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ON THE NUMBER OF JUMPS OF RANDOM WALKS WITH A BARRIER

ALEX IKSANOV,* *National Taras Shevchenko University*
MARTIN MÖHLE,** *University of Düsseldorf*

Abstract

Let $S_0 := 0$ and $S_k := \xi_1 + \cdots + \xi_k$ for $k \in \mathbb{N} := \{1, 2, \ldots\}$, where $\{\xi_k : k \in \mathbb{N}\}$ are independent copies of a random variable ξ with values in $\mathbb{N}$ and distribution $p_k := \mathrm{P}\{\xi = k\}$, $k \in \mathbb{N}$. We interpret the random walk $\{S_k : k = 0, 1, 2, \ldots\}$ as a particle jumping to the right through integer positions. Fix $n \in \mathbb{N}$ and modify the process by requiring that the particle is bumped back to its current state each time a jump would bring the particle to a state larger than or equal to n. This constraint defines an increasing Markov chain $\{R_k^{(n)} : k = 0, 1, 2, \ldots\}$ which never reaches the state n. We call this process a random walk with barrier n. Let M_n denote the number of jumps of the random walk with barrier n. This paper focuses on the asymptotics of M_n as n tends to ∞. A key observation is that, under $p_1 > 0$, $\{M_n : n \in \mathbb{N}\}$ satisfies the distributional recursion $M_1 = 0$ and $M_n \stackrel{\mathrm{D}}{=} M_{n-I_n} + 1$ for $n = 2, 3, \ldots$, where I_n is independent of $M_2, \ldots, M_{n-1}$ with distribution $\mathrm{P}\{I_n = k\} = p_k/(p_1 + \cdots + p_{n-1})$, $k \in \{1, \ldots, n-1\}$. Depending on the tail behavior of the distribution of ξ, several scalings for M_n and corresponding limiting distributions come into play, including stable distributions and distributions of exponential integrals of subordinators. The methods used in this paper are mainly probabilistic. The key tool is to compare (couple) the number of jumps, M_n, with the first time, N_n, when the unrestricted random walk $\{S_k : k = 0, 1, \ldots\}$ reaches a state larger than or equal to n. The results are applied to derive the asymptotics of the number of collision events (that take place until there is just a single block) for $\beta(a, b)$-coalescent processes with parameters $0 < a < 2$ and $b = 1$.

Keywords: Absorption time; beta coalescent; coupling; exponential integral; Mittag–Leffler distribution; random recursive equation; random walk; stable limit; subordinator

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1. Introduction and main results

Fix $n \in \mathbb{N} := \{1, 2, \ldots\}$. By a random walk with the barrier n we mean the sequence $\{R_k^{(n)} : k \in \mathbb{N}_0 := \{0, 1, \ldots\}\}$ defined recursively via $R_0^{(n)} := 0$ and

$$R_k^{(n)} := R_{k-1}^{(n)} + \xi_k \, \mathbf{1}_{\{R_{k-1}^{(n)} + \xi_k < n\}}, \qquad k \in \mathbb{N},$$

where $\{\xi_k : k \in \mathbb{N}\}$ are independent copies of a random variable ξ with some proper and nondegenerate probability distribution

$$p_k := \mathrm{P}\{\xi = k\}, \qquad k \in \mathbb{N},\ p_1 > 0.$$

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* Postal address: Faculty of Cybernetics, National Taras Shevchenko University, 01033 Kiev, Ukraine.
Email address: iksan@unicyb.kiev.ua
** Postal address: Mathematical Institute, University of Düsseldorf, 40225 Düsseldorf, Germany.
Email address: moehle@math.uni-duesseldorf.de

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Note that the sequence $\{R_k^{(n)}: k \in \mathbb{N}_0\}$ is nondecreasing and that $R_k^{(n)} < n$ for all $k \in \mathbb{N}_0$. Let

$$M_n := \#\{k \in \mathbb{N}\colon R_{k-1}^{(n)} \neq R_k^{(n)}\} = \sum_{l=0}^{\infty} \mathbf{1}_{\{R_l^{(n)} + \xi_{l+1} < n\}}$$

denote the number of jumps in the process $\{R_k^{(n)}: k \in \mathbb{N}_0\}$. Note that $M_1 = 0$ and that $1 \leq M_n \leq n-1$ for $n \geq 2$. The aim of the paper is to investigate the asymptotic behavior of M_n as n tends to ∞. Hinderer and Walk [23] investigated processes more general than random walks with a barrier, but the circle of problems they considered was different from ours.

As $p_1 > 0$, it follows from Lemma 1 of [24] that the marginal distributions of $\{M_n: n \in \mathbb{N}\}$ satisfy the distributional recursion $M_1 = 0$ and

$$M_n \stackrel{\mathrm{D}}{=} M_{n-I_n} + 1, \qquad n \in \{2, 3, \ldots\}, \tag{1.1}$$

where I_n is a random variable independent of $M_2, \ldots, M_{n-1}$ with distribution

$$\mathrm{P}\{I_n = k\} = \frac{p_k}{p_1 + \cdots + p_{n-1}}, \qquad k, n \in \mathbb{N},\ k < n. \tag{1.2}$$

Note that I_n is the size of the first jump of $\{R_k^{(n)}: k \in \mathbb{N}_0\}$. It is worth mentioning that the asymptotic results presented later not only apply to the number of jumps in a random walk with a barrier but also to all sequences whose marginal distributions satisfy the distributional recursion (1.1) with the distribution of I_n given by (1.2). The number of jumps of a random walk with barrier n is just one example of a sequence whose marginal distributions satisfy recursion (1.1).

Before we formulate our asymptotic results for M_n, we now briefly discuss closely related and more general models and the corresponding literature. In order to do this, for the moment, assume that the distribution of I_n in (1.1) does not follow (1.2) but rather takes the more general form

$$\mathrm{P}\{I_n = k\} = \pi_{n,n-k}, \qquad k, n \in \mathbb{N},\ k < n, \tag{1.3}$$

where the π_{ij}, $1 \leq j < i$, are some given nonnegative constants satisfying $\sum_{j=1}^{i-1} \pi_{ij} = 1$. Probably the most general description of sequences $\{M_n: n \in \mathbb{N}\}$ satisfying recursion (1.1) with the distribution of I_n given by (1.3) is as follows. Consider a decreasing Markov chain $\{Z_k: k \in \mathbb{N}_0\}$ with state space $\mathbb{N}$ and transition probabilities $\pi_{ij} > 0$ for $i, j \in \mathbb{N}$ with $j < i$ and $\pi_{ij} = 0$ otherwise. For $n \in \mathbb{N}$, let

$$M_n := \inf\{k \geq 1\colon Z_k = 1 \text{ given } Z_0 = n\}$$

denote the absorption time of the Markov chain conditioned on the event that the chain starts in the initial state n. Then the marginal distributions of $\{M_n: n \in \mathbb{N}\}$ satisfy the distributional recursion (1.1) with the distribution of I_n given by (1.3).

We are aware of only two papers, [36] and [39], which address the asymptotic behavior of M_n as n tends to ∞ in the general setting when it is not assumed that π_{ij} takes some particular form. The problem is simpler if either the probabilities π_{ij} are given explicitly, or if they have some particular functional form. In this latter situation some results on the asymptotic behavior of recursion (1.1) with (1.3) are available, for example, in the context of random composition structures [4], [18], [20], [21], of coalescent theory [19], [24], [26] (see also Section 7 of the present work), and in the context of random trees [11], [14], [24], [29], [30]. We also refer the reader to [3] for a number of interpretations of the random recursion (1.1), where I_n satisfies (1.3) with $\pi_{ij} = (i-1)^{-1}$, $i, j \in \mathbb{N}$, $j < i$.

Throughout the paper, $r(\cdot) \sim s(\cdot)$ means that $r(\cdot)/s(\cdot) \to 1$ as the argument tends to ∞. The symbols '$\xrightarrow{\mathrm{D}}$', '$\xrightarrow{\mathrm{W}}$', and '$\xrightarrow{\mathrm{P}}$' respectively denote convergence in law, weak convergence, and convergence in probability, and $X_n \xrightarrow{\mathrm{D}} (\xrightarrow{\mathrm{W}}, \xrightarrow{\mathrm{P}}) X$ means that the limiting relation holds when $n \to \infty$. By L we always denote a function slowly varying at ∞.

We now state our main asymptotic results for sequences of random variables $\{M_n : n \in \mathbb{N}\}$ satisfying the distributional recursion (1.1) with the distribution of I_n given by (1.2). We begin with a weak law of large numbers.

Theorem 1.1. *If $\sum_{j=1}^{n} \sum_{k=j}^{\infty} p_k \sim L(n)$ for some function L slowly varying at ∞ then, as $n \to \infty$,*

$$\frac{M_n}{\mathrm{E}\, M_n} \xrightarrow{\mathrm{P}} 1 \tag{1.4}$$

and $\mathrm{E}\, M_n \sim n/L(n)$. In particular, if

$$m := \mathrm{E}\, \xi < \infty \tag{1.5}$$

then $\mathrm{E}\, M_n \sim n/m$. If (1.5) holds, and if there exists a sequence of positive numbers $\{a_n : n \in \mathbb{N}\}$ such that $M_n/a_n \xrightarrow{\mathrm{P}} 1$ as $n \to \infty$, then $a_n \sim n/m$.

To formulate further results we need some more notation. For $C > 0$ and $\alpha \in [1, 2]$, let μ_α be an α-stable distribution with characteristic function $\psi_\alpha(t)$, $t \in \mathbb{R}$, of the form

$$\begin{gathered} \exp\biggl(-|t|^\alpha C\Gamma(1-\alpha)\biggl(\cos\biggl(\frac{\pi\alpha}{2}\biggr) + \mathrm{i}\sin\biggl(\frac{\pi\alpha}{2}\biggr)\mathrm{sgn}(t)\biggr)\biggr), \qquad 1 < \alpha < 2, \\ \exp\biggl(-|t|C\biggl(\frac{\pi}{2} - \mathrm{i}\log|t|\mathrm{sgn}(t)\biggr)\biggr), \qquad \alpha = 1, \\ \exp\biggl(-\frac{C}{2}t^2\biggr), \qquad \alpha = 2. \end{gathered}$$

In the case when (1.5) holds Theorem 1.2, below, provides necessary and sufficient conditions ensuring that M_n, properly normalized and centered, possesses a weak limit.

Theorem 1.2. *If $m := \mathrm{E}\, \xi < \infty$ then the following assertions are equivalent.*

(i) *There exist sequences of numbers $\{a_n, b_n : n \in \mathbb{N}\}$ with $a_n > 0$ and $b_n \in \mathbb{R}$ such that, as n tends to ∞, $(M_n - b_n)/a_n$ converges weakly to a nondegenerate and proper probability law.*

(ii) *Either $\sigma^2 := \mathrm{var}\, \xi < \infty$ or $\sigma^2 = \infty$ and, for some $\alpha \in [1, 2]$ and some function L slowly varying at ∞,*

$$\sum_{k=1}^{n} k^2 p_k \sim n^{2-\alpha} L(n), \qquad n \to \infty. \tag{1.6}$$

If $\sigma^2 < \infty$ then, with $b_n := n/m$ and $a_n := (m^{-3}C^{-1}\sigma^2 n)^{1/2}$, the limiting law is μ_2 (normal with mean 0 and variance C). If $\sigma^2 = \infty$ and (1.6) holds with $\alpha = 2$ then, with $b_n := n/m$ and $a_n := m^{-3/2}c_n$, where c_n is any sequence satisfying $\lim_{n\to\infty} nL(c_n)/c_n^2 = C$, the limiting law *is μ_2. If $\sigma^2 = \infty$ and (1.6) holds with $\alpha \in [1, 2)$ then, with $b_n := n/m$ and $a_n := m^{-(\alpha+1)/\alpha} c_n$, where c_n is any sequence satisfying*

$$\lim_{n\to\infty} \frac{nL(c_n)}{c_n^\alpha} = \frac{\alpha}{2-\alpha} C,$$

the limiting law is μ_α.

Remark 1.1. For $\sigma^2 < \infty$, the same weak convergence result for M_n was obtained in Theorem 4.1 of [39] in a setting more general than ours. Note that, for $\alpha \in [1, 2)$, (1.6) is equivalent to $\mathrm{P}\{\xi \geq n\} \sim (2-\alpha)n^{-\alpha} L(\alpha)/\alpha$, $n \to \infty$.

If the mean of ξ is infinite, Theorem 1.3 and Theorem 1.4, below, provide conditions ensuring that M_n, properly normalized without centering and centered, respectively, possesses a weak limit.

Theorem 1.3. *Suppose that, for some $\alpha \in (0, 1)$ and some function L slowly varying at ∞,*

$$\mathrm{P}\{\xi \geq n\} = \sum_{k=n}^{\infty} p_k \sim \frac{L(n)}{n^\alpha}, \qquad n \to \infty. \tag{1.7}$$

Then, as $n \to \infty$,

$$\frac{L(n)}{n^\alpha} M_n \xrightarrow{\mathrm{D}} \int_0^\infty \exp(-U_t)\,\mathrm{d}t, \tag{1.8}$$

where $\{U_t : t \geq 0\}$ is a drift-free subordinator with Lévy measure

$$\nu(\mathrm{d}t) = \frac{\mathrm{e}^{-t/\alpha}}{(1-\mathrm{e}^{-t/\alpha})^{\alpha+1}}\,\mathrm{d}t, \qquad t > 0. \tag{1.9}$$

Theorem 1.4. *Suppose that $\mathrm{E}\,\xi = \infty$ and that, for some function L slowly varying at ∞,*

$$\mathrm{P}\{\xi \geq n\} = \sum_{k=n}^{\infty} p_k \sim \frac{L(n)}{n}. \tag{1.10}$$

Let c be any positive function satisfying $\lim_{x\to\infty} x\,\mathrm{P}\{\xi \geq c(x)\} = 1$, and set $\psi(x) := x\int_0^{c(x)} \mathrm{P}\{\xi > y\}\,\mathrm{d}y$. Let b be any positive function satisfying

$$b(\psi(x)) \sim \psi(b(x)) \sim x,$$

and set $a(x) := x^{-1}b(x)c(b(x))$. Then, $(M_n - b(n))/a(n)$ converges weakly to the stable distribution μ_1 with $C = 1$.

In the literature there exist at least two standard approaches to studying distributional recursions. One approach is purely analytic and based on a singularity analysis of generating functions; see, for example, [14] and [30]. Another approach, called the *contraction method*, is more probabilistic; see [28], [34], and [35]. It was remarked in [24] that the recursions (1.1) which satisfy (1.2) can be successfully investigated using probabilistic methods alone (completely different from contraction methods). The present work extends ideas laid down in [24] for the particular case in which

$$\mathrm{P}\{I_n = k\} = \frac{n}{n-1}\frac{1}{k(k+1)}, \qquad k \in \{1, \dots, n-1\}.$$

The basic steps of the technique exploited can be summarized as follows.

Let

$$S_0 := 0, \qquad S_n := \xi_1 + \cdots + \xi_n, \quad \text{and} \quad N_n := \inf\{k \geq 1 \colon S_k \geq n\}, \qquad n \in \mathbb{N}.$$

We may expect that the limiting behavior of M_n and N_n are similar, or at least that the limiting behavior of the latter influences that of the former. Similarity in the limiting behavior of M_n and N_n is well indicated by the asymptotic properties of their difference. In particular, we will prove the following.

(a) If $\mathrm{E}\,\xi < \infty$ then $M_n - N_n$ weakly converges. Therefore, M_n, properly normalized and centered, possesses a weak limit if and only if the same is true for N_n.

(b) Now assume that $\mathrm{E}\,\xi = \infty$.

(b1) If $\sum_{k=n}^{\infty} p_k \sim L(n)/n$ and if $(N_n - b_n)/a_n$ weakly converges to some μ then $(M_n - N_n)/a_n \xrightarrow{\mathrm{P}} 0$, which proves that $(M_n - b_n)/a_n$ weakly converges to μ. Thus, in this case and case (a) the weak behavior of M_n and N_n is the same.

(b2) If, for some $\alpha \in (0, 1)$, $\sum_{k=n}^{\infty} p_k \sim n^{-\alpha} L(n)$ and N_n/a_n weakly converges to some ν_1 then $(M_n - N_n)/a_n$ weakly converges to some ν_2. Even though the argument exploited above does not apply, it will be proved that M_n/a_n weakly converges to $\nu_3 \neq \nu_1$. Thus, in case (b2) a weak behavior of M_n is not completely determined by that of N_n. Now it is influenced by the weak behavior of both N_n and $n - S_{N_n - 1}$ to, approximately, the same extent. This observation can be explained as follows. The probability of one big jump of S_n in comparison to cases (a) and (b1) is higher and, therefore, the epoch N_n comes more 'quickly'. As a consequence, a contribution to M_n of the number of jumps in the sequence $\{R_k^{(n)} \colon k \in \mathbb{N}_0\}$, while $R_k^{(n)}$ is traveling from $R_{N_n-1}^{(n)} = S_{N_n-1}$ to $n - 1$, becomes significant.

The referee pointed out the following interpretation of Theorem 1.3 that can be read from (1.9) in combination with results from [20] on exponential functionals of subordinators. Since N_n is known to be asymptotic to the local time of an unrestricted Bessel process (which has Mittag–Leffler distribution), then M_n is asymptotic to the local time of a modified Bessel process, obtained by recursively peeling the meander of the unrestricted Bessel process (the latter has distribution of the right-hand side of (1.8)).

To close the introduction, it remains to review the structural units of the rest of the paper. In Section 2 we investigate both the univariate and the bivariate weak behavior of $(N_n, n - S_{N_n-1})$, and discuss their relation to exponential integrals of subordinators. The proof of Theorem 1.3 along with some comments explaining the appearance of the limiting law in (1.8) are given in Section 3. Theorems 1.2, 1.1, and 1.4 are proved in Sections 4, 5, and 6, respectively. Finally, in Section 7 we apply our results to derive limiting theorems for the number of collision events that take place in certain beta-coalescent processes until there is just a single block. It turns out that the results are applicable for $\beta(a, b)$-coalescents with $0 < a < 2$ and $b = 1$ because, for that parameter range, the number of collisions satisfy the distributional recursion (1.1) such that I_n has a distribution of the form (1.2).

2. Results on N_n and $n - S_{N_n-1}$: the $m = \infty$ case

2.1. Univariate results

Necessary and sufficient conditions are given below to ensure that the sequence $\{N_n : n \in \mathbb{N}\}$ (a) properly normalized (without centering), weakly converges to a nondegenerate law (Proposition 2.1) and (b) is relatively stable (Proposition 2.2).

It is well known that (1.7) implies (2.1), below (in the case in which $\alpha \in (0, 1)$), and that Proposition 2.2(a) is equivalent to Proposition 2.2(b), below (see [17, Theorem 7] and [9, Corollary 8.1.7], respectively). Although the whole results may seem classic, we have not been able to locate them in the literature in the present form. Therefore, complete proofs of them are provided in [25], which is a preprint version of this work.

We say that a random variable ς_α has a scaled Mittag–Leffler distribution with parameter $\alpha \in [0, 1)$ if

$$\mathrm{E}\,\varsigma_\alpha^n = \frac{n!}{\Gamma^n(1-\alpha)\Gamma(1+n\alpha)}, \qquad n \in \mathbb{N}.$$

Note that the moments $\{\mathrm{E}\,\varsigma_\alpha^n : n \in \mathbb{N}\}$ uniquely determine the distribution.

Proposition 2.1. *If (1.7) holds for some* $\alpha \in [0, 1)$ *then*

$$\lim_{n\to\infty} \frac{L^k(n)}{n^{\alpha k}}\,\mathrm{E}\,N_n^k = \frac{k!}{\Gamma^k(1-\alpha)\Gamma(1+\alpha k)}, \qquad k \in \mathbb{N},$$

and, therefore,

$$\frac{L(n)}{n^\alpha} N_n \xrightarrow{\mathrm{w}} \theta_\alpha, \tag{2.1}$$

where θ_α *is the scaled Mittag–Leffler distribution with parameter* α.

Conversely, assume that there exists a sequence $\{a(n) : n \in \mathbb{N}\}$ *of positive real numbers such that* $N_n/a(n)$ *weakly converges to a nondegenerate and proper law* θ. *Then*

$$a(n) \sim D\left(\sum_{k=n}^{\infty} p_k\right)^{-1} \sim \frac{Dn^\alpha}{L(n)}$$

for some constants $D > 0$, $\alpha \in [0, 1)$, *and some function* L *slowly varying at* ∞, *and (2.1) holds.*

Proposition 2.2. *The following conditions are equivalent.*

(a) $\sum_{m=1}^{n} \sum_{k=m}^{\infty} p_k \sim L(n)$ *for some function* L *slowly varying at* ∞.

(b) $1 - \sum_{n=1}^{\infty} \mathrm{e}^{-sn} p_n \sim sL(1/s)$ *as* $s \downarrow 0$ *for some function* L *slowly varying at* ∞.

(c) *The sequence* $\{N_n : n \in \mathbb{N}\}$ *is relatively stable, i.e. there exists a sequence* $\{a(n) : n \in \mathbb{N}\}$ *of positive real numbers such that* $N_n/a(n) \xrightarrow{\mathrm{P}} 1$.

Moreover, if (a) holds then

$$\lim_{n\to\infty} \frac{L^k(n)}{n^k}\,\mathrm{E}\,N_n^k = 1, \qquad k \in \mathbb{N}, \tag{2.2}$$

and $a(n) \sim \mathrm{E}\,N_n$.

The next result is a corollary of Theorem 1.1 and Proposition 2.2.

Corollary 2.1. *Assume that (1.10) holds. Then,* $\mathrm{E}\,N_n \sim \mathrm{E}\,M_n \sim n/m(n)$*, where* $m(x) := \int_0^x \mathrm{P}\{\xi > y\}\,\mathrm{d}y$, $x > 0$*. Moreover,*

$$\frac{m(n)N_n}{n} \xrightarrow{\mathrm{P}} 1 \quad \textit{and} \quad \frac{m(n)M_n}{n} \xrightarrow{\mathrm{P}} 1.$$

In particular, $M_n/N_n \xrightarrow{\mathrm{P}} 1$.

Proof. Condition (1.10) ensures that $m(\cdot)$ belongs to the de Haan class Π, i.e.

$$\lim_{x\to\infty} \frac{m(\lambda x) - m(x)}{L(x)} = \log\lambda, \qquad \lambda > 0.$$

In particular, $m(\cdot)$ is slowly varying at ∞. Since $\sum_{j=1}^n \sum_{k=j}^\infty p_k \sim m(n)$, Theorem 1.1 and Proposition 2.2 imply the result for M_n and N_n, respectively.

Proposition 2.3, below, is a key ingredient for our proof of Theorem 1.4. Define $Y_n := n - S_{N_n-1}$, $n \in \mathbb{N}$.

Proposition 2.3. *Assume that (1.10) holds. Then, for fixed* $\delta > 0$,

$$\mathrm{E}\,Y_n^\delta = O\biggl(\frac{n^\delta L(n)}{m(n)}\biggr), \tag{2.3}$$

Furthermore, for functions a and b, as used in Theorem 1.4,

$$\frac{b(n)Y_n}{n\,a(n)} \xrightarrow{\mathrm{P}} 0. \tag{2.4}$$

Proof. In the same way as in the proof of Proposition 2.5 it follows that

$$\mathrm{E}\,Y_n^\delta = \sum_{k=0}^{n-1} (n-k)^\delta\,\mathrm{P}\{\xi \ge n-k\}u_k, \qquad n \in \mathbb{N},$$

where $u_k := \sum_{i=0}^k \mathrm{P}\{S_i = k\}$, $k \in \mathbb{N}_0$. By Corollary 2.1, $\mathrm{E}\,N_n \sim n/m(n)$. Moreover, $\mathrm{E}\,N_n \sim \sum_{k=0}^n u_k$, $n \in \mathbb{N}$. Thus, $\sum_{k=0}^n u_k \sim n/m(n)$ and, by Corollary 1.7.3 of [9],

$$U(s) := \sum_{n=0}^\infty s^n u_n \sim \frac{1}{m((1-s)^{-1})(1-s)} \quad \text{as } s \uparrow 1.$$

By the same corollary,

$$V(s) := \sum_{n=1}^\infty s^n n^\delta\,\mathrm{P}\{\xi \ge n\} \sim \frac{\Gamma(\delta)L((1-s)^{-1})}{(1-s)^\delta} \quad \text{as } s \uparrow 1.$$

Therefore,

$$\sum_{n=1}^\infty s^n\,\mathrm{E}\,Y_n^\delta = U(s)V(s) \sim \frac{\Gamma(\delta)}{(1-s)^{\delta+1}}\frac{L((1-s)^{-1})}{m((1-s)^{-1})} \quad \text{as } s \uparrow 1.$$

Therefore, Corollary 1.7.3 of [9] applies and proves (2.3). Recall that $\psi(x) = xm(c(x))$ and that $c(x) \sim xL(c(x))$. Set $v(x) := xa(x)/b(x) = c(b(x))$. Since $m(x)/L(x) \to \infty$, $c(x) \to \infty$, and

$$\frac{\psi(x)}{c(x)} = \frac{xm(c(x))}{c(x)} \sim \frac{m(c(x))}{L(c(x))} \quad \text{as } x \to \infty,$$

we conclude that $\psi(x)/c(x) \to \infty$ as $x \to \infty$. Therefore,

$$\frac{b(x)}{a(x)} = \frac{x}{c(b(x))} \to \infty \quad \text{as } x \to \infty.$$

The latter relation, together with $m(x)/L(x) \to \infty$, implies that

$$\begin{aligned}\frac{L(x)}{m(x)}\frac{b(x)}{a(x)} &= \frac{L(x)}{m(x)}\frac{x}{c(b(x))}\\ &\sim \frac{L(x)}{m(x)}\frac{x}{b(x)L(c(b(x)))}\\ &\sim \frac{L(x)}{m(x)}\frac{\psi(b(x))}{b(x)L(c(b(x)))}\\ &\sim \frac{L(x)}{m(x)}\frac{m(c(b(x)))}{L(c(b(x)))}\end{aligned}$$

remains bounded for large x.

For fixed $\delta \in (0, 1)$ and any $\varepsilon > 0$, we have, by Markov's inequality and (2.3),

$$\mathrm{P}\{Y_n > v(n)\varepsilon\} \le \frac{\mathrm{E}\, Y_n^\delta}{v^\delta(n)\varepsilon^\delta} = O\left(\frac{L(n)b(n)}{m(n)a(n)}\left(\frac{b(n)}{a(n)}\right)^{\delta-1}\right) \to 0 \quad \text{as } n \to \infty.$$

The proof is complete.

2.2. Some results on exponential integrals of subordinators

Let $\{Z_t : t \ge 0\}$ be a drift-free subordinator which is independent of T, an exponentially distributed random variable with mean 1. Set $Q := \int_0^T \exp(-Z_t)\,\mathrm{d}t$, $M := \exp(-Z_T)$, and $A := \int_T^\infty \exp(-Z_t)\,\mathrm{d}t$. As is well known (see, for example, [10, Lemma 6.2]), the following equality of distribution holds:

$$A_\infty \stackrel{\mathrm{D}}{=} MA'_\infty + Q, \tag{2.5}$$

where A'_∞ is a copy of A_∞ which is independent of (M, Q). The latter means that A_∞ is a perpetuity (see [2] for the definition and recent results) generated by the random vector (M, Q).

Our next result generalizes Proposition 3.1 of [10], which deals with the moments of Q, and a number of results concerning the moments of $\int_0^\infty \exp(-Z_t)\,\mathrm{d}t = Q + A$ (see, for example, [38, Proposition 3.3]).

Proposition 2.4. *For $\lambda > 0$ and $\mu \ge 0$,*

$$\mathrm{E}\, Q^\lambda M^\mu = \frac{\lambda}{1+\varphi(\lambda+\mu)}\, \mathrm{E}\, Q^{\lambda-1}M^\mu,$$

where $\varphi(s) := -\log \mathrm{E} \exp(-sZ_1)$, $s \ge 0$. In particular,

$$a_{n,m} := \mathrm{E}\, Q^n M^m = \frac{n!}{\prod_{k=0}^n (1+\varphi(m+k))}, \qquad m, n \in \mathbb{N}_0, \tag{2.6}$$

$$b_{n,m} := \mathrm{E}\, Q^n A^m = \frac{n!\,m!}{\prod_{k=0}^n (1+\varphi(m+k))\varphi(1)\cdots\varphi(m)}, \qquad m, n \in \mathbb{N}_0.$$

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The moment sequences $\{a_{m,n}: m, n \in \mathbb{N}_0\}$ and $\{b_{m,n}: m, n \in \mathbb{N}_0\}$ uniquely determine the laws of the random vectors (M, Q) and (A, Q), respectively.

Proof. For $t > 0$, define $A_t := \int_0^t \exp(-Z_v)\,\mathrm{d}v$. The following is essentially [10, Equation (3.1)]:

$$A_t^\lambda \exp(-\mu Z_t) = \lambda \int_0^t (A_t - A_v)^{\lambda-1} \exp(-\mu(Z_t - Z_v)) \exp(-(\mu+1)Z_v)\,\mathrm{d}v.$$

Since

$$\begin{aligned}(A_t - A_v)^{\lambda-1} &\exp(-\mu(Z_t - Z_v))\\ &= \exp(-(\lambda-1)Z_v)\bigg(\int_0^{t-v} \exp(-(Z_{s+v} - Z_v))\,\mathrm{d}s\bigg)^{\lambda-1} \exp(-\mu(Z_t - Z_v))\end{aligned}$$

and $\{Z_{s+v} - Z_v : s \geq 0\}$ is a subordinator which is independent of $\{Z_v : v \leq t\}$ and has the same law as $\{Z_t : t \geq 0\}$, we conclude that $(\int_0^{t-v} \exp(-(Z_{s+v} - Z_v))\,\mathrm{d}s)^{\lambda-1} \exp(-\mu(Z_t - Z_v))$ has the same law as $A_{t-v}^{\lambda-1} \exp(-\mu Z_{t-v})$ and is independent of $\exp(-(\lambda-1)Z_v)$. Therefore, using Fubini's theorem,

$$\begin{aligned}\mathrm{E}\, A_T^\lambda \exp(-\mu Z_T) &= \int_0^\infty \mathrm{e}^{-t}\, \mathrm{E}\, A_t^\lambda \exp(-\mu Z_t)\,\mathrm{d}t\\ &= \lambda \int_0^\infty \mathrm{e}^{-t} \bigg(\int_0^t \mathrm{e}^{-v\varphi(\lambda+\mu)}\, \mathrm{E}\, A_{t-v}^{\lambda-1} \exp(-\mu Z_{t-v})\,\mathrm{d}v\bigg)\mathrm{d}t\\ &= \lambda \int_0^\infty \mathrm{e}^{-v\varphi(\lambda+\mu)} \bigg(\int_v^\infty \mathrm{e}^{-t}\, \mathrm{E}\, A_{t-v}^{\lambda-1} \exp(-\mu Z_{t-v})\,\mathrm{d}t\bigg)\mathrm{d}v\\ &= \lambda \int_0^\infty \mathrm{e}^{-v(\varphi(\lambda+\mu)+1)}\,\mathrm{d}v \int_0^\infty \mathrm{e}^{-u}\, \mathrm{E}\, A_u^{\lambda-1} \exp(-\mu Z_u)\,\mathrm{d}u\\ &= \frac{\lambda}{1+\varphi(\lambda+\mu)}\, \mathrm{E}\, A_T^{\lambda-1} \exp(-\mu Z_T).\end{aligned}$$

Starting with

$$\mathrm{E} \exp(-\mu Z_T) = \int_0^\infty \mathrm{e}^{-t}\, \mathrm{E} \exp(-\mu Z_t)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-t(1+\varphi(\mu))}\,\mathrm{d}t = \frac{1}{1+\varphi(\mu)}, \tag{2.7}$$

the formula for $a_{n,m}$ follows by induction. To prove that the law of (M, Q) is uniquely determined by $\{a_{n,m}: n, m \in \mathbb{N}_0\}$, it suffices to check that the marginal laws are uniquely determined by the corresponding moment sequences (see [31, Theorem 3]). Since $M \in [0, 1]$ almost surely, the law of M is trivially moment determinate. From (2.6), it follows that

$$\mathrm{E}\, Q^n = \frac{n!}{(1+\varphi(1)) \cdots (1+\varphi(n))}, \qquad n \in \mathbb{N}.$$

Set $f_n := \mathrm{E}\, Q^n/n!$. The limit $f := \lim_{n\to\infty} f_n/f_{n+1}$ exists and is positive (it is finite if Z_t is compound Poisson, otherwise it is infinite). By the Cauchy–Hadamard formula, $f = \sup\{r > 0 : \mathrm{E}\, \mathrm{e}^{rQ} < \infty\}$. Therefore, the law of Q has finite exponential moments of some orders from which we deduce that this law is moment determinate.

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According to Proposition 3.3 of [38], $\mathrm{E}\, A_\infty^m = m!/(\varphi(1)\cdots\varphi(m))$, $m \in \mathbb{N}_0$. In view of (2.5),

$$\begin{aligned}\mathrm{E}\, Q^n A^m &= \mathrm{E}\, Q^n M^m \,\mathrm{E}\, A_\infty^m \\ &= \frac{n!\,m!}{\prod_{k=0}^n (1+\varphi(m+k))\varphi(1)\cdots\varphi(m)}, \qquad m, n \in \mathbb{N}_0.\end{aligned}$$

It can be checked, in the same way as above for (M, Q), that the law of (A, Q) is determined by the moment sequence. We omit the details.

2.3. A bivariate result

Assume that (1.7) holds or, equivalently, that

$$w(n) := \frac{1}{\mathrm{P}\{\xi \ge n\}} = \left(\sum_{k=n}^{\infty} p_k\right)^{-1} \sim \frac{n^\alpha}{L(n)}$$

for some $\alpha \in (0, 1)$. Let T be an exponentially distributed random variable with mean 1, which is independent of a drift-free subordinator $\{U_t : t \ge 0\}$ with Lévy measure (1.9).

From Proposition 2.1, it follows that $N_n/w(n)$ converges in distribution to a random variable ς_α with the scaled Mittag–Leffler distribution with parameter α. From (2.6) or from Proposition 3.1 of [10], we have

$$\mathrm{E}\left(\int_0^T \exp(-U_t)\,\mathrm{d}t\right)^n = \frac{n!}{\Gamma^n(1-\alpha)\Gamma(1+n\alpha)}, \qquad n \in \mathbb{N}_0,$$

which means that $\int_0^T \exp(-U_t)\,\mathrm{d}t \overset{\mathrm{D}}{=} \varsigma_\alpha$. Thus,

$$\frac{N_n}{w(n)} \overset{\mathrm{D}}{\to} \int_0^T \exp(-U_t)\,\mathrm{d}t. \tag{2.8}$$

Let η_α be a beta-distributed random variable with parameters $1-\alpha$ and α, i.e. with density $x \mapsto \pi^{-1}\sin(\pi\alpha)x^{-\alpha}(1-x)^{\alpha-1}$, $x \in (0, 1)$. It is well known that (see, for example, [9, Theorem 8.6.3]) $(1 - S_{N_n-1}/n)^\alpha \overset{\mathrm{D}}{\to} \eta_\alpha^\alpha$. It can be checked that

$$\mathrm{E}\, \eta_\alpha^{n\alpha} = \frac{\Gamma(\alpha(n-1)+1)}{\Gamma(1-\alpha)\Gamma(\alpha n+1)}, \qquad n \in \mathbb{N}_0.$$

From (2.7), it follows that $\exp(-U_T)$ has the same moment sequence. Therefore, since the distribution of $\exp(-U_T)$ is concentrated on $[0, 1]$, it coincides with the distribution of η_α^α. Thus,

$$\left(1 - \frac{S_{N_n-1}}{n}\right)^\alpha \overset{\mathrm{D}}{\to} \exp(-U_T). \tag{2.9}$$

Now we point out a bivariate result generalizing (2.8) and (2.9).

Proposition 2.5. *Suppose that (1.7) holds. Then,*

$$w^{-1}(n)(w(n - S_{N_n-1}), N_n) \overset{\mathrm{D}}{\to} \left(\exp(-U_T), \int_0^T \exp(-U_t)\,\mathrm{d}t\right),$$

where $\{U_t : t \ge 0\}$ is a drift-free subordinator with Lévy measure (1.9).

Remark 2.1. Corollary 3.3 of [33] states that

$$\left(\frac{L(n)}{n^\alpha}(N_{n+1}-1),\, 1-\frac{S_{N_{n+1}-1}}{n}\right) \xrightarrow{\mathrm{D}} (X, Y), \tag{2.10}$$

where the distribution of a random vector (X, Y) was defined by the moment sequence. Our proof of Proposition 2.5 is different from and simpler than Port's [33] proof of (2.10).

Proof of Proposition 2.5. According to Proposition 2.4 it suffices to verify that

$$\lim_{n\to\infty} \frac{\mathrm{E}\, w^i(n-S_{N_n-1})N_n^j}{w^{i+j}(n)} = \frac{j!\,\Gamma(\alpha(i-1)+1)}{\Gamma^{j+1}(1-\alpha)\,\Gamma(\alpha(i+j)+1)}, \qquad i, j \in \mathbb{N}_0. \tag{2.11}$$

By Proposition 2.1,

$$\lim_{n\to\infty} \frac{L^k(n)}{n^{\alpha k}}\,\mathrm{E}\, N_n^k = \frac{k!}{\Gamma^k(1-\alpha)\Gamma(1+\alpha k)}, \qquad k \in \mathbb{N}. \tag{2.12}$$

For $i = 0$, (2.11) follows from (2.12). For $i \in \mathbb{N}$, (2.11) is checked as follows:

$$\begin{aligned}
&\mathrm{E}\, w^i(n-S_{N_n-1})N_n^j \\
&\quad = \sum_{k=1}^{n}\sum_{l=0}^{n-1} w^i(n-l)k^j\, \mathrm{P}\{N_n = k,\ S_{k-1} = l\} \\
&\quad = w^i(n)\,\mathrm{P}\{\xi \ge n\} + \sum_{l=1}^{n-1} w^i(n-l)\,\mathrm{P}\{\xi \ge n-l\}\sum_{k=2}^{l+1} k^j\,\mathrm{P}\{S_{k-1} = l\} \\
&\quad = w^i(n)\,\mathrm{P}\{\xi \ge n\} + \sum_{l=1}^{n-1} w^{i-1}(n-l)\sum_{k=2}^{l+1} k^j\,\mathrm{P}\{S_{k-1} = l\}.
\end{aligned}$$

As in [1, p. 26], define the function $f(x) := 0$ on $[0, 1)$ and $f(x) := (k+1)^j$ on $[k, k+1)$ for $k \in \mathbb{N}$, and set $F(t) := \int_0^t f(x)\,\mathrm{d}x$. Then,

$$\sum_{l=1}^{n-1}\sum_{k=2}^{l+1} k^j\,\mathrm{P}\{S_{k-1} = l\} = \sum_{k=1}^{n-1}(k+1)^j\,\mathrm{P}\{N_n > k\} = \mathrm{E}\,F(N_n).$$

By Karamata's theorem [9, Proposition 1.5.8], $F(t) \sim (j+1)^{-1}t^{j+1}$. Since $\lim_{n\to\infty} N_n = \infty$ almost surely and $(N_n/w(n))^{j+1} \xrightarrow{\mathrm{D}} \varsigma_\alpha^{j+1}$, we have

$$\frac{F(N_n)}{w^{j+1}(n)} \xrightarrow{\mathrm{D}} \frac{\varsigma_\alpha^{j+1}}{j+1}. \tag{2.13}$$

By (2.12),

$$\lim_{n\to\infty} \mathrm{E}\left(\frac{N_n}{w(n)}\right)^{j+2} = \mathrm{E}\,\varsigma_\alpha^{j+2} < \infty.$$

Therefore, the sequence $\{F(N_n)/w^{j+1}(n) \colon n \in \mathbb{N}\}$ is uniformly integrable, which together with (2.13) implies that

$$\mathrm{E}\,F(N_n) \sim \mathrm{E}\,\frac{\varsigma_\alpha^{j+1}}{j+1} w^{j+1}(n) \sim \frac{j!}{\Gamma^{j+1}(1-\alpha)\Gamma(1+(j+1)\alpha)}\,\frac{n^{\alpha(j+1)}}{L^{j+1}(n)}. \tag{2.14}$$

Thus, if $i = 1$, we have

$$\mathrm{E}\, w(n - S_{N_n-1}) N_n^j \sim \frac{j!}{\Gamma^{j+1}(1-\alpha)\Gamma(1+(j+1)\alpha)} \frac{n^{\alpha(j+1)}}{L^{j+1}(n)},$$

and (2.11) follows. Now assume that $i \geq 2$. Since $w^{i-1}(n) \sim n^{\alpha(i-1)}/L^{i-1}(n)$, Corollary 1.7.3 of [9] yields

$$W(s) := \sum_{n=1}^{\infty} s^n w^{i-1}(n) \sim \frac{\Gamma(1+\alpha(i-1))}{(1-s)^{1+\alpha(i-1)} L^{i-1}((1-s)^{-1})}, \qquad s \uparrow 1.$$

By the same corollary, (2.14) implies that

$$\begin{aligned} R(s) &:= \sum_{n=1}^{\infty} s^n \Bigg(\sum_{k=2}^{n+1} k^j\, \mathrm{P}\{S_{k-1} = l\} \Bigg) \\ &\sim \frac{j!}{\Gamma^{j+1}(1-\alpha)} \frac{1}{(1-s)^{\alpha(j+1)} L^{j+1}((1-s)^{-1})}, \qquad s \uparrow 1. \end{aligned}$$

Therefore,

$$W(s)R(s) \sim \frac{\Gamma(1+\alpha(i-1))\, j!}{\Gamma^{j+1}(1-\alpha)} \frac{1}{(1-s)^{1+\alpha(i+j)} L^{i+j}((1-s)^{-1})}, \qquad s \uparrow 1.$$

The sequence $\{w^{i-1}(n) \colon n \in \mathbb{N}\}$ is nondecreasing. Hence, the sequence

$$\Bigg\{ \sum_{l=1}^{n-1} w^{i-1}(n-l) \sum_{k=2}^{l+1} k^j\, \mathrm{P}\{S_{k-1} = l\} \colon n = 2, 3, \dots \Bigg\}$$

is nondecreasing too. Another appeal to Corollary 1.7.3 of [9] gives, as $n \to \infty$,

$$\sum_{l=1}^{n-1} w^{i-1}(n-l) \sum_{k=2}^{l+1} k^j\, \mathrm{P}\{S_{k-1} = l\} \sim \frac{\Gamma(1+\alpha(i-1))\, j!}{\Gamma^{j+1}(1-\alpha)\Gamma(1+\alpha(i+j))} \frac{n^{\alpha(i+j)}}{L^{i+j}(n)}.$$

From this, (2.11) follows.

3. Proof of Theorem 1.3 and some comments

Nothing more than (1.1) and (1.2) is required for the proof given below.
For $k, n \in \mathbb{N}$, set $a_k(n) := \mathrm{E}\, M_n^k$ and $b_k(n) := \mathrm{E}\, N_n^k$. For $x \geq 0$, define

$$\Phi(x) := \frac{\Gamma(1-\alpha)\Gamma(\alpha x + 1)}{\Gamma(\alpha(x-1)+1)} - 1 = \alpha x \mathrm{B}(\alpha x, 1-\alpha) - 1,$$

where B denotes the beta function. Note that

$$\mathrm{B}(\alpha x, 1-\alpha) = \int_0^1 y^{\alpha x - 1}(1-y)^{-\alpha}\, \mathrm{d}y = \alpha^{-1} \int_0^{\infty} \mathrm{e}^{-xy}(1-\mathrm{e}^{-y/\alpha})^{-\alpha}\, \mathrm{d}y,$$

and, hence,

$$\begin{aligned}\Phi(x) &= \int_0^\infty x\mathrm{e}^{-xy}(1-\mathrm{e}^{-y/\alpha})^{-\alpha}\,\mathrm{d}y - 1\\ &= \int_0^\infty (1-\mathrm{e}^{-y/\alpha})^{-\alpha}\,\mathrm{d}(1-\mathrm{e}^{-xy}) - 1\\ &= \int_0^\infty (1-\mathrm{e}^{-xy})\frac{\mathrm{e}^{-y/\alpha}}{(1-\mathrm{e}^{-y/\alpha})^{\alpha+1}}\,\mathrm{d}y. \end{aligned} \tag{3.1}$$

Thus, the function Φ is the Laplace exponent of an infinitely divisible law with zero drift and Lévy measure ν given in (1.9).

Remark 3.1. In [7, p. 102] it was stated that the right-hand side of (3.1) equals $\Phi(x)+1$ (in our notation). Thus, our (3.1) corrects that oversight.

Assuming that (1.7) holds, we will prove that

$$\lim_{n\to\infty}\frac{L^k(n)}{n^{\alpha k}}a_k(n) = \frac{k!}{\Phi(1)\cdots\Phi(k)} =: a_k, \qquad k\in\mathbb{N}. \tag{3.2}$$

This will imply that (see, for example, [7])

(i) $a_k = \mathrm{E}\,\eta^k$, $k\in\mathbb{N}$, where η is a random variable with distribution of the exponential integral of a drift-free subordinator with Lévy measure ν; and

(ii) the moments $\{a_n : n\in\mathbb{N}\}$ uniquely determine the law of η.

Note that the statement in (i) was first obtained in Example 3.4 of [38]. From (i) and (ii), it will follow that (3.2) implies (1.8).

From (1.1) and (1.2), it follows that

$$a_1(n) = 1 + r_n\sum_{i=1}^{n-1} a_1(n-i)p_i,$$

and, for $k\in\{2,3,\dots\}$,

$$\begin{aligned} a_k(n) &= D_k(a_1(n),\dots,a_{k-2}(n)) + ka_{k-1}(n) + r_n\sum_{i=1}^{n-1}a_k(n-i)p_i\\ &=: d_k(n) + r_n\sum_{i=1}^{n-1}a_k(n-i)p_i, \end{aligned}\tag{3.3}$$

where $D_k(\cdot)$ denotes the affine function of $k-2$ positive variables of the form

$$D_k(x_1,x_2,\dots,x_{k-2}) = \gamma_{0,k} + \sum_{i=1}^{k-2}\gamma_{i,k}x_i$$

with coefficients $\gamma_{i,k}\in\mathbb{R}$, $i\in\{0,1,\dots,k-2\}$ (these coefficients can be derived explicitly, but their exact values are of no use here), and $r_n := 1/(p_1+\cdots+p_{n-1})$. Using the equality of distributions,

$$N_1 = 1, \qquad N_n \stackrel{\mathrm{D}}{=} 1 + N'_{n-\xi}\,\mathbf{1}_{\{\xi<n\}}, \quad n=2,3,\dots,$$

where ξ is independent of $\{N'_n : n \in \mathbb{N}\}$, a copy of $\{N_n : n \in \mathbb{N}\}$, we can show that

$$b_k(n) = c_k(n) + \sum_{i=1}^{n-1} b_k(n-i)\, p_i, \qquad k \in \mathbb{N}, \tag{3.4}$$

where $c_1(n) := 1$ and

$$c_k(n) := D_k(b_1(n), \dots, b_{k-2}(n)) + k\, b_{k-1}(n), \qquad k \geq 2.$$

To prove (3.2), we will use induction on k. Suppose that (3.2) holds for $k \in \{1, 2, \dots, j-1\}$. Set

$$\beta_1 := \frac{1}{1-b_1} \quad \text{and} \quad \beta_l := \frac{1}{b_{l-1} - l^{-1} b_l} \prod_{i=1}^{l-1} \frac{b_{i-1}}{b_{i-1} - i^{-1} b_i}, \qquad l \in \{2, 3, \dots\},$$

where $b_l := l!/(\Gamma^l(1-\alpha)\Gamma(1+\alpha l))$, $l \in \mathbb{N}$, and note that

$$a_{l-1} - \beta_l (b_{l-1} - l^{-1} b_l) = 0, \qquad l \in \mathbb{N}. \tag{3.5}$$

In the following we exploit an idea given in the proof of Proposition 3 of [18]. Suppose that there exists an $\varepsilon > 0$ such that $a_j(n) > (\beta_j + \varepsilon) b_j(n)$ for infinitely many n. It is possible to decrease ε so that the inequality $a_j(n) > (\beta_j + \varepsilon) b_j(n) + c$ holds infinitely often for any fixed positive c. Thus, we can define $n_c := \inf\{n \geq 1 : a_j(n) > (\beta_j + \varepsilon) b_j(n) + c\}$. Then

$$a_j(n) \leq (\beta_j + \varepsilon) b_j(n) + c \quad \text{for all } n \in \{1, 2, \dots, n_c - 1\}. \tag{3.6}$$

We have

$$\begin{aligned}
(\beta_j + \varepsilon) & b_j(n_c) + c < a_j(n_c) \\
&= d_j(n_c) + r_{n_c} \sum_{i=1}^{n_c - 1} a_j(n_c - i) p_i \quad \text{by (3.3)} \\
&\leq d_j(n_c) + c + (\beta_j + \varepsilon) r_{n_c} \sum_{i=1}^{n_c - 1} b_j(n_c - i) p_i \quad \text{by (3.6)} \\
&= D_j(a) + j a_{j-1}(n_c) + c \\
&\quad + (\beta_j + \varepsilon)(r_{n_c} - 1)(b_j(n_c) - D_j(b) - j b_{j-1}(n_c)) \\
&\quad + (\beta_j + \varepsilon) b_j(n_c) - (\beta_j + \varepsilon)(D_j(b) + j b_{j-1}(n_c)) \quad \text{by (3.3) and (3.4)},
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
0 < D_j(a) &+ j a_{j-1}(n_c) + (\beta_j + \varepsilon)(r_{n_c} - 1)(b_j(n_c) - D_j(b) - j b_{j-1}(n_c)) \\
&- (\beta_j + \varepsilon)(D_j(b) + j b_{j-1}(n_c)),
\end{aligned}$$

where we have used the abbreviations

$$D_j(a) := D_j(a_1(n_c), \dots, a_{j-2}(n_c)) \quad \text{and} \quad D_j(b) := D_j(b_1(n_c), \dots, b_{j-2}(n_c))$$

for convenience. Divide the latter inequality by $z(c) := n_c^{(j-1)\alpha}/L^{j-1}(n_c)$, and let c go to ∞ (which implies that n_c tends to ∞). Note that, according to (1.7), $r_n - 1 \sim n^{-\alpha}L(n)$ and that, by the induction assumption,

$$\lim_{c\to\infty} \frac{D_j(a_1(n_c), \dots, a_{j-2}(n_c))}{z(c)} = 0 \quad \text{and} \quad \lim_{c\to\infty} \frac{a_{j-1}(n_c)}{z(c)} = a_{j-1}.$$

Using these facts and (2.12), we obtain

$$0 \le ja_{j-1} + (\beta_j + \varepsilon)b_j - (\beta_j + \varepsilon)jb_{j-1}.$$

Since the function Φ defined at the beginning of the proof is positive for $x > 0$ and $jb_{j-1}/b_j - 1 = \Phi(j)$, we conclude that $jb_{j-1} - b_j > 0$. Therefore,

$$\varepsilon(jb_{j-1} - b_j) \le j(a_{j-1} - \beta_j(b_{j-1} - j^{-1}b_j)) = 0$$

by (3.5). This is the desired contradiction. Thus, we have verified that

$$\limsup_{n\to\infty} \frac{a_j(n)}{b_j(n)} \le \beta_j.$$

A symmetric argument proves the converse inequality for the lower bound. Therefore,

$$a_j(n) \sim \beta_j b_j(n) \sim \beta_j b_j \frac{n^{j\alpha}}{L^j(n)} = a_j \frac{n^{j\alpha}}{L^j(n)}.$$

A similar but simpler reasoning yields the result for $k = 1$. We omit the details. The proof is complete.

The above proof only exhibits the limiting law, it does not give any insight into why it is the law of an exponential functional. We intend to explore this issue now in some more detail. Remarkably enough, it seems that we have found a new area where perpetuities appear in a natural way.

Fix $i, j \in \mathbb{N}$. Define $\hat{R}_0^{(j)}(i) := 0$,

$$\hat{R}_k^{(j)}(i) := \hat{R}_{k-1}^{(j)}(i) + \xi_{i+k}\,\mathbf{1}_{\{\hat{R}_{k-1}^{(j)}(i)+\xi_{i+k}<j\}}, \qquad k \in \mathbb{N},$$

and

$$\hat{M}_n(i) := \sum_{l=0}^{\infty} \mathbf{1}_{\{\hat{R}_l^{(n)}(i)+\xi_{i+l+1}<n\}}, \qquad n \in \mathbb{N}.$$

Also set $Y_n := n - S_{N_n-1}$.

The subsequent argument relies upon the following decomposition, (3.8).

Lemma 3.1. *For fixed $n \in \mathbb{N}$ and any $i \in \mathbb{N}$,*

$$\hat{M}_n(i) \overset{\mathrm{D}}{=} M_n \tag{3.7}$$

and

$$M_n - N_n + 1 = \hat{M}_{Y_n}(N_n) \overset{\mathrm{D}}{=} M'_{Y_n}, \tag{3.8}$$

where $\{M'_n : n \in \mathbb{N}\}$ has the same law as $\{M_n : n \in \mathbb{N}\}$ and is independent of (N_n, Y_n).

Proof. We have

$$\begin{aligned} M_n &= \sum_{l=0}^{\infty} \mathbf{1}_{\{R_l^{(n)}+\xi_{l+1}<n\}} \\ &= \sum_{l=0}^{N_n-2} 1 + \sum_{l=N_n}^{\infty} \mathbf{1}_{\{R_l^{(n)}+\xi_{l+1}<n\}} \\ &= N_n - 1 + \sum_{l=0}^{\infty} \mathbf{1}_{\{\hat{R}_l^{(Y_n)}(N_n)+\xi_{N_n+l+1}<Y_n\}} \\ &= N_n - 1 + \hat{M}_{Y_n}(N_n), \end{aligned}$$

and the first equality in (3.8) follows. For any fixed $k \in \mathbb{N}$,

$$\begin{aligned} &\mathrm{P}\{\hat{M}_{Y_n}(N_n) = k\} \\ &\quad = \sum_{i=1}^{n} \sum_{j=0}^{n-1} \mathrm{P}\{\hat{M}_{n-j}(i) = k,\ N_n = i,\ S_{N_n-1} = j\} \\ &\quad = \sum_{i=1}^{n} \sum_{j=0}^{n-1} \mathrm{P}\bigg\{\sum_{l=0}^{\infty} \mathbf{1}_{\{\hat{R}_l^{(n-j)}(i)+\xi_{i+l+1}<n-j\}} = k,\ N_n = i,\ S_{N_n-1} = j\bigg\}. \end{aligned}$$

The sequence $\{\hat{R}_l^{(n-j)}(i) + \xi_{i+l+1} : l \in \mathbb{N}_0\}$ is independent of $\mathbf{1}_{\{N_n=i, S_{N_n-1}=j\}}$ and has the same law as $\{(R_l^{(n-j)})' + \xi'_{l+1} : l \in \mathbb{N}_0\}$, where $\{(R_l^{(\cdot)})' : l \in \mathbb{N}_0\}$ is constructed in the same way as the sequence without the 'prime' by using $\{\xi'_k : k \in \mathbb{N}\}$, an independent copy of $\{\xi_k : k \in \mathbb{N}\}$. This implies (3.7) and

$$\begin{aligned} &\mathrm{P}\{\hat{M}_{Y_n}(N_n) = k\} \\ &\quad = \sum_{i=1}^{n} \sum_{j=0}^{n-1} \mathrm{P}\bigg\{\sum_{l=0}^{\infty} \mathbf{1}_{\{(R_l^{(n-j)})'+\xi'_{l+1}<n-j\}} = k\bigg\} \mathrm{P}\{N_n = i,\ S_{N_n-1} = j\} \\ &\quad = \mathrm{P}\bigg\{\sum_{l=0}^{\infty} \mathbf{1}_{\{(R_l^{(Y_n)})'+\xi'_{l+1}<Y_n\}} = k\bigg\} \\ &\quad = \mathrm{P}\{M'_{Y_n} = k\}, \end{aligned}$$

and the second equality in distribution in (3.8) follows.

Set $t(n) := n^\alpha / L(n)$. From the above proof, we already know that $M_n/t(n)$ converges in law to a random variable Z, say, with a proper law. From $Y_n \xrightarrow{\mathrm{P}} +\infty$ and the result of Lemma 3.1, we conclude that $\hat{M}_{Y_n}/t(Y_n)$ converges in law to a random variable $Z'' \overset{\mathrm{D}}{=} Z$. By Proposition 2.5,

$$\bigg(\frac{t(Y_n)}{t(n)}, \frac{N_n - 1}{t(n)}\bigg) \xrightarrow{\mathrm{D}} (M, Q) := \bigg(\exp(-U_T), \int_0^T \exp(-U_t)\,\mathrm{d}t\bigg).$$

Rewriting (3.8) in the form

$$\frac{M_n}{t(n)} = \frac{\hat{M}_{Y_n}}{t(Y_n)} \frac{t(Y_n)}{t(n)} + \frac{N_n - 1}{t(n)},$$

we conclude that

$$\left(\frac{\hat{M}_{Y_n}}{t(Y_n)}, \frac{t(Y_n)}{t(n)}, \frac{N_n - 1}{t(n)}\right) \xrightarrow{\mathrm{D}} (Z', M, Q),$$

where $Z' \overset{\mathrm{D}}{=} Z$, and using characteristic functions, it can be checked that Z' is independent of (M, Q). Furthermore,

$$Z \overset{\mathrm{D}}{=} MZ' + Q. \tag{3.9}$$

From (2.5), it follows that the distribution of $\int_0^\infty \exp(-U_t)\,\mathrm{d}t$ is a solution of (3.9). By Theorem 1.5(i) of [40], this solution is unique. Therefore,

$$\frac{M_n}{t(n)} \xrightarrow{\mathrm{D}} \int_0^\infty \exp(-U_t)\,\mathrm{d}t.$$

In a similar way, we can prove the following result.

Corollary 3.1. *Suppose that (1.7) holds. Then,*

$$\left(\frac{M_n - N_n}{t(n - S_{N_n - 1})}, \frac{t(n - S_{N_n-1})}{t(n)}, \frac{N_n}{t(n)}\right)$$
$$\xrightarrow{\mathrm{D}} \left(\int_0^\infty \exp(-(U_{t+T} - U_T))\,\mathrm{d}t, \exp(-U_T), \int_0^T \exp(-U_t)\,\mathrm{d}t\right).$$

Furthermore,

$$\frac{M_n - N_n}{t(n - S_{N_n-1})} \quad \text{and} \quad \left(\frac{t(n - S_{N_n-1})}{t(n)}, \frac{N_n}{t(n)}\right)$$

are asymptotically independent, and

$$t_n^{-1}(M_n - N_n, N_n) \xrightarrow{\mathrm{D}} \left(\int_T^\infty \exp(-U_t)\,\mathrm{d}t, \int_0^T \exp(-U_t)\,\mathrm{d}t\right).$$

4. Proof of Theorem 1.2

Our proof essentially relies upon the following classical result:

$$\lim_{n\to\infty} \mathrm{P}\{n - S_{N_n-1} \le k\} = m^{-1} \sum_{i=1}^{k} \mathrm{P}\{\xi \ge i\} =: \mathrm{P}\{W \le k\}, \qquad k \in \mathbb{N}. \tag{4.1}$$

In order to see why (4.1) holds, note that

$$\begin{aligned}\mathrm{P}\{n - S_{N_n-1} = k\} &= \sum_{i=1}^{n} \mathrm{P}\{S_{i-1} = n-k,\ S_i \ge n\} \\ &= \mathrm{P}\{\xi \ge k\} \sum_{i=0}^{n-k} \mathrm{P}\{S_i = n-k\} \\ &\to m^{-1}\,\mathrm{P}\{\xi \ge k\}, \qquad n \to \infty,\end{aligned}$$

by the elementary renewal theorem, and (4.1) follows.

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From (3.8) we conclude that

$$M_n - N_n \xrightarrow{\mathrm{D}} M'_W - 1, \tag{4.2}$$

where W is a random variable with distribution (4.1) which is independent of $\{M'_n : n \in \mathbb{N}\}$. Therefore, for any sequence $\{d_n : n \in \mathbb{N}\}$ such that $\lim_{n\to\infty} d_n = \infty$,

$$\frac{M_n - N_n}{d_n} \xrightarrow{\mathrm{P}} 0. \tag{4.3}$$

Assume that the distribution of ξ does not belong to the domain of attraction of any stable law with index $\alpha \in [1, 2]$. Then, as is well known, it is not possible to find sequences $x_n > 0$ and $y_n \in \mathbb{R}$ such that $(S_n - y_n)/x_n$ converges to a proper and nondegenerate law. In view of the fact that

$$\mathrm{P}\{N_n > m\} = \mathrm{P}\{S_m \le n - 1\},$$

the same is true for N_n (see [17, Theorem 7] and/or [22, Theorem 2] for more details) and, according to (4.3), for M_n.

Assume that the conditions of Theorem 1.2(ii) hold. If $\sigma^2 = \infty$ and (1.6) holds with $\alpha = 2$ then arguing as in the proof of Theorem 2 of [22] we conclude that, with a_n and b_n as defined in our Theorem 1.2,

$$\frac{N_n - b_n}{a_n} \xrightarrow{\mathrm{W}} \mu_2.$$

Theorem 5 of [17] (if $\sigma^2 < \infty$) and Theorem 7 of [17] (if (1.6) holds for some $\alpha \in [1, 2)$) lead to the same limiting relation (with corresponding a_n and b_n, and with μ_2 replaced by μ_α in the latter case).

In view of (4.3), the same limiting relations hold for M_n. The proof of Theorem 1.2 is complete.

5. Proof of Theorem 1.1

First assume that $m = \infty$. According to (2.2), $\mathrm{E}\, N_n^k \sim n^k/L^k(n)$, $k \in \mathbb{N}$. The same argument as in Section 3 yields

$$\mathrm{E}\, M_n^k \sim \frac{n^k}{L^k(n)} \sim (\mathrm{E}\, M_n)^k, \qquad k \in \mathbb{N}.$$

Therefore,

$$\lim_{n\to\infty} \mathrm{E}\left(\frac{M_n}{\mathrm{E}\, M_n}\right)^k = 1, \qquad k \in \mathbb{N},$$

which proves (1.4). In fact, to arrive at (1.4), it suffices to know that $\mathrm{E}\, M_n \sim n/L(n)$ and $\mathrm{E}\, M_n^2 \sim n^2/L^2(n)$, and to exploit Chebyshev's inequality.

Now assume that $m < \infty$. It is well known that

$$\lim_{n\to\infty} \frac{N_n}{n} = \frac{1}{m} \quad \text{almost surely.} \tag{5.1}$$

In view of (4.2), $\lim_{n\to\infty}(M_n - N_n)/n = 0$ almost surely, which yields $\lim_{n\to\infty} M_n/n = 1/m$ almost surely. By the elementary renewal theorem, $\mathrm{E}\, N_n \sim n/m$. Using the same approach as in Section 3, it is straightforward to check that $\mathrm{E}\, M_n \sim n/m$. Conversely, if $M_n/a_n \xrightarrow{\mathrm{P}} 1$ then (4.3) gives $(M_n - N_n)/a_n \xrightarrow{\mathrm{P}} 0$. Therefore, $N_n/a_n \xrightarrow{\mathrm{P}} 1$. An appeal to (5.1) allows us to conclude that $a_n \sim n/m$. The proof is complete.

6. Proof of Theorem 1.4

By Theorem 3(c) and formulae in [8, p. 42] (see also [12]),

$$\frac{N_n - b(n) - 1}{a(n)} \xrightarrow{\mathrm{w}} \mu_1,$$

where μ_1 is the 1-stable law with characteristic function $\int_{-\infty}^{\infty} \mathrm{e}^{itx}\mu_1(\mathrm{d}x) = \mathrm{e}^{it\log|t| - |t|\pi/2}$, $t \in \mathbb{R}$. By Corollary 2.1,

$$\frac{M_n}{N_n - 1} \xrightarrow{\mathrm{P}} 1. \tag{6.1}$$

Therefore,

$$\frac{M_n - b(n)}{a(n)} - \frac{M_n - N_n + 1}{N_n - 1}\frac{b(n)}{a(n)} \xrightarrow{\mathrm{w}} \mu_1.$$

Thus, to prove the theorem it suffices to show that the second summand tends to 0 in probability. Clearly, this can be regarded as a rate of convergence result for (6.1). Recalling the notation $Y_n = n - S_{N_n - 1}$ and using (3.8) gives

$$\begin{aligned}\frac{M_n - N_n + 1}{N_n - 1}\frac{b(n)}{a(n)} &= \frac{\hat{M}_{Y_n}}{Y_n/m(Y_n)}\frac{m(n)}{m(Y_n)}\frac{b(n)Y_n}{na(n)}\frac{n}{m(n)(N_n - 1)}\\ &=: \prod_{i=1}^{4} K_i(n).\end{aligned}$$

By Corollary 2.1, $m(n)M_n/n \xrightarrow{\mathrm{P}} 1$. Using the equality of distributions (3.8) and the fact that $Y_n \xrightarrow{\mathrm{P}} \infty$, allows us to conclude that $K_1(n) \xrightarrow{\mathrm{P}} 1$. By Theorem 6 of [16], $K_2(n) \xrightarrow{\mathrm{D}} 1/R$, where R is a random variable uniformly distributed on [0, 1]. By Proposition 2.3, $K_3(n) \xrightarrow{\mathrm{P}} 0$. Finally, by Corollary 2.1, $K_4(n) \xrightarrow{\mathrm{P}} 1$. The proof is complete.

7. Number of collisions in beta coalescents

In this section the main results presented in Section 1 are applied to the number of collisions that take place in beta-coalescent processes until there is just a single block. Other closely related functionals of coalescent processes such as the total branch length or the number of segregating sites have been studied in [6], [13], [15], and [27] (see also [5]).

Let $\mathcal{E}$ denote the set of all equivalence relations on $\mathbb{N}$. For $n \in \mathbb{N}$, let $\varrho_n : \mathcal{E} \to \mathcal{E}_n$ denote the natural restriction to the set $\mathcal{E}_n$ of all equivalence relations on $\{1, \dots, n\}$. For $\eta \in \mathcal{E}_n$, let $|\eta|$ denote the number of blocks (equivalence classes) of η.

Pitman [32] and Sagitov [37] independently introduced coalescent processes with multiple collisions. These Markovian processes with state space $\mathcal{E}$ are characterized by a finite measure Λ on [0, 1] and are, hence, also called Λ-coalescent processes. For a Λ-coalescent $\{\Pi_t : t \ge 0\}$, it is known that the process $\{|\varrho_n\Pi_t| : t \ge 0\}$ has infinitesimal rates

$$g_{nk} := \lim_{t\downarrow 0}\frac{\mathrm{P}\{|\varrho_n\Pi_t| = k\}}{t} = \binom{n}{k-1}\int_{[0,1]} x^{n-k-1}(1-x)^{k-1}\,\Lambda(\mathrm{d}x) \tag{7.1}$$

for all $k, n \in \mathbb{N}$ with $k < n$. Let $g_n := \sum_{k=1}^{n-1} g_{nk}$, $n \in \mathbb{N}$, denote the total rates. We are interested in the number of collisions (jumps) X_n that take place in the restricted coalescent process $\{\varrho_n\Pi_t : t \ge 0\}$ until there is just a single block. From the structure of the coalescent process, it follows that $(X_n)_{n\in\mathbb{N}}$ satisfies the distributional recursion $X_1 = 0$ and

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$X_n \stackrel{\mathrm{D}}{=} 1+X_{n-I_n}$, $n \in \{2,3,\ldots\}$, where I_n is independent of $X_2,\ldots,X_{n-1}$ with distribution $\mathrm{P}\{I_n = k\} = g_{n,n-k}/g_n$, $k \in \{1,\ldots,n-1\}$. The random variable $n - I_n$ is the (random) state of the process $\{|\varrho_n \Pi_t| : t \geq 0\}$ after its first jump.

We consider beta coalescents, where, by definition, $\Lambda = \beta(a,b)$ is the beta distribution with density $x \mapsto (\mathrm{B}(a,b))^{-1}x^{a-1}(1-x)^{b-1}$ with respect to the Lebesgue measure on $(0,1)$, and $\mathrm{B}(a,b) := \Gamma(a)\Gamma(b)/\Gamma(a+b)$ denotes the beta function, where $a, b > 0$. In this case the rates (7.1) have the form

$$g_{nk} = \binom{n}{k-1}\frac{1}{\mathrm{B}(a,b)}\int_0^1 x^{a+n-k-2}(1-x)^{b+k-2}\,\mathrm{d}x$$
$$= \binom{n}{k-1}\frac{\mathrm{B}(a+n-k-1,b+k-1)}{\mathrm{B}(a,b)}, \qquad k,n \in \mathbb{N},\ k<n. \tag{7.2}$$

From

$$g_{k+1,k} = \frac{k(k+1)}{2}\frac{\mathrm{B}(a,b+k-1)}{\mathrm{B}(a,b)},$$

it follows that

$$g_n = \sum_{k=1}^{n-1}(g_{k+1}-g_k) = \sum_{k=1}^{n-1}\frac{2}{k+1}g_{k+1,k} = \frac{1}{\mathrm{B}(a,b)}\sum_{k=1}^{n-1}k\mathrm{B}(a,b+k-1).$$

In the following it is assumed that $b = 1$ such that the rates (7.2) reduce to

$$g_{nk} = \binom{n}{k-1}\frac{\mathrm{B}(a+n-k-1,k)}{\mathrm{B}(a,1)} = \frac{n!}{(n-k+1)!}a\frac{\Gamma(a+n-k-1)}{\Gamma(a+n-1)}$$

and the total rates reduce to

$$g_n = a\sum_{k=1}^{n-1}kB(a,k) = \begin{cases} \dfrac{a}{a-2}\left(1-\dfrac{\Gamma(a)\Gamma(n+1)}{\Gamma(a+n-1)}\right) & \text{for } a>0,\ a\neq 2,\\[2ex] 2(h_n-1) & \text{for } a=2.\end{cases}$$

Here, $h_n := \sum_{i=1}^n 1/i$ denotes the nth harmonic number. From the last formula, it follows that the parameter $a = 2$ plays a special role in this model. Define

$$p_k := \frac{(2-a)\Gamma(a+k-1)}{\Gamma(a)\Gamma(k+2)}, \qquad k \in \mathbb{N}.$$

Now assume that $0 < a < 2$. In this case (and only in this case) we have $p_k \geq 0$ for $k \in \mathbb{N}$, $\sum_{k=1}^\infty p_k = 1$, and (1.2) holds. Let ξ be a random variable with distribution $\mathrm{P}\{\xi = k\} = p_k$, $k \in \mathbb{N}$. It follows, by induction on n, that

$$\mathrm{P}\{\xi \geq n\} = \frac{\Gamma(a+n-1)}{\Gamma(a)\Gamma(n+1)}, \qquad n \in \mathbb{N}.$$

Using $\Gamma(n+x) \sim \Gamma(n)n^x$ for $n \to \infty$, we conclude that

$$\mathrm{P}\{\xi \geq n\} \sim \frac{n^{a-2}}{\Gamma(a)} = \frac{n^{-\alpha}}{\Gamma(2-\alpha)}, \qquad n \to \infty.$$

Thus, if $1 < a < 2$ or, equivalently, if $0 < \alpha < 1$, Theorem 1.3 is applicable (with $L(n) \equiv 1/\Gamma(a) = 1/\Gamma(2-\alpha)$), and we obtain the following result.

Theorem 7.1. *For the $\beta(a,1)$-coalescent with $1 < a < 2$, i.e. $0 < \alpha := 2-a < 1$, the number of collision events X_n satisfies*

$$\frac{X_n}{\Gamma(2-\alpha)n^\alpha} \xrightarrow{\text{D}} \int_0^\infty \exp(-U_t)\,\mathrm{d}t,$$

where $\{U_t : t \geq 0\}$ is a drift-free subordinator with Lévy measure (1.9).

Note that, for $\Lambda = \beta(a,b)$, we have $\mu_{-1} := \int x^{-1}\,\Lambda(\mathrm{d}x) < \infty$ if and only if $a > 1$. Under the condition $\mu_{-1} < \infty$, limiting results similar to those presented in Theorem 7.1 are known for the number of segregating sites (see, for example, [27, Proposition 5.1]) for general Λ-coalescent processes with mutation.

Now assume that $0 < a < 1$. Then, $m := \mathrm{E}\,\xi = 1/(1-a) < \infty$. It is straightforward to verify that

$$\sum_{k=1}^n k^2 p_k \sim \frac{2-a}{\Gamma(a+1)} n^a, \qquad n \to \infty.$$

In particular, the variance of ξ is infinite. Thus, Theorem 1.2 is applicable (with $L(n) \equiv (2-a)/\Gamma(a+1) = \alpha/\Gamma(3-\alpha)$, $C := 1/\Gamma(a) = 1/\Gamma(2-\alpha)$, $b_n := n(1-a) = n(\alpha-1)$, and $c_n := n^{1/\alpha}$) and yields the following result.

Theorem 7.2. *For the $\beta(a,1)$-coalescent with $0 < a < 1$, i.e. $1 < \alpha := 2-a < 2$, the number of collision events X_n satisfies*

$$\frac{X_n - n(\alpha-1)}{(\alpha-1)^{(\alpha+1)/\alpha} n^{1/\alpha}} \xrightarrow{\text{W}} \mu_\alpha,$$

or, equivalently,

$$\frac{X_n - n(\alpha-1)}{(\alpha-1)n^{1/\alpha}} \xrightarrow{\text{D}} S_\alpha, \tag{7.3}$$

where $\mathrm{E}\exp(\mathrm{i}tS_\alpha) = \exp(|t|^\alpha(\cos(\pi\alpha/2) + \mathrm{i}\sin(\pi\alpha/2)\mathrm{sgn}(t)))$, $t \in \mathbb{R}$.

Gnedin and Yakubovich [19, Theorem 9] used analytic methods to verify the same convergence result (7.3) for Λ-coalescents satisfying $\Lambda([0,x]) = Ax^a + O(x^{a+\zeta})$, $x \to 0$, $0 < a < 1$, and $\zeta > \max\{(2-a)^2/(5-5a+a^2), 1-a\}$.

Theorems 7.1 and 7.2 do not cover the asymptotics of X_n for the Bolthausen–Sznitman coalescent, i.e. the $\beta(a,b)$-coalescent with $a = b = 1$. The limiting behavior of X_n for the Bolthausen–Sznitman coalescent was studied in [24], and also follows from our Theorem 1.4 with $p_k := 1/(k(k+1))$, $L(n) \equiv 1$, $c(x) := x$, $b(x) := x/\log x + x\log\log x/(\log x)^2$, and $a(x) := b^2(x)/x \sim x/(\log x)^2$. Therefore, the asymptotics of X_n for all $\beta(a,1)$-coalescent processes with $0 < a < 2$ is clarified. Unfortunately, our method cannot be used to treat the asymptotics of X_n for $\beta(a,1)$-coalescent processes with $a \geq 2$, as in this case condition (1.2) is not satisfied. Recently, the limiting behavior of X_n for $\beta(2,b)$-coalescents with parameter $b > 0$ was obtained in [26] using a completely different approach based on the asymptotics of moments.

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