the TC-commutator iff it has a (weak) difference term (is neutral) with respect to the linear commutator. We show that a variety V is congruence meet semi-distributive iff V is neutral, iff M_3 is not a sublattice of Con **A**, for $\mathbf{A} \in V$, iff there is a positive integer n such that $V \models_{\mathrm{Con}} \alpha(\beta \circ \gamma) \leq \alpha\beta_n$.

1. **Preliminaries.** In this section we recall some definitions. The rest of the paper is independent from this section, provided the reader is acquainted with the relevant definitions. Familiarity with Part I [Lp1] is welcome, but probably not indispensable for reading the paper. The results presented here and their connection with part I are discussed at the beginning of each section.

Basics about universal algebra can be found, *e.g.*, in [BS], [MKNT].

Let **A** be an algebra, and α, β, γ be congruences on **A**. $M(\alpha, \beta)$ denotes the set of all matrices of the form:

$$\begin{vmatrix} t(\bar{a}, \bar{b}) & t(\bar{a}, \bar{b}') \\ t(\bar{a}', \bar{b}) & t(\bar{a}', \bar{b}') \end{vmatrix}$$

where $\bar{a}, \bar{a}' \in A^n$, $\bar{b}, \bar{b}' \in A^m$, for some $n, m \geq 0$, t is any $m+n$-ary term operation of **A**, and $\bar{a}\alpha\bar{a}'$, $\bar{b}\beta\bar{b}'$.

$C(\alpha, \beta; \gamma)$ means that if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \in M(\alpha, \beta)$, and $a\gamma b$ then also $c\gamma d$.

The *(TC) commutator* $[\alpha, \beta]$ of α and β is the least congruence γ such that $C(\alpha, \beta; \gamma)$ holds.

The *symmetric commutator* $[\alpha, \beta]_{sym}$ of α and β is the least congruence γ such that both $C(\alpha, \beta; \gamma)$ and $C(\beta, \alpha; \gamma)$ hold.

Another commutator will play an important role in this paper, the *linear commutator* $[\ ,\ ]_L$ (Definitions 2.1). Other commutators have some interest [Lp2]; in Part I we used another intermediate commutator $[\ ,\ ]_{2T}$; in the present paper we show that its use can be avoided. We have that $[\ ,\ ]_{sym}$, $[\ ,\ ]_{2T}$ and $[\ ,\ ]_L$ are all symmetric operations, that $[\alpha, \beta] \leq [\alpha, \beta]_{sym} \leq [\alpha, \beta]_{2T} \leq [\alpha, \beta]_L$, and that $[\alpha, \alpha] = [\alpha, \alpha]_{sym}$ (actually, if $[\alpha, \beta] = [\beta, \alpha]$ for given congruences α and β, then $[\alpha, \beta] = [\alpha, \beta]_{sym}$).

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It can be shown that each of the above inequalities may be strict (see, *e.g.*, [Lp2]). However, many of the above commutators coincide, under suitable hypotheses. For example, K. Kearnes and A. Szendrei showed that $[\, ,\,]_L = [\, ,\,]_{sym}$ in every variety satisfying some non-trivial idempotent Mal'cev condition (see Theorem 3.1).

Roughly, a *strong Mal'cev condition* is a finite set of identities, and a variety satisfies the strong Mal'cev condition iff it has terms satisfying these identities. A *Mal'cev condition* is the union of a sequence of strong Mal'cev conditions, each weaker than the preceding ones. The (strong) Mal'cev condition is *idempotent* iff it contains the identity $f(x, x, \ldots, x) = x$, for every operation f involved in the condition.

A (strong, idempotent) *Mal'cev class* is the class of varieties satisfying a (strong, idempotent) Mal'cev condition.

Probably, these definitions are best understood by examples (*e.g.*, in this paper, conditions 3.2(iv) and 4.1(iv)). See *e.g.* [Ta], [Jo] for further details.

If **A** is an algebra, and $[\, ,\,]^*$ is any commutator operation, a ternary term t is a *difference term with respect to* $[\, ,\,]^*$ iff $a = t(a, b, b)$ and $t(a, a, b)[\alpha, \alpha]^* b$, for every $\alpha \in \operatorname{Con} \mathbf{A}$ and $a, b \in A$ such that $a \alpha b$. If we weaken the condition $a = t(a, b, b)$ to $a[\alpha, \alpha]^* t(a, b, b)$, then we say that t is a *weak difference term with respect to* $[\, ,\,]^*$.

A variety V has a (weak) difference term with respect to $[\, ,\,]^*$ iff every algebra in V has a (weak) difference term with respect to $[\, ,\,]^*$ (equivalently, iff there is a term which works for every algebra in V. See Theorem 3.2 and Remark 2.4).

2. **Let the linear commutator come into play.** As we mentioned in Part I, Section 2, there exist many different commutators, each one enjoying particularly interesting features. Among these commutators, the *linear* commutator is particularly interesting, since $[1, 1]_L = 0$ holds in **A** iff **A** is *quasi-affine* (that is, a substructure of the reduct of an affine algebra). Indeed, as shown in [Qu], $[1, 1]_L = 0$ holds iff all quasi-identities valid in affine algebras hold.

Though interesting, the linear commutator seemed really awkward to deal with, because its explicit definition is quite complicated and involves either working in some extension of **A**, or dealing with many matrices at the same time.

In this section, however, we show that the linear commutator has some pleasant properties, and in some respect is the easiest to deal with. For example, we give an easy proof showing that having a weak difference term with respect to the linear commutator is an *idempotent* Mal'cev condition (Corollary 2.3), while we know no such direct proof for the TC (or other) commutators.

Moreover, K. Kearnes and A. Szendrei [KS] show that, in a sense, the linear commutator is not that large, when compared with the symmetrization of the TC-commutator, and that the two commutators coincide in any variety satisfying a non-trivial idempotent Mal'cev condition. Together with Corollary 2.3, Kearnes and Szendrei's result imply that a variety has a (weak) difference term with respect to the TC-commutator iff it has a (weak) difference term with respect to the linear commutator. More generally, the mentioned results enable us to "transfer" results from the linear commutator to the more frequently used TC-commutator (all this will be done in Sections 3 and 4).

Now for the definitions of the linear commutator (of congruences in an algebra **A**).

The simplest one involves taking a particular extension **ZA** of **A**, computing the ordinary TC-commutator there, and then taking its restriction to **A** (see [Qu], [KS]). However, when performing computations, the following definition using matrices is more useful, since it deals with **A** alone, and not with "imaginary" extensions.

2.1 *Definitions.* If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \in M(\alpha,\beta)$, we call a and d *positive* entries, and b, c, *negative* entries.

Then $[\alpha,\beta]_L$, the *linear* commutator, is the set of all pairs (a,b) such that there exists a finite number of matrices in $M(\alpha,\beta)$ having a as a positive entry, b as a negative entry, and with the other positive entries coinciding with the other negative ones (counting multiplicities). See [Qu], [KS] for further information.

THEOREM 2.2. *For every variety V, the following are equivalent:*

(i) V has a weak difference term with respect to the linear commutator.

(ii) For every $\mathbf{A} \in V$, $\alpha \in \operatorname{Con} \mathbf{A}$ and $a\alpha b \in A$ there is a ternary term t such that $a[\alpha,\alpha]_L t(a,b,b)$ and $t(a,a,b)[\alpha,\alpha]_L b$.

(ii)′ If $\mathbf{A} = F_V(x,y)$, and $\alpha = Cg(x,y)$, there is a ternary term t such that $x[\alpha,\alpha]_L t(x,y,y)$ and $t(x,x,y)[\alpha,\alpha]_L y$.

(iii) V has a weak difference term t with respect to the linear commutator, and in addition t is such that there are integers n, m and ternary idempotent terms $q_0,\dots,q_{4n-1}$, $p_0,\dots,p_{4m-1}$ such that if $a\alpha b \in A \in V$, then $a[\alpha,\alpha]_L t(a,b,b)$ is witnessed by the matrices

$$\begin{vmatrix} q_{2i}(a,b,b) & q_{2i}(a,b,a) \\ q_{2i}(a,a,b) & q_{2i}(a,a,a) \end{vmatrix} \quad \begin{vmatrix} q_{2i+1}(a,b,a) & q_{2i+1}(a,b,b) \\ q_{2i+1}(a,a,a) & q_{2i+1}(a,a,b) \end{vmatrix}$$

and symmetrically $t(a,a,b)[\alpha,\alpha]_L b$ is witnessed by the p_i's.

PROOF. (i) ⇒ (ii) ⇒ (ii)′ and (iii) ⇒ (i) are trivial.

(ii)′ ⇒ (iii) Let $\mathbf{A} = F_V(x,y)$, $\alpha = Cg(x,y)$.

The general form for a matrix in $M(\alpha,\alpha)$ is

$$\begin{vmatrix} s(u_1(x,y),\dots,v_1(x,y),\dots) & s(u_1(x,y),\dots,v_1'(x,y),\dots) \\ s(u_1'(x,y),\dots,v_1(x,y),\dots) & s(u_1'(x,y),\dots,v_1'(x,y),\dots) \end{vmatrix},$$

where $u_1(x,x) = u_1'(x,x),\dots,v_1(x,x) = v_1'(x,x),\dots$ are identities of V.

If t satisfies (ii)′, then there are matrices $M_1,\dots,M_n$ as above witnessing $x[\alpha,\alpha]_L t(x,y,y)$ (of course, s, the u_j's, and the v_k's will depend on i). If M_i is as above, let

$$\begin{aligned} q_{4i}(x,y,z) &= s\big(u_1(x,y),u_2(x,y),\dots,v_1(x,z),v_2(x,z),\dots\big), \\ q_{4i+1}(x,y,z) &= s\big(u_1(x,y),u_2(x,y),\dots,v_1'(x,z),v_2'(x,z),\dots\big), \\ q_{4i+2}(x,y,z) &= s\big(u_1'(x,y),u_2'(x,y),\dots,v_1'(x,z),v_2'(x,z),\dots\big), \\ q_{4i+3}(x,y,z) &= s\big(u_1'(x,y),u_2'(x,y),\dots,v_1(x,z),v_2(x,z),\dots\big). \end{aligned}$$

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Since $u_1(x,x) = u_1'(x,x), \ldots$, we have that the following are identities of $V: q_{4i}(x,x,z) = q_{4i+3}(x,x,z)$ and $q_{4i+1}(x,x,z) = q_{4i+2}(x,x,z)$; similarly, $q_{4i}(x,y,x) = q_{4i+1}(x,y,x)$, $q_{4i+3}(x,y,x) = q_{4i+2}(x,y,x)$.

If instead of the n matrices $M_1,\ldots,M_n$ we consider the $4n$ matrices

$$\begin{vmatrix} q_{4i}(x,y,y) & q_{4i}(x,y,x) \\ q_{4i}(x,x,y) & q_{4i}(x,x,x) \end{vmatrix} \quad \begin{vmatrix} q_{4i+1}(x,y,x) & q_{4i+1}(x,y,y) \\ q_{4i+1}(x,x,x) & q_{4i+1}(x,x,y) \end{vmatrix}$$
$$\begin{vmatrix} q_{4i+3}(x,x,y) & q_{4i+3}(x,x,x) \\ q_{4i+3}(x,y,y) & q_{4i+3}(x,y,x) \end{vmatrix} \quad \begin{vmatrix} q_{4i+2}(x,x,x) & q_{4i+2}(x,x,y) \\ q_{4i+2}(x,y,x) & q_{4i+2}(x,y,y) \end{vmatrix},$$

these matrices still witness $x[\alpha,\alpha]_L t(x,y,y)$, since we have added couples of equal positive and negative entries (we have disposed the matrices in such a way that adjacent entries of contiguous matrices reciprocally annihilate).

Since we are working in $F_V(x,y)$, the identities we get hold throughout the variety, so that if $\mathbf{A} \in V$, $\alpha \in \operatorname{Con} \mathbf{A}$ and $a\alpha b \in A$ are arbitrary, it is enough to replace x by a and y by b in the above $4n$ matrices, thus witnessing $a[\alpha,\alpha]_L t(a,b,b)$.

Now rotate by π those matrices in which the index of q is $4i+2$ or $4i+3$, and all matrices have the desired form.

The argument for showing the existence of terms p_i's witnessing $t(a,a,b)[\alpha,\alpha]_L b$ is entirely similar.

Finally, some entry in the original matrices $M_1,\ldots,M_n$ is x, so that $q_{i_0}(x,y,y) = x$, for some i_0, but trivially $q_i(x,x,x) = q_j(x,x,x)$ for all i,j (wlog we can throw out "unconnected" matrices), so that all q_i's are idempotent. ∎

COROLLARY 2.3. *The class of varieties with a weak difference term with respect to the linear commutator is an idempotent Mal'cev class.*

PROOF. In the first part of the proof of Theorem 2.2 (ii)$'$ ⇒ (iii) we show that if V has a weak difference term with respect to the linear commutator then there are idempotent terms t, q_i, p_i satisfying certain equations (which will be explicitated in condition (iv) of Theorem 3.2).

In the second part of the proof we show that these equations are enough to imply that t is a weak difference term with respect to the linear commutator. ∎

REMARK 2.4. A direct proof of 2.2 (ii)$'$ ⇒ (i) is quite simple: as in the proof of (ii)$'$ ⇒ (iii), consider the matrices M_i and replace x by a and y by b. This argument is sufficient to show that the class of varieties with a weak difference term with respect to the linear commutator is a Mal'cev class, but we have no guarantee that the terms s, $u_1,\ldots,v_1,\ldots$ are idempotent.

Of course, a similar argument works for varieties having a weak difference term with respect to the TC-commutator (as sketched in [Lp3, Proposition 1]) and, actually, with respect to any other commutator which can be defined using matrices (see [Lp2]). Such a simple argument also gives a direct proof for the equivalence (for any fixed commutator) of conditions (i), (ii) and (ii)$'$ in Theorem 4.1.

However, we see no direct argument showing that having a weak difference term (or being neutral) with respect to the TC-commutator is an *idempotent* Mal'cev condition (anyway, this is a consequence of Theorems 3.2 and 4.1).

REMARK 2.5. Actually, what we use in Remark 2.4 is only a homomorphism property of commutators defined by matrices: if $\phi\colon \mathbf{A} \longrightarrow \mathbf{B}$ is a homomorphism and $\alpha, \beta \in \operatorname{Con} \mathbf{A}$ then $\phi[\alpha,\beta]_L \leq [\phi\alpha, \phi\beta]_L$.

Indeed, if $a[\alpha,\beta]_L b$ is witnessed by the matrices

$$\begin{vmatrix} a_i & b_i \\ c_i & d_i \end{vmatrix} \quad (i = 1, \ldots, n)$$

then the matrices

$$\begin{vmatrix} \phi a_i & \phi b_i \\ \phi c_i & \phi d_i \end{vmatrix} \quad (i = 1, \ldots, n)$$

witness $\phi a[\phi\alpha, \phi\beta]_L \phi b$. The conclusion follows from the fact that, for every congruence θ, $\phi(\theta)$ is generated by $\{(\phi a, \phi b) | a\theta b\}$ (compare also [Ke, Lemma 2.6]).

Of course, the same argument works for the TC commutator, and for all the commutators introduced in Section 1 and [Lp2].

3. **More characterizations of varieties with a weak difference term.** Given α, β, γ congruences, define recursively:

$$\beta_0 = \gamma_0 = 0; \quad \beta_{n+1} = \beta + \alpha\gamma_n; \quad \gamma_{n+1} = \gamma + \alpha\beta_n.$$

In Part I, Theorem 3.1, we showed that V has a weak difference term iff $V \models_{\mathrm{Con}} \alpha(\beta \circ \gamma) \leq \alpha\beta_n \circ \gamma \circ \beta \circ \alpha\beta_n$, for some n; but we used the additional assumption that in V $[\alpha,\alpha] = 0$ iff $[\alpha,\alpha]_{2T} = 0$. Under the same assumption, we showed in Theorem 4.3 that V is neutral iff $V \models_{\mathrm{Con}} \alpha(\beta \circ \gamma) \leq \alpha\beta_n$, for some n.

In Problems 3.2 and 4.4 we asked whether this extra assumption is necessary. The following theorem by Kearnes and Szendrei (together with Part I) immediately solves these Problems. Further, we shall use Kearnes and Szendrei's result together with Theorem 2.2 in order to provide more characterizations of varieties with a (weak) difference term and of neutral varieties.

THEOREM 3.1 [KS]. *If V satisfies a non-trivial idempotent Mal'cev condition then* $V \models_{\mathrm{Con}} [\alpha,\beta]_{sym} = [\alpha,\beta]_L$.

Notice that, under the hypothesis of 3.1, $[\alpha,\alpha] = [\alpha,\alpha]_L$; so that, in particular, $[\alpha,\alpha]_{2T} = [\alpha,\alpha]$. This shows that the hypothesis $[\alpha,\alpha] = 0$ iff $[\alpha,\alpha]_{2T} = 0$ is unnecessary in Part I, Theorems 3.1 and 4.3, since the congruence identities used there imply a non-trivial idempotent Mal'cev condition.

Let us denote *solvable series* as follows:

$[\alpha,\beta]^{(1)} = [\alpha,\beta]$;

$[\alpha,\beta]^{(h+1)} = \big[[\alpha,\beta]^{(h)}, [\alpha,\beta]^{(h)}\big]$;

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$[\alpha,\beta]_L^{(h)}$ is defined similarly.

If B is a subset of the algebra $\mathbf{A}$, the *restriction* of $\mathbf{A}$ to B is the algebra whose base set is B, and whose operations are all the polynomials of $\mathbf{A}$ whose restriction to B are total functions. In conditions (x), (xi) below we consider blocks of congruences as algebras, under restriction.

If $[\,,\,]^*$ is a commutator, a congruence α is *abelian in the sense of* $[\,,\,]^*$ iff $[\alpha,\alpha]^* = 0$.

THEOREM 3.2. *For every variety V, the following are equivalent:*

(i) *V has a weak difference term with respect to* $[\,,\,]$.

(ii) *$F_V(2)$ has a weak difference term with respect to* $[\,,\,]$.

(iiia) *There is a positive integer n such that* $V \models_{\mathrm{Con}} \alpha(\beta\circ\gamma) \le \alpha\beta_n \circ \gamma \circ \beta \circ \alpha\beta_n$.

(iiib) *There is a positive integer n such that* $V \models_{\mathrm{Con}} \alpha(\beta\circ\gamma) \le \gamma_n \circ \beta_n$.

(iiic) *For every (equivalently, some) even $m \ge 2$ there is a positive integer n such that* $V \models_{\mathrm{Con}} \alpha(\underbrace{\beta\circ\gamma\circ\beta\circ\ldots}_{m\ \text{factors}}) \le \alpha\beta_n \circ \underbrace{\gamma\circ\beta\circ\ldots}_{m\ \text{factors}} \circ \alpha\beta_n$.

(iv) *There exist integers m, n, ternary idempotent terms $t, q_0, \ldots, q_{4n-1}, p_0, \ldots, p_{4m-1}$, and bijections $\sigma\colon \{0,2,\ldots,4n-2\} \longrightarrow \{1,3,\ldots,4n-1\}$, $\tau\colon \{0,2,\ldots,4m-2\} \longrightarrow \{1,3,\ldots,4m-1\}$ such that the following identities hold throughout V:*

$q_{4i}(x,x,z) = q_{4i+3}(x,x,z)$, $p_{4i}(x,x,z) = p_{4i+3}(x,x,z)$,
$q_{4i+1}(x,x,z) = q_{4i+2}(x,x,z)$, $p_{4i+1}(x,x,z) = p_{4i+2}(x,x,z)$,
$q_{4i}(x,y,x) = q_{4i+1}(x,y,x)$, $p_{4i}(x,y,x) = p_{4i+1}(x,y,x)$,
$q_{4i+3}(x,y,x) = q_{4i+2}(x,y,x)$, $p_{4i+3}(x,y,x) = p_{4i+2}(x,y,x)$,
$q_i(x,y,y) = q_{\sigma(i)}(x,y,y)$ (i even > 0), $p_i(x,y,y) = p_{\tau(i)}(x,y,y)$ (i even > 0),
$x = q_0(x,y,y)$, $p_{\tau(0)}(x,y,y) = y$,
$q_{\sigma(0)}(x,y,y) = t(x,y,y)$, $t(x,x,y) = p_0(x,y,y)$.

(v) $V \models_{\mathrm{Con}} \alpha\circ\beta \le [\alpha,\alpha]\circ\beta\circ\alpha\circ[\beta,\beta]$.

(vi) $V \models_{\mathrm{Con}} \alpha(\beta\circ\gamma) \le [\alpha,\alpha]\circ\gamma\circ\beta\circ[\gamma,\gamma]$.

(vii) $V \models_{\mathrm{Con}} \alpha(\beta\circ\gamma) \le [\alpha,\alpha]\circ\gamma\circ\beta\circ[\alpha,\alpha]$.

(viii) $V \models_{\mathrm{Con}} \alpha+\beta \le ([\alpha,\alpha]+[\beta,\beta])\circ\beta\circ\alpha\circ([\alpha,\alpha]+[\beta,\beta])$.

(ix) $V \models_{\mathrm{Con}} \alpha\circ\beta \le [\alpha+\beta,\alpha+\beta]\circ\beta\circ\alpha\circ[\alpha+\beta,\alpha+\beta]$.

(x) *Within V, if $[\alpha,\alpha] = 0$ then every block of α has a term satisfying $x = t(x,y,y)$ and $t(x,x,y) = y$.*

(xi) *Within V, every block of every abelian (in the sense of* $[\,,\,]$*) congruence is affine.*

In all the above conditions we can equivalently replace any occurrence of $[\,,\,]$ *by* $[\,,\,]_L$*, or (except that in (xi)) by* $[\,,\,]^{(h)}$*, or by* $[\,,\,]_L^{(h)}$*, and hence also by any intermediate commutator.*

PROOF. If $(*)$ is any one of the above conditions, let $(*)_L$ denote the respective condition with respect to $[\,,\,]_L$.

If $(*)$ holds, then clearly $(*)_L$ holds, since $[\,,\,]_L \ge [\,,\,]$.

The equivalence of $(\mathrm{i})_L$, $(\mathrm{ii})_L$ and $(\mathrm{iv})_L$ follows from the proof of Theorem 2.2.

If $(\mathrm{i})_L$ holds then $(\mathrm{v})_L$-$(\mathrm{ix})_L$ hold because of [Lp4, Lemma 3.1].

$(\mathrm{ix})_L \Rightarrow (\mathrm{i})_L$. Let $\alpha = Cg(x,y)$ and $\beta = Cg(y,z)$ in $F_V(x,y,z)$. By $(\mathrm{ix})_L$, and since $x\alpha\circ\beta z$, there is a ternary term t such that $x([\alpha+\beta,\alpha+\beta]_L\circ\beta)t(x,y,z)(\alpha\circ[\alpha+\beta,\alpha+\beta]_L)z$. Let $\phi\colon F_V(x,y,z)\longrightarrow F_V(x,y)$ be the homomorphism which sends z to y and leaves x and y fixed; by Remark 2.5, $\phi[\alpha+\beta,\alpha+\beta]_L \leq [\phi(\alpha+\beta),\phi(\alpha+\beta)]_L = [\alpha,\alpha]_L$, and hence $x = \phi x[\alpha,\alpha]_L\phi t(x,y,z) = t(x,y,y)$. Similarly, $t(y,y,z)[\beta,\beta]_L z$ and, since $F_V(x,y)$ and $F_V(y,z)$ are isomorphic, we obtain that condition (ii)$'$ in Theorem 2.2 is satisfied, and hence $(\mathrm{i})_L$ holds.

The proof that each of $(\mathrm{v})_L$–$(\mathrm{viii})_L$ implies $(\mathrm{i})_L$ is similar and easier (compare also [Ke, Lemma 2.7]).

$(\mathrm{i})_L \Rightarrow (\mathrm{x})_L$ is trivial.

$(\mathrm{x})_L \Rightarrow (\mathrm{xi})_L$ is immediate from Herrmann's theorem [He].

$(\mathrm{xi})_L \Rightarrow (\mathrm{i})_L$ If $a\alpha b$, $\alpha \in \mathrm{Con}\,\mathbf{A} \in V$ then $\alpha/[\alpha,\alpha]_L$ is abelian (in $\mathbf{A}/[\alpha,\alpha]_L$), hence every block of $\alpha/[\alpha,\alpha]_L$ is affine; in particular, there is a term t such that if $a' = a/[\alpha,\alpha]_L$ and $b' = b/[\alpha,\alpha]_L$ then $a' = t(a',b',b')$ and $t(a',a',b') = b'$, and hence $a[\alpha,\alpha]_L t(a,b,b)$ and $t(a,a,b)[\alpha,\alpha]_L b$.

So we get that every algebra in V has a weak difference term (which is slightly less than $(\mathrm{i})_L$, since the term t might depend on $\mathbf{A}\in V$). But if we apply Theorem 2.2(ii) $\Rightarrow$ (i) we get a term working uniformly through V, and $(\mathrm{i})_L$ follows.

Hence $(\mathrm{i})_L$ and $(\mathrm{v})_L$–$(\mathrm{xi})_L$ are all equivalent.

Whence, by Corollary 2.3, each $(*)_L$ (except possibly (iiia)–(iiic)) is an idempotent Mal'cev condition, so that, by Theorem 3.1, $[\alpha,\alpha] = [\alpha,\alpha]_L$, and hence (*) holds.

(i) $\Rightarrow$ (iiia) is from Part I, Theorem 3.1; (iiia) $\Rightarrow$ (iiib) is trivial; (iiib) $\Rightarrow (\mathrm{i})_L$ is as in the last part of the proof of Part I, Theorem 3.1 (compare also Part I, Theorem 3.4, or see below). (i) $\Rightarrow$ (iiic) $\Rightarrow (\mathrm{i})_L$ are from Part I, Remark 3.3(d).

Hence all are equivalent.

We can equivalently replace $[\,,\,]$ by $[\,,\,]^{(h)}$, and $[\,,\,]_L$ by $[\,,\,]_L^{(h)}$, because of [Lp4, Lemma 3.1]. ∎

We sketch below an alternative proof of (iiib) $\Rightarrow (\mathrm{i})_L$.

Let $\delta = [Cg(x,y), Cg(x,y)]_L$ in $F_V(x,y)$, and let $x' = x/\delta$, $y' = y/\delta$, $\mathbf{B} = F_V(x,y)/\delta \times F_V(x,y)/\delta$, and α, β, γ be, respectively, the diagonal congruence on $\mathbf{B}$, and the kernels of the first and of the second projection. Since $F_V(x,y)/\delta$ is abelian in the sense of the linear commutator, $\alpha\beta = \alpha\gamma = 0$, and hence $\beta_n = \beta$ and $\gamma_n = \gamma$ for all n.

If $\mathbf{C}$ is the subalgebra of $\mathbf{B}$ generated by (x',y'), (x',x'), (y',y'), and α', β', γ' are the restrictions of α, β, γ to $\mathbf{C}$, then still $\beta'_n = \beta'$ and $\gamma'_n = \gamma'$, so that $(x',x')\gamma'\circ\beta'(y',y')$, by (iiib), since $(x',x')\beta'(x',y')\gamma'(y',y')$.

If $(x',x')\gamma'(a,b)\beta'(y',y')$, then necessarily $a = y'$ and $b = x'$, and, because of the way we have constructed $\mathbf{C}$, there must be a ternary term t such that $t\big((x',x'),(x',y'),(y',y')\big) = (y',x')$. Translating this condition in terms of $F_V(x,y)$, we get Condition 2.2(ii)$'$, and so $(\mathrm{i})_L$ holds.

(iiib)(iiic) $\Rightarrow (\mathrm{i})_L$ can be proved also using the method of [He, proof of Lemma 4]. ∎

REMARKS 3.3. (a) From the proof of Theorem 2.2 we can extract many other conditions equivalent to 3.2(iv) involving the existence of certain terms.

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For example, the request $q_{4i}(x,x,z) = q_{4i+3}(x,x,z)$ can be weakened to: *there exists a permutation σ' of n such that $q_{4i}(x,x,z) = q_{4\sigma'(i)+3}(x,x,z)$ are identities of V*.

More equivalent conditions can be built using identities involving the terms in the matrices M_i, or similar identities relative to other matrices assumed to witness $x[\alpha,\alpha]t(x,y,y)$ (the TC-commutator rather than the linear one).

(b) A parallel remark holds for condition 4.1(iv).

(c) There are other congruence identities different from 3.2(iii), and involving more than 3 congruences, which can be used to characterize varieties with a weak difference term. Details shall be given elsewhere.

(d) Clearly, Condition 3.2(xi) cannot be improved to "*Every block of every solvable congruence is affine*", as the example of a solvable non abelian group shows.

(e) If $V \models_{\mathrm{Con}} \alpha(\beta \circ \gamma) \leq (\gamma + \alpha\beta) \circ (\beta + \alpha\gamma)$ then V has a weak difference term with respect to $[\ ,\ |1]$, as noticed in [Lp3, p. 163] (see [Lp3] for the definition of $[\alpha,\beta|1]$).

By [HMK, Chapter 9], every locally finite n-permutable variety satisfies the above identity; hence, for locally finite varieties, the subscripts $n(n+2)$ in [Lp3, Theorem 1] can be improved to $n+2$. Other corresponding results admit a similar strengthening.

We do not know whether every n-permutable variety satisfies $\alpha(\beta \circ \gamma) \leq (\gamma + \alpha\beta) \circ (\beta + \alpha\gamma)$.

4. **Congruence neutrality is the same as meet semi-distributivity.** In Part III we shall give the analogue of Theorem 3.2 for varieties with a difference term (together with further characterizations). Now we shall give the version of 3.2 (and slightly more) for neutral varieties.

If $[\ ,\ ]^*$ is a commutator, a variety V is *neutral in the sense of* $[\ ,\ ]^*$ iff $\alpha = [\alpha,\alpha]^*$ is an identity in V.

A variety V is *congruence meet semi-distributive* iff $V \models_{\mathrm{Con}} \alpha\beta = \alpha\gamma \Rightarrow \alpha(\beta+\gamma) = \alpha\beta$.

THEOREM 4.1. *For every variety V, the following are equivalent:*

(i) *V is neutral with respect to* $[\ ,\ ]$.

(ii) $F_V(2)$ *is neutral with respect to* $[\ ,\ ]$.

(ii)′ *If* $\mathbf{A} = F_V(x,y)$, *and* $\alpha = Cg(x,y)$ *then* $x[\alpha,\alpha]y$.

(iiia) *There is a positive integer n such that* $V \models_{\mathrm{Con}} \alpha(\beta \circ \gamma) \leq \alpha\beta_n$.

(iiib) *For every* $j \geq 2$ *there is a positive integer n such that*

$$V \models_{\mathrm{Con}} \alpha(\underbrace{\beta \circ \gamma \circ \ldots}_{j\ factors}) \leq \alpha\beta_n.$$

(iiic) $V \models_{\mathrm{Con}} \alpha(\beta+\gamma) \leq \bigcup_{n\in\omega} \alpha\beta_n$.

(iv) *There exist an integer n, ternary idempotent terms* $q_0, \ldots, \ldots, q_{4n-1}$, *and a bijection* $\sigma : \{0, 2, \ldots, 4n-2\} \longrightarrow \{1, 3, \ldots, 4n-1\}$, *such that the following identities hold throughout V :*

$q_{4i}(x,x,z) = q_{4i+3}(x,x,z)$

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$$q_{4i+1}(x,x,z) = q_{4i+2}(x,x,z)$$
$$q_{4i}(x,y,x) = q_{4i+1}(x,y,x)$$
$$q_{4i+3}(x,y,x) = q_{4i+2}(x,y,x),$$
$$q_i(x,y,y) = q_{\sigma(i)}(x,y,y) \ (i \text{ even} > 0),$$
$$x = q_0(x,y,y),$$
$$q_{\sigma(0)}(x,y,y) = y.$$

(v) V is congruence meet semi-distributive.

(vi) For no $\mathbf{A} \in V$ M_3 is a sublattice of Con $\mathbf{A}$.

In conditions (i), (ii), (ii)′ we can equivalently replace any occurrence of $[\ ,\ ]$ *by* $[\ ,\ ]_L$, $[\ ,\ ]^{(h)}$, *or* $[\ ,\ ]^{(h)}_L$, *and hence also by any intermediate commutator.*

PROOF. The equivalence of (i)–(iv) (with respect to any commutator) is proved as Theorem 3.2, using Part I, Theorem 4.3 (notice that (i) implies that V has a weak difference term, hence by Corollary 2.3 one can apply directly Theorem 3.1, instead of repeating all the arguments in Section 2. See also Remark 2.4).

(i) ⇒ (v) follows immediately from the semi-distributivity of $[\ ,\ ]$.

(v) ⇒ (vi) is trivial.

We now show that (vi) implies that V is neutral with respect to the linear commutator.

Suppose by contradiction $\alpha \in \text{Con}\,\mathbf{A}$, $\mathbf{A} \in V$ and $[\alpha,\alpha]_L < \alpha$. Wlog, working in $\mathbf{A}/[\alpha,\alpha]_L$, we can suppose $[\alpha,\alpha]_L = 0$. But then M_3 is a sublattice of the congruence α, thought of as an algebra.

This concludes the proof. ∎

For locally finite varieties, the equivalence of some of the conditions in Theorem 4.1 has been obtained in [HMK].

The equivalence of conditions (iiib), (iiic) and (v) has first been shown in [Cz], thus giving an explicit weak Mal'cev condition for meet semi-distributivity (actually, (iiic) and (v) are equivalent for single algebras rather than varieties, and, more generally, for elements of algebraic lattices).

Condition (iiia) shows more:

COROLLARY 4.2. *The class of congruence meet semi-distributive varieties is an idempotent Mal'cev class.*

PROBLEM 4.3. In [HMK, Theorem 9.11] many conditions are given characterizing locally finite varieties all whose finite algebras have join semi-distributive congruence lattices. We do not know whether some of these conditions are equivalent without assuming local finiteness (except for the trivial equivalence (3)⇔(4)). However, it is possible to show that if $\mathbf{A}$ is an algebra whose congruence lattice is join semi-distributive, then $\alpha + \beta\gamma \geq \beta(\alpha \circ \gamma \circ \alpha)$ holds in Con $\mathbf{A}$.

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