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BOUNDEDNESS OF GENERALIZED RIESZ POTENTIALS ON THE VARIABLE HARDY SPACES

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Abstract

We study the boundedness from $H^{p(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$ of certain generalized Riesz potentials and the boundedness from $H^{p(\cdot)}(\mathbb{R}^n)$ into $H^{q(\cdot)}(\mathbb{R}^n)$ of the Riesz potential, both results are achieved via the finite atomic decomposition developed in Cruz-Uribe and Wang ['Variable Hardy spaces', *Indiana University Mathematics Journal* **63**(2) (2014), 447–493].

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1. Introduction

The Lebesgue spaces with variable exponents are a generalization of the classical Lebesgue spaces, replacing the constant exponent p with a variable exponent function $p(\cdot)$. For more than 25 years, the variable Lebesgue spaces have received considerable attention for their applications in fluid dynamics, elasticity theory and differential equations with nonstandard growth conditions, see [6] and references therein.

In the celebrated paper [7], Fefferman and Stein defined the Hardy spaces $H^p(\mathbb{R}^n)$ for $0 < p < \infty$. One of the principal interests of H^p theory is that the L^p boundedness of certain integral operators proved for $p > 1$ extend to the context of H^p, for all $p > 0$. In many cases, this is achieved by means of the atomic decomposition of elements in H^p (see [8, 11, 20]).

The theory of variable exponent Hardy spaces was developed independently by Nakai and Sawano in [15] and by Cruz-Uribe and Wang in [4]. Both theories prove equivalent definitions in terms of maximal operators using different approaches. In [4, 15], one of their main goals is the atomic decomposition of elements in $H^{p(\cdot)}(\mathbb{R}^n)$, as an application of the atomic decomposition they proved that singular integrals are bounded on $H^{p(\cdot)}$.

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Let $0 \leq \alpha < n$ and $m > 1$, $(m \in \mathbb{N})$, we consider the following generalization of Riesz potential

$$T_{\alpha,m} f(x) = \int_{\mathbb{R}^n} |x - A_1 y|^{-\alpha_1} \dots |x - A_m y|^{-\alpha_m} f(y)\, dy, \tag{1.1}$$

where $\alpha_1 + \cdots + \alpha_m = n - \alpha$, and $A_1, \dots, A_m$ are $n \times n$ orthogonal matrices such that $A_i - A_j$ are invertible if $i \neq j$. We observe that in the case $\alpha > 0$, $m = 1$ and $A_1 = I$, $T_{\alpha,1}$ is the classical fractional integral operator (also known as the Riesz potential) I_α. A interesting survey about fractional integrals can be found in [14].

With respect to classical Lebesgue or Hardy spaces, for the case $0 \leq \alpha < n$ and $m > 1$, in the paper [17], the author jointly with Urciuolo proved the $H^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)$ boundedness of the operator $T_{\alpha,m}$ and we also showed that the $H^p(\mathbb{R}) - H^q(\mathbb{R})$ boundedness does not hold for $T_{\alpha,m}$ with $0 \leq \alpha < 1$, $m = 2$, $A_1 = 1$, and $A_2 = -1$. This is an important difference with the case $0 < \alpha < n$ and $m = 1$. Indeed, in the paper [22], Taibleson and Weiss, using molecular characterization of the real Hardy spaces, obtained the boundedness of I_α from $H^p(\mathbb{R}^n)$ into $H^q(\mathbb{R}^n)$, for $0 < p \leq 1$ and $1/q = 1/p - \alpha/n$.

In [1], it gives the boundedness of the Riesz potential from $L^{p(\cdot)}$ into $L^{q(\cdot)}$ where $1/q(\cdot) = 1/p(\cdot) - \alpha/n$. Sawano obtains in [19] the $H^{p(\cdot)} - H^{q(\cdot)}$ boundedness of the Riesz potential using a finite atomic decomposition different from that given in [4]. In [18], the author jointly with Urciuolo proved the $H^{p(\cdot)} - L^{q(\cdot)}$ boundedness of the operator $T_{\alpha,m}$ and the $H^{p(\cdot)} - H^{q(\cdot)}$ boundedness of the Riesz potential via the infinite atomic and molecular decomposition developed in [15].

The purpose of this article is to give another proof of the results obtained in [18], but by using the finite atomic decomposition developed in [4]. Here, a key tool is a weighted vector-valued inequality for the fractional maximal operator. We also rely on the theory of weighted Hardy spaces and on the Rubio de Francia iteration algorithm.

This method allows us to avoid the more delicate convergence arguments that are often necessary when utilizing the infinite atomic decomposition.

The main results of this work are contained in the following theorem.

Theorem 1.1. *For $0 \leq \alpha < n$ and $m > 1$, let $T_{\alpha,m}$ be the operator defined by (1.1). Given a measurable function $q(\cdot) : \mathbb{R}^n \to (0, \infty)$ such that $0 < q_0 < q_- \leq q_+ < +\infty$, with $0 < q_0 < 1$, define the function $p(\cdot)$ by $1/p(\cdot) = 1/q(\cdot) + \alpha/n$. If $q(\cdot) \in M\mathcal{P}_0$, and $q(A_i x) = q(x)$ for all x and all $i = 1, \dots, m$, then $T_{\alpha,m}$ can be extended to an $H^{p(\cdot)}(\mathbb{R}^n) - L^{q(\cdot)}(\mathbb{R}^n)$ bounded operator.*

Theorem 1.2. *Let $0 < \alpha < n$. Given a measurable function $q(\cdot) : \mathbb{R}^n \to (0, \infty)$ such that $0 < q_0 < q_- \leq q_+ < +\infty$, with $0 < q_0 < 1$, define the function $p(\cdot)$ by $1/p(\cdot) = 1/q(\cdot) + \alpha/n$. If $q(\cdot) \in M\mathcal{P}_0$, then the Riesz potential I_α can be extended to a bounded operator from $H^{p(\cdot)}(\mathbb{R}^n)$ into $H^{q(\cdot)}(\mathbb{R}^n)$.*

In Theorem 1.1 we assume that the exponent $q(\cdot)$ is invariant under the m orthogonal matrices A_i, this hypothesis allows us to recover the norm $\|\cdot\|_{q(\cdot)}$, that is $\|f_{A_i}\|_{q(\cdot)} = \|f\|_{q(A_i\, \cdot)} = \|f\|_{q(\cdot)}$, where $f_{A_i}(x) = f(A_i^{-1} x)$. This identity is used in the proof of Theorem 1.1.

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We give two examples of variable exponents which satisfy the hypothesis of Theorem 1.1.

EXAMPLE 1.3. Suppose $h : \mathbb{R} \to (0, \infty)$ that satisfies the log-Hölder continuity on $\mathbb{R}$ (see [2]) and $0 < h_- \leq h_+ < +\infty$. Let $q(x) = h(|x|)$ for $x \in \mathbb{R}^n$ and for $m > 1$ let $A_1, \ldots, A_m$ be $n \times n$ orthogonal matrices such that $A_i - A_j$ is invertible for $i \neq j$. It is easy to check that the function $q(\cdot)$ satisfies $0 < q_- \leq q_+ < +\infty$ and the log-Hölder continuity on $\mathbb{R}^n$. So $q(\cdot) \in M\mathcal{P}_0$ and $q(A_i x) = q(x)$ for $1 \leq i \leq m$.

Another nontrivial example of exponent functions and orthogonal matrices satisfying the hypothesis of the theorem is given below.

We consider $m = 2$, $q(\cdot) : \mathbb{R}^n \to (0, \infty)$ that satisfies the log-Hölder continuity on $\mathbb{R}^n$, $0 < q_- \leq q_+ < +\infty$, and then we take $q_e(x) = q(x) + q(-x)$, $A_1 = I$ and $A_2 = -I$.

In Section 2 we give some basic results about the variable Lebesgue spaces and the theory of weights. We also recall the definition and atomic decomposition of the variable Hardy spaces given in [4]. In Section 3 we state some auxiliary lemmas and propositions to get the main results. In Section 4 we prove Theorems 1.1 and 1.2.

NOTATION 1.4. We denote by $B(x_0, r)$ the ball centered at $x_0 \in \mathbb{R}^n$ of radius r. For a measurable subset $E \subset \mathbb{R}^n$ we denote $|E|$ and χ_E the Lebesgue measure of E and the characteristic function of E, respectively. Given a real number $s \geq 0$, we write $\lfloor s \rfloor$ for the integer part of s. As usual we denote with $\mathcal{S}(\mathbb{R}^n)$ the space of Schwartz functions, with $\mathcal{S}'(\mathbb{R}^n)$ the dual space. If β is the multiindex $\beta = (\beta_1, \ldots, \beta_n)$, then $|\beta| = \beta_1 + \cdots + \beta_n$.

Throughout this paper, C will denote a positive constant, not necessarily the same at each occurrence.

2. Preliminaries

In this section, we give some definitions and some basic results about the variable Lebesgue spaces and the theory of weights.

Given a measurable function $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ such that

$$0 < ess \inf_{x \in \mathbb{R}^n} p(x) \leq ess \sup_{x \in \mathbb{R}^n} p(x) < \infty,$$

let $L^{p(\cdot)}(\mathbb{R}^n)$ denote the space of all measurable functions f such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx < \infty.$$

We set

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leq 1\right\}.$$

We see that $(L^{p(\cdot)}(\mathbb{R}^n), \|.\|_{p(\cdot)})$ is a quasi normed space.

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Here we adopt the standard notation in variable exponents. We write

$$p_- = ess \inf_{x\in\mathbb{R}^n} p(x), \quad p_+ = ess \sup_{x\in\mathbb{R}^n} p(x), \quad \text{and} \quad \underline{p} = \min\{p_-, 1\}.$$

From now on we assume $0 < p_- \leq p_+ < \infty$. It is not so hard to see the following lemma.

Lemma 2.1. *The following statements hold:*

(1) $\|f\|_{p(\cdot)} \geq 0$, *and* $\|f\|_{p(\cdot)} = 0$ *if and only if* $f \equiv 0$.
(2) $\|cf\|_{p(\cdot)} = |c|\,\|f\|_{p(\cdot)}$ *for* $c \in \mathbb{C}$.
(3) $\|f+g\|_{p(\cdot)}^{\underline{p}} \leq \|f\|_{p(\cdot)}^{\underline{p}} + \|g\|_{p(\cdot)}^{\underline{p}}$.
(4) $\||f|^s\|_{p(\cdot)} = \|f\|_{sp(\cdot)}^s$, *for all* $s > 0$.
(5) *If* A *is an* $n \times n$ *orthogonal matrix and* $p(Ax) = p(x)$ *for all* $x \in \mathbb{R}^n$, *then* $\|f_A\|_{p(.)} = \|f\|_{p(.)}$, *where* $f_A(x) = f(A^{-1}x)$.

Let $\mathcal{P}_0$ denote the collection of all measurable functions $p(.) : \mathbb{R}^n \to (0, \infty)$ such that $0 < p_- \leq p_+ < \infty$.

Given $p(\cdot) \in \mathcal{P}_0$ with $p_- > 1$, define the conjugate exponent $p'(\cdot)$ by the equation $1/p(\cdot) + 1/p'(\cdot) = 1$.

Lemma 2.2 (See [2, Theorem 2.34]). *If* $p(\cdot) \in \mathcal{P}_0$ *with* $p_- > 1$, *then for all* $f \in L^{p(\cdot)}$,

$$\|f\|_{p(\cdot)} \leq C \sup \int_{\mathbb{R}^n} |f(x)g(x)|\, dx,$$

where the supremum is taken over all $g \in L^{p'(\cdot)}$ *such that* $\|g\|_{p'(\cdot)} \leq 1$.

Let f be a locally integrable function on $\mathbb{R}^n$. The function

$$M(f)(x) = \sup_{B\ni x} \frac{1}{|B|} \int_B |f(y)|\, dy,$$

where the supremum is taken over all balls B containing x, is called the uncentered Hardy–Littlewood maximal function of f.

Throughout, we will make use of the following class of exponents.

Definition 2.3. Given $p(\cdot) \in \mathcal{P}_0$, we say $p(\cdot) \in M\mathcal{P}_0$ if there exists p_0, $0 < p_0 < p_-$, such that $\|Mf\|_{p(\cdot)/p_0} \leq C\|f\|_{p(\cdot)/p_0}$.

Lemma 2.4. *Given* $p(\cdot) \in \mathcal{P}_0$, *with* $p_- > 1$, *if the maximal operator is bounded on* $L^{p(\cdot)}$, *then for every* $s > 1$ *it is bounded on* $L^{sp(\cdot)}$.

Proof. From Hölder's inequality and Lemma 2.1(4),

$$\|Mf\|_{sp(\cdot)} = \|(Mf)^s\|_{p(\cdot)}^{1/s} \leq \|(M(|f|^s)\|_{p(\cdot)}^{1/s} \leq C\||f|^s\|_{p(\cdot)}^{1/s} = C\|f\|_{sp(\cdot)}. \qquad \square$$

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The following lemma is a remarkable result in the theory of the variable Lebesgue spaces (see [5, Theorem 8.1]).

LEMMA 2.5. *Given $p(\cdot) \in \mathcal{P}_0$, with $p_- > 1$, the maximal operators are bounded on $L^{p(\cdot)}$ if and only if they are bounded on $L^{p'(\cdot)}$.*

It is well known that a useful sufficient condition for the boundedness of the maximal operator is a log-Hölder continuity, (see [2, 6]). In our main results, we will only assume that the exponent $p(\cdot)$ belongs to $M\mathcal{P}_0$.

In the paper [4], Cruz-Uribe and Wang give a variety of distinct approaches, based on differing definitions, all leading to the same notion of the variable Hardy space $H^{p(.)}$.

We recall the definition and the atomic decomposition of the Hardy spaces with variable exponents.

Topologize $\mathcal{S}(\mathbb{R}^n)$ by the collection of semi-norms $\|\cdot\|_{\alpha,\beta}$, with α and β multi-indices, given by

$$\|\varphi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)|.$$

For each $N \in \mathbb{N}$, we set $\mathcal{S}_N = \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \|\varphi\|_{\alpha,\beta} \le 1, |\alpha|, |\beta| \le N\}$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, we denote by $\mathcal{M}_N$ the grand maximal operator given by

$$\mathcal{M}_N f(x) = \sup_{t>0} \sup_{\varphi\in\mathcal{S}_N} |(t^{-n}\varphi(t^{-1}\cdot) * f)(x)|.$$

DEFINITION 2.6. Let $p(\cdot) \in M\mathcal{P}_0$. For $N > n/p_0 + n + 1$, define the Hardy space with variable exponents $H^{p(\cdot)}(\mathbb{R}^n)$ as the set of all $f \in S'(\mathbb{R}^n)$ for which $\mathcal{M}_N f \in L^{p(\cdot)}(\mathbb{R}^n)$. In this case, we write $\|f\|_{H^{p(\cdot)}} = \|\mathcal{M}_N f\|_{p(\cdot)}$.

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a function such that $\int \phi(x)\,dx \neq 0$. For $f \in \mathcal{S}'(\mathbb{R}^n)$, we define the maximal function $M_\phi f$ by

$$M_\phi f(x) = \sup_{t>0} |(t^{-n}\phi(t^{-1}\cdot) * f)(x)|.$$

Theorem 3.1 in [4] asserts that the quantities $\|M_\phi f\|_{p(\cdot)}$ and $\|\mathcal{M}_N f\|_{p(\cdot)}$ are comparable, with bounds that depend only on $p(\cdot)$ and n and not on f, if $N > n/p_0 + n + 1$.

Now, we give the definition of atoms.

DEFINITION 2.7. Given $p(\cdot) \in M\mathcal{P}_0$, and $1 < q \le \infty$, a function $a(\cdot)$ is a $(p(\cdot), q)$-atom if there exists a ball $B = B(x_0, r)$ such that

(a_1) $\operatorname{supp}(a) \subset B$,

(a_2) $\|a\|_q \le |B|^{1/q}\|\chi_B\|_{p(\cdot)}^{-1}$,

(a_3) $\int a(x)x^\alpha\,dx = 0$ for all $|\alpha| \le \lfloor n(1/p_0 - 1)\rfloor$.

REMARK 2.8. Let $a(\cdot)$ be a $(p(\cdot), q)$-atom and $1 < s < q$, then Hölder's inequality implies $\|a\|_s \le |B|^{1/s}/\|\chi_B\|_{p(\cdot)}$.

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Given $1 < q < \infty$, let $H^{p(\cdot),q}_{\text{fin}}(\mathbb{R}^n)$ be the subspace of $H^{p(\cdot)}(\mathbb{R}^n)$ consisting of all f that have decompositions as finite sums of $(p(\cdot), q)$-atoms. By [4, Theorem 7.1], if q is sufficiently large, $H^{p(\cdot),q}_{\text{fin}}(\mathbb{R}^n)$ is dense in $H^{p(\cdot)}(\mathbb{R}^n)$.

For $f \in H^{p(\cdot),q}_{\text{fin}}(\mathbb{R}^n)$, define

$$\|f\|_{H^{p(\cdot),q}_{\text{fin}}} = \inf\bigg\{\bigg\|\sum_{j=1}^{k} \lambda_j \frac{\chi_{B_j}}{\|\chi_{B_j}\|_{p(\cdot)}}\bigg\|_{p(\cdot)} : f = \sum_{j=1}^{k} \lambda_j a_j\bigg\},$$

where the infimum is taken over all finite decompositions of f using $(p(\cdot), q)$-atoms. Theorem 7.8 in [4] asserts that $\|f\|_{H^{p(\cdot),q}_{\text{fin}}} \simeq \|f\|_{H^{p(\cdot)}}$ for all $f \in H^{p(\cdot),q}_{\text{fin}}(\mathbb{R}^n)$.

A weight is a nonnegative locally integrable function on $\mathbb{R}^n$ that takes values in $(0, \infty)$ almost everywhere, that is: the weights are allowed to be zero or infinity only on a set of measure zero with respect to Lebesgue measure.

Given a weight w and a measurable set E, we use the notation $w(E) = \int_E w(x)\,dx$. Let f be a locally integrable function on $\mathbb{R}^n$. The function

$$\widetilde{M}(f)(x) = \sup_{\delta>0} \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} |f(y)|\,dy,$$

is called the centered Hardy–Littlewood maximal function of f. It is easy to check that

$$2^{-n} M(f)(x) \le \widetilde{M}(f)(x) \le M(f)(x) \quad \text{for all } x \in \mathbb{R}^n. \tag{2.1}$$

We say that a weight $w \in \mathcal{A}_1$ if there exists $C > 0$ such that

$$M(w)(x) \le C w(x) \quad \text{a.e. } x \in \mathbb{R}^n, \tag{2.2}$$

the best possible constant is denoted by $[w]_{\mathcal{A}_1}$. Equivalently, a weight $w \in \mathcal{A}_1$ if there exists $C > 0$ such that for every ball B

$$\frac{1}{|B|} \int_B w(x)\,dx \le C\, ess \inf_{x \in B} w(x). \tag{2.3}$$

Remark 2.9. The orthogonal group $O(n)$[1] induces an action on functions by $f_A(x) = f(A^{-1}x)$, where $A \in O(n)$. Since $\widetilde{M}(w_A)(x) = [\widetilde{M}(w)]_A(x)$ for all $x \in \mathbb{R}^n$, and taking account (2.1) and (2.2), it follows that $w \in \mathcal{A}_1$ if and only if $w_A \in \mathcal{A}_1$ for all $A \in O(n)$. Therefore, the space of weights $\mathcal{A}_1$ is invariant under the orthogonal group, and the $\mathcal{A}_1$ constant is preserved.

Remark 2.10. If $w \in \mathcal{A}_1$ and $0 < r < 1$, then by the Hölder inequality, $w^r \in \mathcal{A}_1$.

For $1 < p < \infty$, we say that a weight $w \in \mathcal{A}_p$ if there exists $C > 0$ such that for every ball B

$$\bigg(\frac{1}{|B|} \int_B w(x)\,dx\bigg)\bigg(\frac{1}{|B|} \int_B [w(x)]^{-1/(p-1)}\,dx\bigg)^{p-1} \le C.$$

It is well known that $\mathcal{A}_{p_1} \subset \mathcal{A}_{p_2}$ for all $1 \le p_1 < p_2 < \infty$.

[1] $O(n) = \{A \in GL_n(\mathbb{R}) : A^t = A^{-1}\}$.

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Given $1 < p \leq q < \infty$, we say that a weight $w \in \mathcal{A}_{p,q}$ if there exists $C > 0$ such that for every ball B

$$\left(\frac{1}{|B|}\int_B [w(x)]^q\,dx\right)^{1/q}\left(\frac{1}{|B|}\int_B [w(x)]^{-p'}\,dx\right)^{1/p'} \leq C < \infty.$$

For $p = 1$, we say that a weight $w \in \mathcal{A}_{1,q}$ if there exists $C > 0$ such that for every ball B

$$\left(\frac{1}{|B|}\int_B [w(x)]^q\,dx\right)^{1/q} \leq C\, ess\inf_{x\in B} w(x).$$

When $p = q$, this definition is equivalent to $w^p \in \mathcal{A}_p$.

Remark 2.11. From the inequality in (2.3) it follows that if a weight $w \in \mathcal{A}_1$, then $0 < ess\inf_{x\in B} w(x) < \infty$ for each ball B. Thus, $w \in \mathcal{A}_1$ implies that $w^{1/q} \in \mathcal{A}_{p,q}$, for each $1 \leq p \leq q < \infty$.

A weight satisfies the reverse Hölder inequality with exponent $s > 1$, denoted by $w \in RH_s$, if there exists $C > 0$ such that for every ball B,

$$\left(\frac{1}{|B|}\int_B [w(x)]^s\,dx\right)^{1/s} \leq C\frac{1}{|B|}\int_B w(x)\,dx;$$

the best possible constant is denoted by $[w]_{RH_s}$. We observe that if $w \in RH_s$, then by Hölder's inequality, $w \in RH_t$ for all $1 < t < s$, and $[w]_{RH_t} \leq [w]_{RH_s}$.

Lemma 2.12. *Given $w \in \mathcal{A}_1$, then $w \in RH_s$, where $s = 1 + (2^{n+1}[w]_{\mathcal{A}_1})^{-1}$.*

This result was proved for cubes in [10]. However, since $w \in \mathcal{A}_1$ is doubling, the lemma holds for balls with same exponent.

Given $0 < \alpha < n$, we define the fractional maximal operator M_α by

$$M_\alpha f(x) = \sup_B \frac{1}{|B|^{1-\alpha/n}}\int_B |f(y)|\,dy,$$

where f is a locally integrable function and the supremum is taken over all the balls B which contain x. In the case $\alpha = 0$, the fractional maximal operator reduces to the Hardy–Littlewood maximal operator.

The fractional maximal operator satisfies the following weighted vector-valued inequality.

Lemma 2.13. *Given $0 < \alpha < n$, let p and q such that $1 < p \leq q < \infty$ and $1/q = 1/p - \alpha/n$. Then, for all $w \in \mathcal{A}_1$ and all $1 < \theta < \infty$,*

$$\left\|\left\{\sum_{j=1}^{\infty}[M_\alpha(f_j)]^\theta\right\}^{1/\theta}\right\|_{L^q(w)} \leq C\left\|\left\{\sum_{j=1}^{\infty}|f_j|^\theta\right\}^{1/\theta}\right\|_{L^p(w^{p/q})},$$

for all sequences of functions $\{f_j\}_{j=1}^\infty \subset L^p(w^{p/q})$.

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PROOF. Given $w \in \mathcal{A}_1$, from Remark 2.11, $w^{1/q} \in \mathcal{A}_{p,q}$. So, the lemma follows from [3, Theorem 3.23]. □

We conclude these preliminaries with the definition of the weighted Hardy spaces. Given a weight $w \in \mathcal{A}_1$ and $p_0 > 0$, the weighted Hardy space $H^{p_0}(w)$ consists of all tempered distributions f such that

$$\|f\|_{H^{p_0}(w)} = \|M_\phi f\|_{L^{p_0}(w)} = \left(\int_{\mathbb{R}^n} [M_\phi f(x)]^{p_0} w(x)\, dx\right)^{1/p_0} < \infty.$$

For a sufficiently large N, $\|M_\phi f\|_{L^{p_0}(w)} \simeq \|\mathcal{M}_N f\|_{L^{p_0}(w)}$, (see [21]).

Let $p(\cdot) \in M\mathcal{P}_0$, and $q > 1$. Given $w \in \mathcal{A}_1$, define $H^{p_0,q}_{\text{fin}}(w)$ as the set of all finite sums of $(p(\cdot), q)$-atoms. If q sufficiently large, then by [4, Lemma 7.6], $H^{p_0,q}_{\text{fin}}(w) \subset H^{p_0}(w)$. Moreover, by [4, Lemma 7.3], $H^{p(\cdot),q}_{\text{fin}}(\mathbb{R}^n) = H^{p_0,q}_{\text{fin}}(w)$ as sets. (We introduce this notation involving w to stress that it is a subset of $H^{p_0}(w)$.)

For $f \in H^{p_0,q}_{\text{fin}}(w)$, define

$$\|f\|_{H^{p_0,q}_{\text{fin}}(w)} = \inf\left\{\left\|\sum_{j=1}^{k} \lambda_j^{p_0} \frac{\chi_{B_j}}{\|\chi_{B_j}\|^{p_0}_{p(\cdot)}}\right\|^{1/p_0}_{L^1(w)} : f = \sum_{j=1}^{k} \lambda_j a_j\right\},$$

where the infimum is taken over all finite decompositions of f using $(p(\cdot), q)$-atoms. If $w \in \mathcal{A}_1 \cap L^{(p(\cdot)/p_0)'}$, then [4, Lemma 7.11] asserts that $\|f\|_{H^{p_0,q}_{\text{fin}}(w)} \leq C\|f\|_{H^{p_0}(w)}$ for all $f \in H^{p_0,q}_{\text{fin}}(w)$.

3. Auxiliary results

The following lemmas are crucial to get the main results.

LEMMA 3.1. *For $0 \leq \alpha < n$ and $m > 1$, let $T_{\alpha,m}$ be the operator defined by (1.1). If $w \in \mathcal{A}_1$, then*

$$w(\{x : |T_{\alpha,m} f(x)| \geq \lambda\}) \leq C\lambda^{-n/(n-\alpha)} \sum_{i=1}^{m}\left(\int_{\mathbb{R}^n} |f(x)|[w_{A_i^{-1}}(x)]^{(n-\alpha)/n}\, dx\right)^{n/(n-\alpha)} \tag{3.1}$$

for all integrable function f with compact support.

PROOF. We study separately the cases $0 < \alpha < n$ and $\alpha = 0$. For $0 < \alpha < n$, $|T_{\alpha,m} f(x)| \leq \sum_{i=1}^{m} (I_\alpha|f|)_{A_i}(x)$, so

$$\{x : |T_{\alpha,m} f(x)| \geq \lambda\} \subset \bigcup_{i=1}^{m} A_i(\{x : (I_\alpha|f|)(x) \geq \lambda/m\}).$$

Since $w \in \mathcal{A}_1$, from Remarks 2.9 and 2.11, it follows that $[w_{A_i^{-1}}]^{(n-\alpha)/n} \in \mathcal{A}_{1,n/(n-\alpha)}$ for each $i = 1, 2, \ldots, m$. Now [13, Theorem 5] gives (3.1).

The proof for $\alpha = 0$ is analogous to [16, Proof of Theorem 1(b)]. □

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Lemma 3.2. *For $0 \leq \alpha < n$ and $m > 1$, let $T_{\alpha,m}$ be the operator defined by (1.1) and let $a(\cdot)$ be a $(p(\cdot), q/p_0)$-atom supported on a ball B.*

(a) *If $0 < \alpha < n$, $w \in \mathcal{A}_1$ and $q > np_0/\alpha$, then for $1/q_0 = 1/p_0 - \alpha/n$*

$$\int_{\mathbb{R}^n} |T_{\alpha,m}a(x)|^{q_0} w(x)\,dx \leq C|B|^{\alpha/nq_0} \|\chi_B\|_{p(.)}^{-q_0} \sum_{i=1}^{m} w_{A_i^{-1}}(B).$$

(b) *If $\alpha = 0$ and $w \in \mathcal{A}_1 \cap RH_{(q/p_0)'}$, then*

$$\int_{\mathbb{R}^n} |T_{0,m}a(x)|^{p_0} w(x)\,dx \leq C\|\chi_B\|_{p(.)}^{-p_0} \sum_{i=1}^{m} w_{A_i^{-1}}(B).$$

Proof. (a) Let $B = B(x_0, r)$ be the ball which a is supported, we put $B_i^* = B(A_i x_0, 2r)$ with $i = 1, \ldots, m$. We decompose $\mathbb{R}^n = \bigcup_{i=1}^m B_i^* \cup R$, where $R = \mathbb{R}^n \backslash (\bigcup_{i=1}^m B_i^*)$ and write

$$\begin{aligned}\int_{\mathbb{R}^n} |T_{\alpha,m}a(x)|^{q_0} w(x)\,dx &= \int_{\bigcup_{i=1}^m B_i^*} |T_{\alpha,m}a(x)|^{q_0} w(x)\,dx + \int_R |T_{\alpha,m}a(x)|^{q_0} w(x)\,dx \\ &= I_1 + I_2.\end{aligned}$$

We first estimate I_1,

$$I_1 \leq \sum_{i=1}^{m} \int_{B_i^*} |T_{\alpha,m}a(x)|^{q_0} w(x)\,dx = \sum_{i=1}^{m} \int_0^{\infty} q_0 \lambda^{q_0-1} w(\{x \in B_i^* : |T_{\alpha,m}a(x)| > \lambda\}) d\lambda$$

for $i = 1, \ldots, m$, we write $w_i(x) = [w_{A_i^{-1}}(x)]^{(n-\alpha)/n}$, from Lemma 3.1 and the generalized Kolmogorov's inequality (see [9, Theorem 3.3.1, page 51])

$$I_1 \leq C\Big(\sum_{i=1}^{m} w(B_i^*)\Big)^{1-(n-\alpha)/nq_0} \Big(\sum_{i=1}^{m} \|a\|_{L^1(w_i)}\Big)^{q_0}.$$

Now we estimate $\sum_{i=1}^m \|a\|_{L^1(w_i)}$, by Hölder's inequality and since $a(.)$ is a $(p(.), q/p_0)$ atom

$$\begin{aligned}&\|a\|_{L^1(w_i)} \\ &= \int |a(x)| [w_{A_i^{-1}}(x)]^{(n-\alpha)/n}\,dx \\ &\leq \Big(\int_B |a(x)|^{q/p_0}\,dx\Big)^{p_0/q} \Big(\int_B [w_{A_i^{-1}}(x)]^{(n-\alpha)/n(q/p_0)'}\,dx\Big)^{1/(q/p_0)'} \\ &\leq |B|^{1/(q/p_0)'} |B|^{p_0/q} \|\chi_B\|_{p(.)}^{-1} \Big(\frac{1}{|B|} \int_B [w_{A_i^{-1}}(x)]^{(n-\alpha)/n(q/p_0)'}\,dx\Big)^{1/(q/p_0)'}\end{aligned}$$

the condition $q > np_0/\alpha$ implies $(n-\alpha)/n(q/p_0)' < 1$, then from Hölder's inequality

$$\leq |B| \|\chi_B\|_{p(.)}^{-1} \Big(\frac{1}{|B|} \int_B w_{A_i^{-1}}(x)\,dx\Big)^{(n-\alpha)/n} = |B|^{\alpha/n} \|\chi_B\|_{p(.)}^{-1} [w_{A_i^{-1}}(B)]^{(n-\alpha)/n},$$

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so

$$\Big(\sum_{i=1}^{m}\|a\|_{L^1(w_i)}\Big)^{q_0} \le |B|^{\alpha/nq_0}\|\chi_B\|_{p(.)}^{-q_0}\Big(\sum_{i=1}^{m}[w_{A_i^{-1}}(B)]^{(n-\alpha)/n}\Big)^{q_0}$$
$$\le C|B|^{\alpha/nq_0}\|\chi_B\|_{p(\cdot)}^{-q_0}\Big(\sum_{i=1}^{m}[w_{A_i^{-1}}(B)]\Big)^{(n-\alpha)/nq_0}.$$

Since $w_{A_i^{-1}} \in \mathcal{A}_1$ for each $i = 1, \ldots, m$ (see Remark 2.9) and the weights in $\mathcal{A}_1$ are doubling measures (see [4, Remark 7.7]), it follows that

$$I_1 \le C|B|^{\alpha/nq_0}\|\chi_B\|_{p(\cdot)}^{-q_0}\sum_{i=1}^{m} w_{A_i^{-1}}(B).$$

To estimate I_2, we start with a pointwise estimate. Let $d = \lfloor n(1/p_0 - 1)\rfloor$. We denote $k(x, y) = |x - A_1 y|^{-\alpha_1} \cdots |x - A_m y|^{-\alpha_m}$. In view of the moment condition of $a(.)$

$$T_{\alpha,m}a(x) = \int_{B(x_0,r)} k(x, y)a(y)\,dy = \int_{B(x_0,r)} (k(x, y) - q_d(x, y))a(y)\,dy,$$

where q_d is the degree d Taylor polynomial of the function $y \to k(x, y)$ expanded around x_0. By the standard estimate of the remainder term of the Taylor expansion, there exists ξ between y and x_0 such that

$$|k(x, y) - q_d(x, y)| \lesssim |y - x_0|^{d+1}\sum_{k_1+\cdots+k_n=d+1}\Big|\frac{\partial^{d+1}}{\partial y_1^{k_1}\cdots\partial y_n^{k_n}}k(x, \xi)\Big|$$
$$\le C|y - x_0|^{d+1}\Big(\prod_{i=1}^{m}|x - A_i\xi|^{-\alpha_i}\Big)\Big(\sum_{l=1}^{m}|x - A_l\xi|^{-1}\Big)^{d+1}.$$

Now, we decompose $R = \bigcup_{k=1}^{m} R_k$ where

$$R_k = \{x \in R : |x - A_k x_0| \le |x - A_i x_0| \text{ for all } i \ne k\}.$$

If $x \in R$ then $|x - A_i x_0| \ge 2r$, since $\xi \in B$ it follows that $|A_i x_0 - A_i\xi| \le r \le \frac{1}{2}|x - A_i x_0|$ so

$$|x - A_i\xi| = |x - A_i x_0 + A_i x_0 - A_i\xi| \ge |x - A_i x_0| - |A_i x_0 - A_i\xi| \ge \tfrac{1}{2}|x - A_i x_0|.$$

If $x \in R$, then $x \in R_k$ for some k and since $\alpha_1 + \cdots + \alpha_m = n - \alpha$

$$|k(x, y) - q_d(x, y)| \le C|y - x_0|^{d+1}\Big(\prod_{i=1}^{m}|x - A_i x_0|^{-\alpha_i}\Big)\Big(\sum_{l=1}^{m}|x - A_l x_0|^{-1}\Big)^{d+1}$$
$$\le Cr^{d+1}|x - A_k x_0|^{-n+\alpha-d-1},$$

this inequality allows us to conclude that

$$|T_{\alpha,m}a(x)| \le C\|a\|_1 r^{d+1}|x - A_k x_0|^{-n+\alpha-d-1}$$
$$\le C|B|^{1-p_0/q}\|a\|_{q/p_0} r^{d+1}|x - A_k x_0|^{-n+\alpha-d-1},$$

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since $\|a\|_{q/p_0} \le |B|^{p_0/q}\|\chi_B\|_{p(\cdot)}^{-1}$,

$$|T_{\alpha,m}a(x)| \le C\frac{r^{n+d+1}}{\|\chi_B\|_{p(\cdot)}}|x - A_k x_0|^{-n+\alpha-d-1}$$
$$\le C\frac{(M_{\alpha n/(n+d+1)}(\chi_B)(A_k^{-1}x))^{(n+d+1)/n}}{\|\chi_B\|_{p(\cdot)}} \quad \text{if } x \in R_k. \tag{3.2}$$

This pointwise estimate gives

$$I_2 = \int_R |T_{\alpha,m}a(x)|^{q_0} w(x)\,dx \le C\sum_{k=1}^m \int_{\mathbb{R}^n} \frac{(M_{\alpha n/(n+d+1)}(\chi_B)(A_k^{-1}x))^{q_0(n+d+1)/n}}{\|\chi_B\|_{p(\cdot)}^{q_0}} w(x)\,dx$$
$$= C\|\chi_B\|_{p(\cdot)}^{-q_0}\sum_{k=1}^m \int_{\mathbb{R}^n} (M_{\alpha n/(n+d+1)}(\chi_B)(x))^{q_0(n+d+1)/n} w_{A_k^{-1}}(x)\,dx.$$

Since $d = \lfloor n(1/p_0 - 1)\rfloor$, $q_0(n+d+1)/n > 1$. We write $\widetilde{q} = q_0(n+d+1)/n$ and let $1/(\widetilde{p}) = 1/\widetilde{q} + \alpha/(n+d+1)$, so $\widetilde{p}/\widetilde{q} = p_0/q_0$ and $(w_{A_k^{-1}})^{1/\widetilde{q}} \in \mathcal{A}_{\widetilde{p},\widetilde{q}}$. From [13, Theorem 3],

$$\int_{\mathbb{R}^n} (M_{\alpha n/(n+d+1)}(\chi_B)(x))^{q_0(n+d+1)/n} w_{A_k^{-1}}(x)\,dx \le \Big(\int_{\mathbb{R}^n} \chi_B(x)[w_{A_k^{-1}}(x)]^{p_0/q_0}\,dx\Big)^{q_0/p_0}$$
$$\le |B|^{\alpha/nq_0} w_{A_k^{-1}}(B),$$

where Hölder's inequality gives the last inequality.

(b) As in (a) we decompose $\mathbb{R}^n = \bigcup_{i=1}^m B_i^* \cup R$, where $R = \mathbb{R}^n \backslash (\bigcup_{i=1}^m B_i^*)$, but now we write

$$\int_{\mathbb{R}^n} |T_{0,m}a(x)|^{p_0} w(x)\,dx = \int_{\bigcup_{i=1}^m B_i^*} |T_{0,m}a(x)|^{p_0} w(x)\,dx + \int_R |T_{0,m}a(x)|^{p_0} w(x)\,dx$$
$$= I_1 + I_2.$$

A similar computation as done in (a) allows us to obtain the following equation:

$$I_1 \le C\Big(\sum_{i=1}^m w(B_i^*)\Big)^{1-p_0}\Big(\sum_{i=1}^m \int_B |a(x)| w_{A_i^{-1}}(x)\,dx\Big)^{p_0}.$$

To estimate the last integral we use that $a(\cdot)$ is an $(p(\cdot), q/p_0)$ atom and $w \in RH_{(q/p_0)'}$, so

$$\int_B |a(x)| w_{A_i^{-1}}(x)\,dx \le |B|^{1/(q/p_0)'}|B|^{p_0/q}\|\chi_B\|_{p(\cdot)}^{-1}\Big(\frac{1}{|B|}\int_B [w_{A_i^{-1}}(x)]^{(q/p_0)'}\,dx\Big)^{1/(q/p_0)'}$$
$$\le C\|\chi_B\|_{p(\cdot)}^{-1} w_{A_i^{-1}}(B).$$

Therefore,

$$I_1 \le C\|\chi_B\|_{p(\cdot)}^{-p_0}\sum_{i=1}^m w_{A_i^{-1}}(B).$$

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To estimate I_2, following a similar argument to that used in (a),

$$I_2 = \int_R |T_{0,m}a(x)|^{p_0} w(x)\,dx \le C\|\chi_B\|_{p(\cdot)}^{-p_0} \sum_{i=1}^m \int_{\mathbb{R}^n} [M(\chi_B)(x)]^{p_0(n+d+1)/n} w_{A_i^{-1}}(x)\,dx,$$

where $d = \lfloor n(1/p_0 - 1)\rfloor$, so $p_0(n+d+1)/n > 1$. Finally, since $w_{A_i^{-1}} \in \mathcal{A}_1 \subset \mathcal{A}_{p_0(n+d+1)/n}$ for each $i = 1, \ldots, m$, from [12, Theorem 9], it follows that

$$I_2 \le C\|\chi_B\|_{p(.)}^{-p_0} \sum_{i=1}^m w_{A_i^{-1}}(B).$$

The proof is therefore concluded. □

Proposition 3.3. *For $0 \le \alpha < n$ and $m > 1$, let $T_{\alpha,m}$ be the operator defined by (1.1). Let $p(\cdot) \in M\mathcal{P}_0$, with $0 < p_0 < n/(n+\alpha)$, such that $p(A_i x) = p(x)$ for all $i = 1, \ldots, m$. If $w \in \mathcal{A}_1 \cap L^{p_0/q_0(p(\cdot)/p_0)'}(\mathbb{R}^n)$ and $0 < \alpha < n$ or $w \in \mathcal{A}_1 \cap L^{(p(\cdot)/p_0)'}(\mathbb{R}^n) \cap RH_{(q/p_0)'}$ and $\alpha = 0$, then for $1/q_0 = 1/p_0 - \alpha/n$*

$$\|T_{\alpha,m} f\|_{L^{q_0}(w)} \le C \sum_{i=1}^m \|f\|_{H^{p_0}([w_{A_i^{-1}}]^{p_0/q_0})},$$

for all $f \in H_{\text{fin}}^{p_0,q/p_0}(w)$, where q is sufficiently large.

Proof. Given $f \in H_{\text{fin}}^{p_0,q/p_0}(w)$, $f = \sum_{j=1}^k \lambda_j a_j$, where a_j is a $(p(\cdot), q/p_0)$-atom supported on a ball B_j. Since $0 < q_0 < 1$, Lemma 3.2 gives

$$\begin{aligned}
\|T_{\alpha,m} f\|_{L^{q_0}(w)}^{q_0} = \int_{\mathbb{R}^n} |T_{\alpha,m} f(x)|^{q_0} w(x)\,dx &\le \sum_{j=1}^k \lambda_j^{q_0} \int_{\mathbb{R}^n} |T_{\alpha,m} a_j(x)|^{q_0} w(x)\,dx \\
&\le C \sum_{i=1}^m \sum_{j=1}^k \lambda_j^{q_0} |B_j|^{\alpha/nq_0} \|\chi_{B_j}\|_{p(\cdot)}^{-q_0} w_{A_i^{-1}}(B_j) \\
&= C \sum_{i=1}^m \int_{\mathbb{R}^n} \Big(\sum_{j=1}^k \lambda_j^{q_0} |B_j|^{\alpha/nq_0} \|\chi_{B_j}\|_{p(\cdot)}^{-q_0} \chi_{B_j}(x)\Big) w_{A_i^{-1}}(x)\,dx.
\end{aligned}$$

The embedding $l^{p_0} \hookrightarrow l^{q_0}$ gives

$$\le C \sum_{i=1}^m \int_{\mathbb{R}^n} \bigg\{\sum_{j=1}^k \bigg(\frac{\lambda_j |B_j|^{\alpha/n} \chi_{B_j}(x)}{\|\chi_{B_j}\|_{p(\cdot)}}\bigg)^{p_0}\bigg\}^{q_0/p_0} w_{A_i^{-1}}(x)\,dx. \tag{3.3}$$

It is clear that if $\alpha = 0$, then the proposition follows from [4, Lemma 7.11], since $w_{A_i^{-1}} \in \mathcal{A}_1 \cap L^{(p(\cdot)/p_0)'}(\mathbb{R}^n)$ and $H_{\text{fin}}^{p_0,q/p_0}(w) = H_{\text{fin}}^{p_0,q/p_0}(w_{A_i^{-1}})$ as sets. For the case $0 < \alpha < n$, a computation gives $|B_j|^{\alpha/n}\chi_{B_j}(x) \le (M_{\alpha p_0/2}(\chi_{B_j})(x))^{2/p_0}$, so (3.3)

$$\begin{aligned}
&\le C \sum_{i=1}^m \int_{\mathbb{R}^n} \bigg\{\sum_{j=1}^k \bigg(\frac{\lambda_j (M_{\alpha p_0/2}(\chi_{B_j})(x))^{2/p_0}}{\|\chi_{B_j}\|_{p(\cdot)}}\bigg)^{p_0}\bigg\}^{q_0/p_0} w_{A_i^{-1}}(x)\,dx \\
&= C \sum_{i=1}^m \bigg\|\bigg\{\sum_{j=1}^k \frac{\lambda_j^{p_0} (M_{\alpha p_0/2}(\chi_{B_j})(\cdot))^2}{\|\chi_{B_j}\|_{p(\cdot)}^{p_0}}\bigg\}^{1/2}\bigg\|_{L^{2q_0/p_0}(w_{A_i^{-1}})}^{2q_0/p_0}
\end{aligned}$$

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because $p_0/2q_0 = \frac{1}{2} - \alpha p_0/2n$ and $[w_{A_i^{-1}}]^{p_0/2q_0} \in \mathcal{A}_{2,2q_0/p_0}$, by Lemma 2.13

$$\leq C \sum_{i=1}^{m} \bigg\| \bigg\{ \sum_{j=1}^{k} \frac{\lambda_j^{p_0} \chi_{B_j}(\cdot)}{\|\chi_{B_j}\|_{p(\cdot)}^{p_0}} \bigg\}^{1/2} \bigg\|_{L^2([w_{A_i^{-1}}]^{p_0/q_0})}^{2q_0/p_0}$$

$$= C \sum_{i=1}^{m} \bigg\| \sum_{j=1}^{k} \frac{\lambda_j^{p_0} \chi_{B_j}(\cdot)}{\|\chi_{B_j}\|_{p(\cdot)}^{p_0}} \bigg\|_{L^1([w_{A_i^{-1}}]^{p_0/q_0})}^{q_0/p_0}.$$

Since, for each $i = 1, \ldots, m$, $[w_{A_i^{-1}}]^{p_0/q_0} \in \mathcal{A}_1 \cap L^{(p(\cdot)/p_0)'}(\mathbb{R}^n)$ (see Lemma 2.1, Remarks 2.9 and 2.10), and $H_{\text{fin}}^{p_0,q/p_0}(w) = H_{\text{fin}}^{p_0,q/p_0}([w_{A_i^{-1}}]^{p_0/q_0})$ as sets, by [4, Lemma 7.11], we can take the infimum over all such decompositions to get

$$\|T_{\alpha,m} f\|_{L^{q_0}(w)} \leq C \sum_{i=1}^{m} \|f\|_{H^{p_0}([w_{A_i^{-1}}]^{p_0/q_0})},$$

for all $f \in H_{\text{fin}}^{p_0,q/p_0}(w)$. □

For $0 < \alpha < n$, let I_α be the Riesz potential given by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \tag{3.4}$$

where $f \in L^s(\mathbb{R}^n)$, and $1 \leq s < n/\alpha$.

We introduce the following discrete maximal, given $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$, we define

$$M_\phi^d f(x) = \sup_{j \in \mathbb{Z}} |(\phi^j * f)(x)|,$$

where $\phi^j(x) = 2^{jn}\phi(2^j x)$. From [15, Lemma 3.2 and Proof of Theorem 3.3], it follows that for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and all $0 < \theta < 1$

$$\mathcal{M}_N f(x) \leq C[M((M_\phi^d f)^\theta)(x)]^{1/\theta} \quad \text{for all } x \in \mathbb{R}^n, \tag{3.5}$$

if N is sufficiently large. This inequality gives the following lemma.

Lemma 3.4. *If $w \in \mathcal{A}_1$ and $0 < q_0 < 1$, then $\|f\|_{H^{q_0}(w)} \leq C\|M_\phi^d f\|_{L^{q_0}(w)}$.*

Proof. Let $0 < \theta < q_0$. Since $\mathcal{A}_1 \subset \mathcal{A}_{q_0/\theta}$, then the lemma follows from the inequality in (3.5) and [12, Theorem 9]. □

Proposition 3.5. *Let $0 < \alpha < n$. If I_α is the Riesz potential defined in (3.4) and $a(\cdot)$ is a $(p(\cdot), q/p_0)$-atom, $q/p_0 > n/\alpha$, such that $\int x^\beta a(x)\, dx = 0$ for all $|\beta| \leq 2\lfloor n(1/q_0 - 1)\rfloor + 3 + \lfloor \alpha \rfloor + n$, where $1/q_0 = 1/p_0 - \alpha/n$, then*

$$M_\phi^d(I_\alpha a)(x) \leq C|B|^{\alpha/n}\|\chi_B\|_{p(\cdot)}^{-1}[M(\chi_B)(x)]^{(n+k+1)/n} \quad \text{if } x \in \mathbb{R}^n \backslash B(x_0, 2r), \tag{3.6}$$

where $B = B(x_0, r)$ is the ball which $a(\cdot)$ is supported and $k = \lfloor n(1/q_0 - 1)\rfloor$.

PROOF. We observe that $2\lfloor n(1/q_0 - 1)\rfloor + 3 + \lfloor\alpha\rfloor + n > \lfloor n(1/p_0 - 1)\rfloor$; thus, $a(\cdot)$ is an atom with additional vanishing moments.

The same argument utilized to obtain the pointwise estimate that appears in (3.2) works if we consider the operator I_α instead $T_{m,\alpha}$, so

$$|I_\alpha a(x)| \leq C_{|\beta|} \frac{r^{n+|\beta|+1}}{\|\chi_B\|_{p(\cdot)}} |x - x_0|^{-n+\alpha-|\beta|+1}, \tag{3.7}$$

for all $x \in \mathbb{R}^n \backslash B(x_0, 2r)$, and all $0 \leq |\beta| \leq 2\lfloor n(1/q_0 - 1)\rfloor + 3 + \lfloor\alpha\rfloor + n$. Taking $|\beta| = 2\lfloor n(1/q_0 - 1)\rfloor + 3 + \lfloor\alpha\rfloor + n$ in (3.7), a simple computation gives

$$|I_\alpha a(x)| \leq C \frac{r^\alpha}{\|\chi_B\|_{p(\cdot)}} \left(1 + \frac{|x - x_0|}{r}\right)^{-2n-2k-3} \quad \forall x \in \mathbb{R}^n \backslash B(x_0, 2r) \tag{3.8}$$

where $k = \lfloor n(1/q_0 - 1)\rfloor$.

Let $1 < s < n/\alpha$, from the $L^s - L^{sn/(n-s\alpha)}$ boundedness of I_α and Remark 2.8,

$$\|I_\alpha a\|_{L^{sn/(n-s\alpha)}(B(x_0,2r))} \leq C\|a\|_s \leq C \frac{|B|^{(n-s\alpha)/sn}|B|^{\alpha/n}}{\|\chi_B\|_{p(\cdot)}}. \tag{3.9}$$

Taibleson and Weiss in [22] proved that

$$\int_{\mathbb{R}^n} x^\beta I_\alpha a(x)\, dx = 0, \tag{3.10}$$

for $0 \leq |\beta| \leq \lfloor n(1/q_0 - 1)\rfloor$.

Finally, we observe that the argument utilized in [15, Proof of Theorem 5.2] works in this setting, but considering now the estimates (3.8), (3.9) and the moment condition (3.10). Therefore, we get (3.6). □

REMARK 3.6. If $a(\cdot)$ is a $(p(\cdot), q/p_0)$-atom such that $\int x^\beta a(x)\, dx = 0$ for all $|\beta| \leq 2\lfloor n(1/q_0 - 1)\rfloor + 3 + \lfloor\alpha\rfloor + n$, where $1/q_0 = 1/p_0 - \alpha/n$, then from the inequality in (3.7), it follows that

$$|I_\alpha a(x)| \leq C \frac{|B|^{\alpha/n}}{\|\chi_B\|_{p(\cdot)}} [M(\chi_B)(x)]^{(n+k+1)/n},$$

for all $x \in \mathbb{R}^n \backslash B(x_0, 2r)$, and $k = \lfloor n(1/q_0 - 1)\rfloor$.

PROPOSITION 3.7. *For $0 < \alpha < n$, let I_α be the Riesz potential given by (3.4). Let $p(\cdot) \in M\mathcal{P}_0$, with $0 < p_0 < n/(n + \alpha)$. If $w \in \mathcal{A}_1 \cap L^{p_0/q_0(p(\cdot)/p_0)'}(\mathbb{R}^n) \cap RH_{(q/p_0)'}$, where $1/q_0 = 1/p_0 - \alpha/n$, then*

$$\|I_\alpha f\|_{H^{q_0}(w)} \leq C\|f\|_{H^{p_0}(w^{p_0/q_0})} \quad \text{for all } f \in H_{\text{fin}}^{p_0,q/p_0}(w),$$

where $q > \max\{1, p_+, p_0(1 + 2^{n+3}(\|M\|_{(p(\cdot)/p_0)'} + \|M\|_{(q(\cdot)/q_0)'})), p_0 n/\alpha\}$.

PROOF. We recall that in the atomic decomposition, we can always choose atoms with additional vanishing moments. This is, if l is any fixed integer with $l > \lfloor n(1/p_0 - 1)\rfloor$, then in the definition of the space $H_{\text{fin}}^{p(\cdot),q}(\mathbb{R}^n)$ we can assume that all moments up to order l of our atoms are zero. Thus, given $f \in H^{p(\cdot),q/p_0}_{\text{fin}}(\mathbb{R}^n)$, $f = \sum_{j=1}^k \lambda_j a_j$, where a_j are atoms with a moment condition up to order $2\lfloor n(1/q_0 - 1)\rfloor + 3 + \lfloor \alpha \rfloor + n$.

By Lemma 3.4 and since $0 < q_0 < 1$

$$\int_{\mathbb{R}^n} (\mathcal{M}_N(I_\alpha f)(x))^{q_0} w(x)\,dx \le C \int_{\mathbb{R}^n} (M^d_\phi(I_\alpha f)(x))^{q_0} w(x)\,dx \le C \sum_{j=1}^k \lambda_j^{q_0} \int_{\mathbb{R}^n} (M^d_\phi(I_\alpha a_j)(x))^{q_0} w(x)\,dx.$$

Thus, we estimate the last integral for an arbitrary atom $a(\cdot)$ supported on a ball $B = B(x_0, r)$.

$$\int_{\mathbb{R}^n} (M^d_\phi(I_\alpha a)(x))^{q_0} w(x)\,dx = \int_{B(x_0,2r)} (M^d_\phi(I_\alpha a)(x))^{q_0} w(x)\,dx + \int_{(B(x_0,2r))^c} (M^d_\phi(I_\alpha a)(x))^{q_0} w(x)\,dx = J_1 + J_2.$$

We first estimate J_1 and we use the fact that $M^d_\phi(I_\alpha a)(x) \le M(I_\alpha a)(x)$, for all $x \in \mathbb{R}^n$. $w \in \mathcal{A}_1$, and since the Hardy–Littlewood maximal operator satisfies Kolmogorov's inequality,

$$J_1 \le C w(B(x_0, 2r))^{1-q_0} \Big(\int_{\mathbb{R}^n} |I_\alpha a(x)| w(x)\,dx \Big)^{q_0}.$$

To get the desired estimate for J_1, it will suffice to show that

$$L = \int_{\mathbb{R}^n} |I_\alpha a(x)| w(x)\,dx \le C \frac{|B|^{\alpha/n}}{\|\chi_B\|_{p(\cdot)}} w(B(x_0, 2r)).$$

To prove this, we split the integral

$$L = \int_{B(x_0,2r)} |I_\alpha a(x)| w(x)\,dx + \int_{B(x_0,2r)^c} |I_\alpha a(x)| w(x)\,dx = L_1 + L_2.$$

To estimate L_1, we take $1 < s < n/\alpha$ such that $0 < 1/s - \alpha/n < p_0/q$, so if $\widetilde{s}$ is defined by $1/\widetilde{s} = 1/s - \alpha/n$, Hölder's inequality and the $L^s - L^{\widetilde{s}}$ boundedness of I_α give

$$L_1 \le \|I_\alpha a\|_{\widetilde{s}} \Big(\int_{B(x_0,2r)} [w(x)]^{\widetilde{s}'}\,dx \Big)^{1/\widetilde{s}'} \le C\|a\|_s \Big(\int_{B(x_0,2r)} [w(x)]^{\widetilde{s}'}\,dx \Big)^{1/\widetilde{s}'}$$

since $1 < s < q/p_0$, Remark 2.8 gives

$$\le C \frac{|B|^{1/s}}{\|\chi_B\|_{p(\cdot)}} \Big(\int_{B(x_0,2r)} [w(x)]^{\widetilde{s}'}\,dx \Big)^{1/\widetilde{s}'} = C \frac{|B|^{1+\alpha/n}}{\|\chi_B\|_{p(\cdot)}} \Big(\frac{1}{|B|} \int_{B(x_0,2r)} [w(x)]^{\widetilde{s}'}\,dx \Big)^{1/\widetilde{s}'}$$

a computation gives $1 < \widetilde{s}' < (q/p_0)'$, since $w \in RH_{(q/p_0)'}$ it follows that $w \in RH_{\widetilde{s}'}$ and thus

$$L_1 \le C \frac{|B|^{\alpha/n}}{\|\chi_B\|_{p(\cdot)}} w(B(x_0, 2r)).$$

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To estimate L_2, we use [12, Remark 3.6 and Theorem 9] to obtain

$$L_2 \leq C \frac{|B|^{\alpha/n}}{\|\chi_B\|_{p(\cdot)}} w(B(x_0, 2r)).$$

From the estimates of L_1, L_2 and since the weight w is doubling,

$$J_1 \leq C \frac{|B|^{q_0\alpha/n}}{\|\chi_B\|_{p(\cdot)}^{q_0}} w(B).$$

Now we estimate J_2. By Proposition 3.5 and since $w \in \mathcal{A}_1 \subset \mathcal{A}_{q_0(n+k+1)/n}$, once again by [12, Theorem 9],

$$\begin{aligned} J_2 &\leq C \frac{|B|^{q_0\alpha/n}}{\|\chi_B\|_{p(\cdot)}^{q_0}} \int_{\mathbb{R}^n} [M(\chi_B)(x)]^{q_0(n+k+1)/n} w(x)\, dx \\ &\leq C \frac{|B|^{q_0\alpha/n}}{\|\chi_B\|_{p(\cdot)}^{q_0}} \int_{\mathbb{R}^n} \chi_B(x) w(x)\, dx = C \frac{|B|^{q_0\alpha/n}}{\|\chi_B\|_{p(\cdot)}^{q_0}} w(B). \end{aligned}$$

Then

$$\int_{\mathbb{R}^n} (M_\phi^d(I_\alpha a)(x))^{q_0} w(x)\, dx = J_1 + J_2 \leq C \frac{|B|^{q_0\alpha/n}}{\|\chi_B\|_{p(\cdot)}^{q_0}} w(B).$$

So

$$\|I_\alpha f\|_{H^{q_0}(w)}^{q_0} \leq C \int_{\mathbb{R}^n} \left\{ \sum_{j=1}^k \left(\frac{\lambda_j |B_j|^{\alpha/n} \chi_{B_j}(x)}{\|\chi_{B_j}\|_{p(.)}} \right)^{q_0} \right\} w(x)\, dx$$

the embedding $l^{p_0} \hookrightarrow l^{q_0}$ gives

$$\leq C \int_{\mathbb{R}^n} \left\{ \sum_{j=1}^k \left(\frac{\lambda_j |B_j|^{\alpha/n} \chi_{B_j}(x)}{\|\chi_{B_j}\|_{p(.)}} \right)^{p_0} \right\}^{q_0/p_0} w(x)\, dx \tag{3.11}$$

a computation allows us to obtain $|B_j|^{\alpha/n}\chi_{B_j}(x) \leq (M_{\alpha p_0/2}(\chi_{B_j})(x))^{2/p_0}$, so (3.11)

$$\begin{aligned} &\leq C \int_{\mathbb{R}^n} \left\{ \sum_{j=1}^k \left(\frac{\lambda_j (M_{\alpha p_0/2}(\chi_{B_j})(x))^{2/p_0}}{\|\chi_{B_j}\|_{p(\cdot)}} \right)^{p_0} \right\}^{q_0/p_0} w(x)\, dx \\ &= C \left\| \left\{ \sum_{j=1}^k \frac{\lambda_j^{p_0} (M_{\alpha p_0/2}(\chi_{B_j})(\cdot))^2}{\|\chi_{B_j}\|_{p(\cdot)}^{p_0}} \right\}^{1/2} \right\|_{L^{2q_0/p_0}(w)}^{2q_0/p_0} \end{aligned}$$

because $p_0/2q_0 = \frac{1}{2} - \alpha p_0/2n$ and $w^{p_0/2q_0} \in \mathcal{A}_{2,2q_0/p_0}$, by Lemma 2.13

$$\leq C \left\| \left\{ \sum_{j=1}^k \frac{\lambda_j^{p_0} \chi_{B_j}(\cdot)}{\|\chi_{B_j}\|_{p(\cdot)}^{p_0}} \right\}^{1/2} \right\|_{L^2(w^{p_0/q_0})}^{2q_0/p_0} = C \left\| \sum_{j=1}^k \frac{\lambda_j^{p_0} \chi_{B_j}(.)}{\|\chi_{B_j}\|_{p(\cdot)}^{p_0}} \right\|_{L^1(w^{p_0/q_0})}^{q_0/p_0}.$$

Since $w^{p_0/q_0} \in \mathcal{A}_1 \cap L^{(p(\cdot)/p_0)'}(\mathbb{R}^n)$ (see Lemma 2.1 and Remark 2.10), and $H_{\text{fin}}^{p_0,q/p_0}(w^{p_0/q_0}) = H_{\text{fin}}^{p_0,q/p_0}(w) = H_{\text{fin}}^{p(\cdot),q/p_0}(\mathbb{R}^n)$ as sets, by [4, Lemma 7.11], we can take the infimum over all such decompositions to get

$$\|I_\alpha f\|_{H^{q_0}(w)} \leq C \|f\|_{H^{p_0}(w^{p_0/q_0})},$$

for all $f \in H_{\text{fin}}^{p_0,q/p_0}(w)$. □

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4. Main results

In the sequel, we will consider a measurable function $q(\cdot) : \mathbb{R}^n \to (0, \infty)$ such that $0 < q_0 < q_- \le q_+ < +\infty$, with $0 < q_0 < 1$. Given $0 \le \alpha < n$, we define $p(\cdot)$ by $1/p(\cdot) := 1/q(\cdot) + \alpha/n$ and p_0 by $1/p_0 := 1/q_0 + \alpha/n$, it is easy to check that $0 < p_0 < p_- \le p_+ \le n/\alpha$ and $0 < p_0 < n/(n + \alpha)$. If $q(\cdot)$ is such that the maximal operator is bounded on $L^{q(\cdot)/q_0}$ (that is $q(\cdot) \in M\mathcal{P}_0$), then by Lemma 2.5 we have that it is bounded on $L^{(q(\cdot)/q_0)'}$, now by Lemma 2.4 it follows that the maximal operator is bounded on $L^{(p(\cdot)/p_0)'}$ because $(p(\cdot)/p_0)' = q_0/p_0(q(\cdot)/q_0)'$. In the definition of the space $H^{p(\cdot),q/p_0}_{\text{fin}}(\mathbb{R}^n)$ we will assume $q > \max\{1, p_+, p_0(1 + 2^{n+3}\|M\|_{(p(\cdot)/p_0)'})\}$ if $\alpha = 0$, or $q > \max\{1, p_+, p_0(1 + 2^{n+3}(\|M\|_{(p(\cdot)/p_0)'} + \|M\|_{(q(\cdot)/q_0)'})), p_0 n/\alpha\}$ if $0 < \alpha < n$.

Proof of Theorem 1.1. The operator $T_{\alpha,m}$ is well defined on the elements of $H^{p(\cdot),q/p_0}_{\text{fin}}(\mathbb{R}^n)$. So, given $f \in H^{p(\cdot),q/p_0}_{\text{fin}}(\mathbb{R}^n)$, from Lemma 2.2,

$$\|T_{\alpha,m} f\|^{q_0}_{q(\cdot)} = \||T_{\alpha,m} f|^{q_0}\|_{q(\cdot)/q_0} \le C \sup \int |T_{\alpha,m} f(x)|^{q_0} |g(x)|\, dx$$

where the supremum is taken over all $g \in L^{(q(\cdot)/q_0)'}$ such that $\|g\|_{(q(\cdot)/q_0)'} \le 1$. Now we utilize the Rubio de Francia iteration algorithm with respect to $L^{(q(\cdot)/q_0)'}$. Given a function g, define

$$\mathcal{R}g(x) = \sum_{i=0}^{\infty} \frac{M^i g(x)}{2^i \|M\|^i_{(q(\cdot)/q_0)'}},$$

where $M^0 g = g$ and, for $i \ge 1$, $M^i g = M \circ \cdots \circ M g$ denotes i iterates of the Hardy–Littlewood maximal operator. The function $\mathcal{R}g$ satisfies:

(1) $|g(x)| \le \mathcal{R}g(x)$ for all $x \in \mathbb{R}^n$;
(2) $\|\mathcal{R}g\|_{(q(\cdot)/q_0)'} \le C\|g\|_{(q(\cdot)/q_0)'}$;
(3) $\mathcal{R}g \in \mathcal{A}_1$ and $[\mathcal{R}g]_{\mathcal{A}_1} \le 2\|M\|_{(q(\cdot)/q_0)'}$.

By Lemma 2.12 and our assumption on q $\mathcal{R}g \in RH_{(q/p_0)'}$; by these properties and since $H^{p(\cdot),q/p_0}_{\text{fin}}(\mathbb{R}^n) = H^{p_0,q/p_0}_{\text{fin}}(\mathcal{R}g)$ as sets, according to the ideas in [2, Proof of Theorem 5.28], Proposition 3.3 and taking account that $(p(\cdot)/p_0)' = q_0/p_0(q(\cdot)/q_0)'$ and $q(A_i x) = q(x)$ for each $i = 1, \ldots, m$

$$\|T_{\alpha,m} f\|_{L^{q(\cdot)}} \le C\|f\|_{H^{p(\cdot)}},$$

for all $f \in H^{p(\cdot),q/p_0}_{\text{fin}}(\mathbb{R}^n)$, so the theorem follows from the density of $H^{p(\cdot),q/p_0}_{\text{fin}}(\mathbb{R}^n)$ in $H^{p(\cdot)}(\mathbb{R}^n)$. □

Remark 4.1. Observe that Theorem 1.1 still holds for $m = 1$ and $0 < \alpha < n$. In particular, if $A_1 = I$, then the Riesz potential is bounded from $H^{p(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$.

Proof of Theorem 1.2. The operator I_α is well defined on $H^{p(\cdot),q/p_0}_{\text{fin}}(\mathbb{R}^n)$. So given $f \in H^{p(\cdot),q/p_0}_{\text{fin}}(\mathbb{R}^n)$, from Lemma 2.2,

$$\begin{aligned}\|I_\alpha f\|^{q_0}_{H^{q(\cdot)}} &\le C\|\mathcal{M}_N(I_\alpha f)\|^{q_0}_{L^{q(\cdot)}} = \|(\mathcal{M}_N(I_\alpha f))^{q_0}\|_{L^{q(\cdot)/q_0}} \\ &\le C \sup \int_{\mathbb{R}^n} (\mathcal{M}_N(I_\alpha f)(x))^{q_0} |g(x)|\, dx\end{aligned}$$

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where the supremum is taken over all $g \in L^{(q(\cdot)/q_0)'}$ such that $\|g\|_{(q(\cdot)/q_0)'} \leq 1$. From Proposition 3.7 and proceeding as in the proof of Theorem 1.1 we apply the Rubio de Francia iteration algorithm with respect to $L^{(q(\cdot)/q_0)'}$ to obtain

$$\|I_\alpha f\|_{H^{q(\cdot)}} \leq C\|f\|_{H^{p(\cdot)}},$$

for all $f \in H_{\text{fin}}^{p(\cdot),q/p_0}(\mathbb{R}^n)$, so the theorem follows from the density of $H_{\text{fin}}^{p(\cdot),q/p_0}(\mathbb{R}^n)$ in $H^{p(\cdot)}(\mathbb{R}^n)$. □

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