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A NOTE ON THE RANDOM WALK MODEL ARISING IN DOUBLE DIFFUSION

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Abstract

The discrete random walk problem for the unrestricted particle formulated in the double diffusion model given in Hill [2] is solved explicitly. In this model it is assumed that a particle moves along two distinct horizontal paths, say the upper path 1 and lower path 2. For $i = 1, 2$, when the particle is in path i, it can move at each jump in one of four possible ways, one step to the right with probability p_i, one step to the left with probability q_i, remains in the same position with probability r_i or exchanges paths but remains in the same horizontal position with probability s_i ($p_i + q_i + r_i + s_i = 1$). Using generating functions, the probability distribution of the position of an unrestricted particle is derived. Finally some special cases are discussed to illustrate the general result.

1. Introduction

The classical random walk model is generalized by Hill [2], so that a particle moves along one of two distinct paths 1 and 2 as follows. For $i = 1, 2$, the particle is in path i and at each jump it moves one step to the right with probability p_i, one to the left with probability q_i, remains in the same position with probability r_i or exchanges paths but remains in the same position with probability s_i ($p_i + q_i + r_i + s_i = 1$). These probabilities are assumed to be independent of the position of the particle. The purpose of this article is to deduce explicit expressions for the probability distribution of the position of an unrestricted particle. This problem is formulated in Hill [2], but not solved, although means and variances are given. As noted in the review paper Hill [3] the corresponding problems for the two models continuous in time but with discrete space, and continuous in both space and

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time, can however be solved explicitly. Here we give the corresponding expressions for the fully discrete formulation of the model.

Let $u^i_{k,n}$, $v^i_{k,n}$ ($i = 1, 2$) denote the probabilities that the particle is at position k in paths 1 and 2 respectively at the nth step given that its initial position is at the origin in path i. For the unrestricted particle we have the following forward equations

$$\begin{aligned} u^i_{k,n+1} &= p_1 u^i_{k-1,n} + r_1 u^i_{k,n} + q_1 u^i_{k+1,n} + s_2 v^i_{k,n}, \\ v^i_{k,n+1} &= p_2 v^i_{k-1,n} + r_2 v^i_{k,n} + q_2 v^i_{k+1,n} + s_1 u^i_{k,n}, \end{aligned} \tag{1.1}$$

for $i = 1, 2$ with the initial conditions

$$u^1_{k,0} = \delta_{k,0}, \qquad v^1_{k,0} = 0, \qquad u^2_{k,0} = 0, \qquad v^2_{k,0} = \delta_{k,0}, \tag{1.2}$$

for all integers k, for integers $n \geqslant 0$ and where δ_{ij} is the usual Kronecker delta. This formulation differs slightly from the approach of Hill [2]. We solve (1.1) subject to the initial conditions (1.2) in Sections 2 and 3 using generating functions. In the final section, the formulae obtained are illustrated with special cases.

In this section, for comparison we note the solution of the forward equation for the standard random walk model with three possibilities. Given

$$u_{k,n+1} = p u_{k-1,n} + r u_{k,n} + q u_{k+1,n}, \tag{1.3}$$

for all integers k and for integers $n \geqslant 0$, the generating function

$$U_n(z) = \sum_{k=-\infty}^{\infty} u_{k,n} z^k, \tag{1.4}$$

satisfies

$$U_{n+1}(z) = (pz + r + q/z) U_n(z), \tag{1.5}$$

which gives

$$U_n(z) = \pi(z)^n, \tag{1.6}$$

where $\pi(z) = pz + r + q/z$. Expanding $\pi(z)^n$ as a power series of z, and taking the coefficient of z^k, we obtain the probability distribution $u_{k,n}$ given by

$$u_{k,n} = \sum_{m=m_0}^{(n-k)/2} \frac{n! p^{m+k} q^m r^{n-2m-k}}{(m+k)! m! (n-2m-k)!}, \tag{1.7}$$

where $m_0 = \max(0, -k)$ (see Cox and Miller [1], page 26).

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2. Formulae for generating functions

Let $U_n^i(z)$ and $V_n^i(z)$ denote the k-generating functions of the probability distributions $u_{k,n}^i$ and $v_{k,n}^i$ respectively. For $i = 1$ we have from (1.1) and (1.2)

$$\begin{aligned} U_{n+1}^1(z) - U_n^1(z) &= (\omega_1(z) - s_1)U_n^1(z) + s_2 V_n^1(z), \\ V_{n+1}^1(z) - V_n^1(z) &= (\omega_2(z) - s_2)V_n^1(z) + s_1 U_n^1(z), \end{aligned} \tag{2.1}$$

where the functions $\omega_i(z)$ are defined by

$$\omega_i(z) = p_i z - (p_i + q_i) + q_i/z \qquad (i = 1, 2) \tag{2.2}$$

and the initial conditions are

$$U_0^1(z) = 1, \qquad V_0^1(z) = 0. \tag{2.3}$$

Define the double generating functions

$$U^1(z,\xi) = \sum_{n=0}^{\infty} U_n^1(z)\xi^n, \qquad V^1(z,\xi) = \sum_{n=0}^{\infty} V_n^1(z)\xi^n. \tag{2.4}$$

Then from (2.1) and (2.3) we obtain

$$\begin{aligned} \xi U^1(z,\xi) &= \frac{[(1/\xi - 1) - \omega_2(z) + s_2]}{\{[(1/\xi - 1) - \omega_1(z) + s_1][(1/\xi - 1) - \omega_2(z) + s_2] - s_1 s_2\}}, \\ \xi V^1(z,\xi) &= \frac{s_1}{\{[(1/\xi - 1) - \omega_1(z) + s_1][(1/\xi - 1) - \omega_2(z) + s_2] - s_1 s_2\}}. \end{aligned} \tag{2.5}$$

Let $\pi_+(z) = [\pi_1(z) + \pi_2(z)]/2$ and $\pi_-(z) = [\pi_1(z) - \pi_2(z)]/2$ where $\pi_i(z)$ is given by

$$\begin{aligned} \pi_i(z) &= p_i z + r_i + q_i/z \\ &= 1 + \omega_i(z) - s_i \qquad (i = 1, 2). \end{aligned} \tag{2.6}$$

Now

$$\pi_\pm(z) = P_\pm z + R_\pm + Q_\pm/z, \tag{2.7}$$

where

$$P_\pm = (p_1 \pm p_2)/2, \qquad Q_\pm = (q_1 \pm q_2)/2, \qquad R_\pm = (r_1 \pm r_2)/2, \tag{2.8}$$

so that from (2.5) we have

$$\begin{aligned} \xi U^1(z,\xi) &= \frac{[1/\xi - \pi_+(z)] + \pi_-(z)}{\{[1/\xi - \pi_+(z)]^2 - [\pi_-(z)^2 + s_1 s_2]\}}, \\ \xi V^1(z,\xi) &= \frac{s_1}{\{[1/\xi - \pi_+(z)]^2 - [\pi_-(z)^2 + s_1 s_2]\}}. \end{aligned} \tag{2.9}$$

On expanding (2.9) as a power series in ξ we have

$$U_n^1(z) = \frac{1}{2}\left\{(\theta_+^n + \theta_-^n) + \frac{\pi_-(z)}{\left[\pi_-(z)^2 + s_1 s_2\right]^{1/2}}(\theta_+^n - \theta_-^n)\right\},$$
$$V_n^1(z) = \frac{1}{2}\left\{\frac{s_1}{\left[\pi_-(z)^2 + s_1 s_2\right]^{1/2}}(\theta_+^n - \theta_-^n)\right\}, \tag{2.10}$$

where

$$\theta_+ = \pi_+(z) + \left[\pi_-(z)^2 + s_1 s_2\right]^{1/2}, \qquad \theta_- = \pi_+(z) - \left[\pi_-(z)^2 + s_1 s_2\right]^{1/2}. \tag{2.11}$$

The appearance of the square root term in (2.10) makes it difficult to proceed any further in obtaining the probabilities $u_{k,n}^1$ and $v_{k,n}^1$. In order to overcome this situation we make use of the identities

$$\theta_+^n \pm \theta_-^n = [(\theta_+ + \theta_-)/2]^n[(1+x)^n \pm (1-x)^n], \tag{2.12}$$

where

$$x = (\theta_+ - \theta_-)/(\theta_+ + \theta_-) = \left[\pi_-(z)^2 + s_1 s_2\right]^{1/2}/\pi_+(z). \tag{2.13}$$

From the above expression we obtain

$$\frac{(\theta_+^n - \theta_-^n)}{\left[\pi_-(z)^2 + s_1 s_2\right]^{1/2}} = 2\pi_+(z)^n \sum_{l=0}^{L_0} \binom{n}{2l+1} \frac{\left[\pi_-(z)^2 + s_1 s_2\right]^l}{\pi_+(z)^{2l+1}}, \tag{2.14}$$

and

$$(\theta_+^n + \theta_-^n) = 2\pi_+(z)^n \sum_{l=0}^{L_1} \binom{n}{2l} \frac{\left[\pi_-(z)^2 + s_1 s_2\right]^l}{\pi_+(z)^{2l}}, \tag{2.15}$$

where

$$L_0 = n/2 - 1, \qquad L_1 = n/2 \quad \text{for } n \text{ even},$$
$$L_0 = L_1 = (n-1)/2 \quad \text{for } n \text{ odd}. \tag{2.16}$$

These results permit writing $U_n^1(z)$, $V_n^1(z)$ as double series in $\pi_+(z)$ and $\pi_-(z)$. These in turn are expressible as series in z of the form

$$\pi_+(z)^{n-2l} = \sum_{f=-(n-2l)}^{n-2l} \sum_{i=I_0}^{(n-2l-f)/2} \frac{(n-2l)!P_+^{f+i}Q_+^i R_+^{n-2l-2i-f} z^f}{(f+i)!i!(n-2l-2i-f)!},$$
$$\pi_-(z)^{2m} = \sum_{g=-2m}^{2m} \sum_{j=J_0}^{(2m-g)/2} \frac{2m!P_-^{g+j}Q_-^j R_-^{2m-2j-g} z^g}{(g+j)!j!(2m-2j-g)!}, \tag{2.17}$$

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where $P_\pm$, $Q_\pm$ and $R_\pm$ are given by (2.8) and

$$I_0 = \max(0, -f), \quad J_0 = \max(0, -g). \tag{2.18}$$

Thus

$$\pi_+(z)^{n-2l}\pi_-(z)^{2m} = \sum_{k=-(n-2l+2m)}^{n-2l+2m} \gamma_{k,n-2l,2m} z^k, \tag{2.19}$$

where the coefficients $\gamma_{a,b,c}$ are defined as the convolution

$$\gamma_{a,b,c} = \sum_{H=H_0}^{H_1} \alpha^+_{a-H,b}\alpha^-_{H,c}, \tag{2.20}$$

where $H_0 = \max(a-b, -c)$, $H_1 = \min(a+b, c)$, and

$$\alpha^\pm_{a,b} = \sum_{k=K_0}^{(b-a)/2} \frac{b!P_\pm^{k+a}Q_\pm^k R_\pm^{b-2k-a}}{(k+a)!k!(b-2k-a)!}, \tag{2.21}$$

and $K_0 = \max(0, -a)$. Substituting these results in the series obtained for $U_n^1(z)$ and $V_n^1(z)$ and using the double sum

$$\sum_{j=0}^{N}\sum_{i=0}^{j} a_{j,i} = \sum_{j=0}^{N}\sum_{i=j}^{N} a_{i,i-j}, \tag{2.22}$$

we obtain

$$\begin{aligned} U_n^1(z) = {} & \sum_{l=0}^{L_1}\sum_{m=l}^{L_1}\binom{n}{2m}\binom{m}{l}(s_1s_2)^l \sum_{k=-(n-2l)}^{n-2l} \gamma_{k,n-2m,2(m-l)}z^k \\ & + \sum_{l=0}^{L_0}\sum_{m=l}^{L_0}\binom{n}{2m+1}\binom{m}{l}(s_1s_2)^l \sum_{k=-(n-2l)}^{n-2l} \gamma_{k,n-2m-1,2(m-l)+1}z^k, \end{aligned} \tag{2.23}$$

$$V_n^1(z) = s_1\sum_{l=0}^{L_0}\sum_{m=l}^{L_0}\binom{n}{2m+1}\binom{m}{l}(s_1s_2)^l \sum_{k=-(n-2l-1)}^{n-2l-1} \gamma_{k,n-2m-1,2(m-l)}z^k.$$

Similar expressions can be obtained for $U_n^2(z)$ and $V_n^2(z)$ from $V_n^1(z)$ and $U_n^1(z)$ respectively by simply interchanging p_1 and p_2, q_1 and q_2, r_1 and r_2 and s_1 and s_2.

3. The probability distribution of the position of the particle

In this section, the probability distribution of the position of the particle is given. The probabilities $u^i_{k,n}$ and $v^i_{k,n}$ are obtained from the coefficients of z^k in

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the generating functions $U_n^i(z)$ and $V_n^i(z)$ respectively. The final results for $i = 1$ are

$$u_{k,n}^1 = \sum_{l=0}^{n/2-|k/2|} \sum_{m=l}^{[n/2]} \binom{n}{2m}\binom{m}{l}(s_1s_2)^l \gamma_{k,n-2m,2(m-l)} + \sum_{l=0}^{h} \sum_{m=l}^{[n/2-1/2]} \binom{n}{2m+1}\binom{m}{l}(s_1s_2)^l \gamma_{k,n-2m-1,2(m-l)+1}, \tag{3.1}$$

for $-n \leqslant k \leqslant n$ and where $[y]$ denotes the largest integer less than or equal to y and $h = n/2 - |k/2| - \delta_{k,0}/2$. Further

$$v_{k,n}^1 = s_1 \sum_{l=0}^{h} \sum_{m=l}^{[n/2-1/2]} \binom{n}{2m+1}\binom{m}{l}(s_1s_2)^l \gamma_{k,n-2m-1,2(m-l)}, \tag{3.2}$$

for $-(n-1) \leqslant k \leqslant (n-1)$. Similarly for $i = 2$ we have

$$u_{k,n}^2 = s_2 \sum_{l=0}^{h} \sum_{m=l}^{[n/2-1/2]} \binom{n}{2m+1}\binom{m}{l}(s_1s_2)^l \gamma_{k,n-2m-1,2(m-l)}, \tag{3.3}$$

for $-(n-1) \leqslant k \leqslant (n-1)$. Further

$$v_{k,n}^2 = \sum_{l=0}^{n/2-|k/2|} \sum_{m=l}^{[n/2]} \binom{n}{2m}\binom{m}{l}(s_1s_2)^l \gamma_{k,n-2m,2(m-l)} - \sum_{l=0}^{h} \sum_{m=l}^{[n/2-1/2]} \binom{n}{2m+1}\binom{m}{l}(s_1s_2)^l \gamma_{k,n-2m-1,2(m-l)+1}, \tag{3.4}$$

for $-n \leqslant k \leqslant n$. Thus the probability that the particle is at position k, of path 1 at the nth step, $u_{k,n}$, is given by

$$u_{k,n} = u_0 u_{k,n}^1 + v_0 u_{k,n}^2, \tag{3.5}$$

and the probability that the particle is at position k of path 2 at the nth step, $v_{k,n}$, is given by

$$v_{k,n} = u_0 v_{k,n}^1 + v_0 v_{k,n}^2, \tag{3.6}$$

where u_0 and v_0 are the probabilities the particle is initially at the origin in paths 1 and 2 respectively.

4. Special cases and examples

In this section we illustrate the results obtained with some special cases and examples. Firstly we check the case where $p_1 = p_2 = p$, $q_1 = q_2 = q$, $r_1 = r_2 = r$

and $s_1 = s_2 = s$. This should give a simple random walk model with the probability of the particle being in the same position to be $r + s$. Thus equation (2.8) becomes

$$P_+ = p, \qquad Q_+ = q, \qquad R_+ = r, \qquad P_- = Q_- = R_- = 0, \tag{4.1}$$

which gives together with (2.19), (2.20) and (2.21) that

$$\alpha^-_{a,b} = \delta_{a,0}\delta_{b,0}, \qquad \gamma_{a,b,c} = \alpha^+_{a,b}\delta_{c,0}. \tag{4.2}$$

Hence equations (3.1) and (3.2) reduce to, for $-n \leqslant k \leqslant n$,

$$u^1_{k,n} = \sum_{l=0}^{(n-|k|)/2} \binom{n}{2l} s^{2l}\alpha^+_{k,n-2l}, \tag{4.3}$$

while for $-(n-1) \leqslant k \leqslant (n-1)$

$$v^1_{k,n} = \sum_{l=0}^{h} \binom{n}{2l+1} s^{2l+1}\alpha^+_{k,n-(2l+1)}. \tag{4.4}$$

Thus the probability that the particle is in position k at the nth step, starting from the origin is given by

$$\begin{aligned} u^1_{k,n} + v^1_{k,n} &= u^2_{k,n} + v^2_{k,n}, \\ &= \sum_{i=0}^{n-|k|} \binom{n}{i} s^i \alpha^+_{k,n-i}, \\ &= \sum_{m=m_0}^{(n-k)/2} \frac{n!p^{m+k}q^m(r+s)^{n-k-2m}}{(m+k)!m!(n-k-2m)!}, \end{aligned} \tag{4.5}$$

where $m_0 = \max(0, -k)$. Notice that this result does agree with that of the classical random walk model (see (1.7)).

Secondly, consider the case $s_1 = 0$, in which downward transitions are forbidden. Then the equations (3.1) and (3.2) give

$$u^1_{k,n} = \sum_{m=0}^{[n/2]} \binom{n}{2m} \gamma_{k,n-2m,2m} + \sum_{m=0}^{[n/2-1/2]} \binom{n}{2m+1} \gamma_{k,n-(2m+1),2m+1}, \tag{4.6}$$

and $v^1_{k,n} = 0$ for all k such that $-n \leqslant k \leqslant n$. Further (4.6) reduces to

$$u^1_{k,n} = \sum_{i=0}^{n} \binom{n}{i} \gamma_{k,n-i,i}, \tag{4.7}$$

which can be shown to take the form of a simple random walk model with parameters p_1, q_1 and r_1. Now consider the probability the particle is in position n

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at the nth step. From (3.1) and (3.2) we have

$$u^1_{n,n} = \sum_{m=0}^{[n/2]} \binom{n}{2m} \gamma_{n,n-2m,2m} + \sum_{m=0}^{[n/2-1/2]} \binom{n}{2m+1} \gamma_{n,n-(2m+1),2m+1}$$
$$= p_1^n,$$
$$v^1_{n,n} = 0. \tag{4.8}$$

Hence,

$$u_{n,n} = u_0 p_1^n, \qquad v_{n,n} = v_0 p_2^n. \tag{4.9}$$

Similarly we can show that

$$u_{-n,n} = u_0 q_1^n, \qquad v_{-n,n} = v_0 q_2^n. \tag{4.10}$$

Finally we give explicitly the probability distributions for $n = 2$ and $n = 3$ from (3.1) and (3.2). Using (4.8) we can easily obtain the probabilities $u_{\pm 2,2}$, $v_{\pm 2,2}$, $u_{\pm 3,3}$, $v_{\pm 3,3}$. In order to obtain the remaining probabilities, the necessary values for the γ and $\alpha^{\pm}$ functions are given by Tables 1 and 2. For the case $n = 2$, which represents an even number of steps, we have after simplification

$$u^1_{0,2} = r_1^2 + s_1 s_2 + 2 p_1 q_1,$$
$$v^1_{0,2} = s_1 r_1 + s_1 r_2,$$
$$u^1_{1,2} = 2 p_1 r_1, \quad u^1_{-1,2} = 2 q_1 r_1,$$
$$v^1_{1,2} = s_1 p_1 + s_1 p_2, \quad v^1_{-1,2} = s_1 q_1 + s_1 q_2. \tag{4.11}$$

TABLE 1

$\gamma_{a,b,c}$ in terms of $\alpha^{\pm}$ functions for special values of a, b and c.

$\gamma_{i,j,0}$	$\alpha^+_{i,j}$ $\quad (i, j = 0, 1, 2, \ldots)$
$\gamma_{i,0,j}$	$\alpha^-_{i,j}$ $\quad (i, j = 0, 1, 2, \ldots)$
$\gamma_{0,1,1}$	$\alpha^+_{1,1}\alpha^-_{-1,1} + \alpha^+_{0,1}\alpha^-_{0,1} + \alpha^+_{-1,1}\alpha^-_{1,1}$
$\gamma_{0,1,2}$	$\alpha^+_{1,1}\alpha^-_{-1,2} + \alpha^+_{0,1}\alpha^-_{0,2} + \alpha^+_{-1,1}\alpha^-_{1,2}$
$\gamma_{0,2,1}$	$\alpha^+_{1,2}\alpha^-_{-1,1} + \alpha^+_{0,2}\alpha^-_{0,1} + \alpha^+_{-1,2}\alpha^-_{1,1}$
$\gamma_{1,1,1}$	$\alpha^+_{1,1}\alpha^-_{0,1} + \alpha^+_{0,1}\alpha^-_{1,1}$
$\gamma_{1,1,2}$	$\alpha^+_{1,1}\alpha^-_{0,2} + \alpha^+_{0,1}\alpha^-_{1,2} + \alpha^+_{-1,1}\alpha^-_{2,2}$
$\gamma_{1,2,1}$	$\alpha^+_{2,2}\alpha^-_{-1,1} + \alpha^+_{1,2}\alpha^-_{0,1} + \alpha^+_{0,2}\alpha^-_{1,1}$
$\gamma_{2,1,2}$	$\alpha^+_{1,1}\alpha^-_{1,2} + \alpha^+_{0,1}\alpha^-_{2,2}$
$\gamma_{2,2,1}$	$\alpha^+_{1,2}\alpha^-_{1,1} + \alpha^+_{2,2}\alpha^-_{0,1}$

TABLE 2

$\alpha^{\pm}$ functions appearing in Table 1, in terms of $P_{\pm}$, $Q_{\pm}$ and $R_{\pm}\left(\alpha_{0,0}^{\pm}=1\right)$.

$\alpha_{0,1}^{\pm}$	$R_{\pm}$	$\alpha_{-1,2}^{\pm}$	$2Q_{\pm}R_{\pm}$
$\alpha_{1,1}^{\pm}$	$P_{\pm}$	$\alpha_{2,2}^{\pm}$	$P_{\pm}^{2}$
$\alpha_{-1,1}^{\pm}$	$Q_{\pm}$	$\alpha_{0,3}^{\pm}$	$R_{\pm}^{3}+6P_{\pm}Q_{\pm}R_{\pm}$
$\alpha_{0,2}^{\pm}$	$R_{\pm}^{2}+2P_{\pm}Q_{\pm}$	$\alpha_{1,3}^{\pm}$	$3P_{\pm}R_{\pm}^{2}+3P_{\pm}^{2}Q_{\pm}$
$\alpha_{1,2}^{\pm}$	$2P_{\pm}R_{\pm}$	$\alpha_{2,3}^{\pm}$	$3P_{\pm}^{2}R_{\pm}$

For $n = 3$, which represents an odd number of steps, simplification yields

$$\begin{aligned}
u_{0,3}^{1} &= r_1^3 + 6p_1q_1r_1 + 2s_1s_2r_1 + s_1s_2r_2,\\
v_{0,3}^{1} &= s_1\left(r_1^2 + r_2^2 + r_1r_2 + 2p_1q_1 + 2p_2q_2 + p_2q_1 + p_1q_2\right),\\
u_{1,3}^{1} &= 3p_1r_1^2 + 3q_1p_1^2 + 2s_1s_2p_1 + s_1s_2p_2,\\
v_{1,3}^{1} &= s_1(2p_1r_1 + 2p_2r_2 + p_2r_1 + p_1r_2),\\
u_{-1,3}^{1} &= 3q_1r_1^2 + 3p_1q_1^2 + 2s_1s_2q_1 + s_1s_2q_2,\\
v_{-1,3}^{1} &= s_1(2q_1r_1 + 2q_2r_2 + q_2r_1 + q_1r_2),\\
u_{2,3}^{1} &= 3r_1p_1^2, \quad v_{2,3}^{1} = s_1\left(p_1^2 + p_1p_2 + p_2^2\right),\\
u_{-2,3}^{1} &= 3r_1q_1^2, \quad v_{-2,3}^{1} = s_1\left(q_1^2 + q_1q_2 + q_2^2\right).
\end{aligned} \tag{4.12}$$

Such expressions might be expected, and provide a check on the general result.

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