Compositio Mathematica **135:** 49–101, 2003.

b-Functions of Prehomogeneous Vector Spaces of Dynkin–Kostant Type for Exceptional Groups

KIMITOSHI UKAI
Graduate School of Mathematics, Nagoya University, Nagoya, 464-8602, Japan.
e-mail: k_ukai@hkg.odn.ne.jp

(Received: 1 January 2001; accepted in final form: 3 May 2001)

Abstract. The purpose of this paper is to determine the b-functions of all the prehomogeneous vector spaces associated to nilpotent orbits in the Dynkin–Kostant theory for all the complex simple algebraic groups of exceptional type. Our method to calculate b-functions is based on the structure theorem for b-functions of several variables and the functional equations for them.

Mathematics Subject Classifications (2000). Primary: 20G05; Secondary: 20G20, 11S90.

Key words. b-functions; prehomogeneous vector spaces; nilpotent orbits.

Introduction

The purpose of this paper is to determine the b-functions of all the prehomogeneous vector spaces of Dynkin–Kostant type for all the complex simple algebraic groups of exceptional type.

Let G be a complex reductive algebraic group and let $\mathfrak{g}$ be the Lie algebra of G. For each nilpotent element N of $\mathfrak{g}$, the ad(G)-orbit of N gives a prehomogeneous vector space via the Dynkin–Kostant theory. The obtained prehomogeneous vector spaces are called prehomogeneous vector spaces of Dynkin–Kostant type. Their b-functions play an important role in the harmonic analysis involving the ad(G)-orbit of the original nilpotent element, and therefore we want to know their explicit form. For classical groups, the b-functions of such prehomogeneous vector spaces have been studied by F. Sato [19] and Y. Kaneko [9]. In this paper we determine them for exceptional groups.

For exceptional groups, the list of all the weighted Dynkin diagrams associated to nilpotent orbits via the Dynkin–Kostant theory is given by A. J. Alekseevsky [1] and A. G. Elašvili [3]. We give the explicit form of b-functions for all such weighted Dynkin diagrams. Our method of calculating b-functions is as follows. First we determine the characters of the irreducible relative invariants. Next we calculate the a-functions of several variables defined in Lemma 1.3.1. Then the b-functions of relative invariants are expressed with some positive rational numbers $\alpha_{j,r}$ in Corollary 1.3.6. Hence, it is enough to determine these numbers $\alpha_{j,r}$ for our purpose. We mainly use the following fact to determine them. It is known that a certain functional equation holds for a b-function of several variables defined in Lemma 1.3.4, which is due to M. Sato [23]. This equation gives a relation among positive rational numbers $\alpha_{j,r}$.

The plan of this paper is as follows. In Section 1, we review some basic results on prehomogeneous vector spaces. In Section 2, we exhibit our method of calculating the b-functions by taking up the prehomogeneous vector space represented by the diagram $\underset{0}{020020}$. We determine the b-functions of all the prehomogeneous vector spaces of Dynkin–Kostant type for all the complex simple algebraic groups of exceptional type. We summarize these results in the tables of Section 4.

1. Preliminaries

In this section we review basic notions about the theory of prehomogeneous vector spaces and give some properties used in the later sections.

1.1. PREHOMOGENEOUS VECTOR SPACES

Let G be a connected reductive group defined over the complex number field $\mathbb{C}$ and let $\rho: G \to GL(V)$ be a finite dimensional rational representation. The triple (G, ρ, V) (or simply the pair (G, V)) is called a *prehomogeneous vector space* if V has an open dense G-orbit, say $O_0 = Gv_0$. Let f be a nonzero polynomial function on V and $\phi \in \mathrm{Hom}(G, \mathbb{C}^\times)$. We call f a *relative invariant* whose character is ϕ if $f(gv) = \phi(g)f(v)$ for all $g \in G$ and $v \in V$. Then (G, V) and f have the following properties given in Propositions 1.1.1–1.1.6. The proofs are found in the literature assigned there.

PROPOSITION 1.1.1 ([21, §4, Proposition 3]). *Let f_1 and f_2 be relative invariants which have the same character. Then f_1 is a constant multiple of f_2.*

PROPOSITION 1.1.2 ([21, §4, Proposition 5]). *Let $S_1, \ldots, S_l$ be the irreducible components of $V \backslash O_0$ with codimension one and suppose each S_i is the zeros of some irreducible polynomial f_i, namely $S_i = \{v \in V; f_i(v) = 0\}$. Then $f_1, \ldots, f_l$ are algebraically independent relative invariants. Every relative invariant f is of the form $f = cf_1^{m_1} \cdots f_l^{m_l}$ $(c \in \mathbb{C}, m_i \in \mathbb{Z})$.*

We call the polynomials $f_1, \ldots, f_l$ the *fundamental system of relative invariants.*

PROPOSITION 1.1.3 ([21, §4, Proposition 19]). *Let $X^*(G, V)$ be the totality of the characters associated to some relative invariants of (G, V) and let G_{v_0} be the isotropy group at the point v_0 of the open orbit. Then*

$$X^*(G, V) = \{\phi \in \mathrm{Hom}(G, \mathbb{C}^\times); \phi|_{G_{v_0}} \equiv 1\}.$$

PROPOSITION 1.1.4 ([4, Lemma 1.5]). *Let $\rho^\vee\colon G \to GL(V^\vee)$ be the contragradient representation of ρ. Then the triple $(G, \rho^\vee, V^\vee)$ is a prehomogeneous vector space. It has a relative invariant of degree $d := \deg f$, say $f^\vee$, whose character is ϕ^{-1}.*

PROPOSITION 1.1.5 ([4, Lemmas 1.6 and 1.7]). *There exists a polynomial*

$$b_f(s) = b_0 s^d + b_1 s^{d-1} + \cdots + b_d$$

with $b_0 \neq 0$ such that

$$f^\vee(\mathrm{grad}_x) f(x)^{s+1} = b_f(s) f(x)^s,$$
$$f(\mathrm{grad}_y) f^\vee(y)^{s+1} = b_f(s) f^\vee(y)^s,$$

for all $s \in \mathbb{Z}_{\geqslant 0}$. Here we put

$$\mathrm{grad}_x := \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) \quad \text{and} \quad \mathrm{grad}_y := \left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right).$$

We call b_f the *b-function* of f. We put $a_f := b_0$. The following fact about the b-functions is known.

PROPOSITION 1.1.6 ([11, Corollary 5.2]). *Let $b_f(s) = b_0 \prod_{j=1}^d (s + \alpha_j)$. Then each α_j is a positive rational number.*

DEFINTION 1.1.7 (Direct sum of prehomogeneous vector spaces). Let V be a direct sum of G-submodules $V_i \subset V$ and assume that any relative invariant $f \in \mathbb{C}[V]$ is of the form $f(v) = c \prod_i h_i(v_i)$ for $v = \sum_i v_i \in V$ with $c \in \mathbb{C}^\times$ and $h_i \in \mathbb{C}[V_i]$. Then we call V *a direct sum of subspaces $V_i \subset V$ as prehomogeneous vector spaces*. It is clear that $b_f(s) = \prod_i b_{h_i}(s)$.

1.2. PREHOMOGENEOUS VECTOR SPACES ASSOCIATED TO NILPOTENT ORBITS

Let G be a reductive group defined over the complex number field $\mathbb{C}$ and let $\mathfrak{g}$ be the Lie algebra of G. The reference of this subsection is [26, Chapter III, Section 4].

LEMMA 1.2.1. *Let N be a nilpotent element of $\mathfrak{g}$. Then there exists a filtration $\{W_{\leqslant m}\mathfrak{g}\}_{m \in \mathbb{Z}}$ of $\mathfrak{g}$ satisfying following conditions.*

(1) $W_{\leqslant m-1}\mathfrak{g} \subset W_{\leqslant m}\mathfrak{g}$.
(2) $W_{\leqslant m}\mathfrak{g} = 0$ *if* $m \ll 0$, *and* $W_{\leqslant m}\mathfrak{g} = \mathfrak{g}$ *if* $m \gg 0$.
(3) $\mathrm{ad}(N)(W_{\leqslant m}\mathfrak{g}) \subset W_{\leqslant m-2}\mathfrak{g}$.
(4) $\mathrm{ad}(N)^m$ *is an isomorphism from* $\mathrm{gr}_m\mathfrak{g}$ *to* $\mathrm{gr}_{-m}\mathfrak{g}$, *where* $\mathrm{gr}_m\mathfrak{g} = W_{\leqslant m}\mathfrak{g}/W_{\leqslant m-1}\mathfrak{g}$.

Let $W_{\leqslant m}G$ ($m \leqslant 0$) be the subgroup of G whose Lie algebra is $W_{\leqslant m}\mathfrak{g}$, and put $\mathrm{gr}_0 G := W_{\leqslant 0}G/W_{\leqslant -1}G$.

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

PROPOSITION 1.2.2. *Consider the action of* $\mathrm{gr}_0 G$ *on* $\mathrm{gr}_{-2}\mathfrak{g}$ *induced by the adjoint action of* G *on* $\mathfrak{g}$. *Then the pair* $(\mathrm{gr}_0 G, \mathrm{gr}_{-2}\mathfrak{g})$ *is a prehomogeneous vector space.*

We call the prehomogeneous vector space obtained as $(\mathrm{gr}_0 G, \mathrm{gr}_{-2}\mathfrak{g})$ a *prehomogeneous vector space of Dynkin–Kostant type*. Since it is known that the generic isotropy group of $\mathrm{gr}_0 G$ is reductive, these prehomogeneous vector spaces are regular.

It is known that there exists a homomorphism $\pi\colon SL_2 \to G$ such that $(d\pi)\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) = N$. Let T be a maximal torus of G containing $\left\{\pi\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right); t \in \mathbb{C}^\times\right\}$. Put $\mathfrak{h} := \mathrm{Lie}(T)$, and choose a root basis $\{\alpha_1, \ldots, \alpha_l\} \subset \mathfrak{h}^\vee$ as in [26, III, 4.24]. Consider the Dynkin diagram whose vertices correspond to the root basis. It is known that

$$\alpha_i\left(\pi\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) = t^{m_i} \quad (m_i = 0, 1, 2).$$

Write the number m_i on the vertex associated to α_i. The resulting diagram is called the *weighted Dynkin-diagram*.

1.3. *a*-FUNCTIONS AND *b*-FUNCTIONS

We give here the definitions of *a*-functions and *b*-functions of several variables and some properties of them. The references of this subsection are [8, §8] and [22, §3, 4].

Let $f_1, \ldots, f_l$ be the fundamental system of the irreducible relative invariants of a prehomogeneous vector space (G, V) and let $f_1^\vee, \ldots, f_l^\vee$ the irreducible relative invariants of $(G, V^\vee)$ such that the characters of f_i and $f_i^\vee$ are the inverse of each other. We put $\underline{f} := (f_1, \ldots, f_l)$, $\underline{f}^\vee := (f_1^\vee, \ldots, f_l^\vee)$, and $V_{\underline{f}} := \bigcap_{i=1}^l V_{f_i}$. For each l-tuple $\underline{s} = (s_1, \ldots, s_l)$, we put $\underline{f}^{\underline{s}} := \prod_{i=1}^l f_i^{s_i}$ and $\underline{f}^{\vee\underline{s}} := \prod_{i=1}^l f_i^{\vee s_i}$.

First we define the *a*-functions of several variables and give their properties.

LEMMA 1.3.1 ([22, Proposition 10]). *For any l-tuple $\underline{m} = (m_1, \ldots, m_l)$, we have*

$$\underline{f}^{\underline{m}}(v)\underline{f}^{\vee\underline{m}}(\mathrm{grad}\log\underline{f}^{\underline{s}}(v)) = a_{\underline{m}}(\underline{s}),$$

for all $v \in V_{\underline{f}}$ with some nonzero homogeneous polynomial $a_{\underline{m}}(\underline{s})$ which is independent of v.

We call $a_{\underline{m}}(\underline{s})$ the *a*-functions of $\underline{f}$. When $\underline{m} = \varepsilon_i := (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 appears only at the ith place, we write $a_i(\underline{s})$ instead of $a_{\varepsilon_i}(\underline{s})$ for an abbreviation. We can easily see that $a_{\underline{m}} = \prod_{i=1}^l a_i^{m_i}$ by definition. The following theorem about the structures of *a*-functions $a_{\underline{m}}(\underline{s})$ is given in [22, §3, 4].

THEOREM 1.3.2 ([22, Theorem 1]). *The a-function $a_{\underline{m}}(\underline{s})$ is expressed as*

$$a_{\underline{m}}(\underline{s}) = \underline{A}^{\underline{m}} \prod_{j=1}^N \left(\gamma_j(\underline{s})^{\gamma_j(\underline{m})}\right)^{\mu_j}.$$

Here $\underline{A}^{\underline{m}} := \prod_{i=1}^{l} A_i^{m_i}$ *with* $A_i \in \mathbb{C}^{\times}$, $N \in \mathbb{Z}_{>0}$, *and* $\mu_j \in \mathbb{Z}_{>0}$, *while each* $\gamma_j(\underline{s})$ *is a* $\mathbb{Z}$*-linear function* $\sum_{i=1}^{l} \gamma_{ij} s_i$ *with* $\gamma_{ij} \in \mathbb{Z}_{\geqslant 0}$ *and* $\mathrm{GCD}(\gamma_{1j}, \ldots, \gamma_{lj}) = 1$.

It is clear that $a_{\underline{f}^{\underline{m}}} = a_{\underline{m}}(\underline{m})$ by definition. Hence we have the following corollary by Theorem 1.3.2.

COROLLARY 1.3.3. *The leading coefficient* $a_{\underline{f}^{\underline{m}}}$ *of the* b*-function* $b_{\underline{f}^{\underline{m}}}(s)$ *of the* relative invariant $\underline{f}^{\underline{m}}$ $(\underline{m} \in (\mathbb{Z}_{\geqslant 0})^l)$ is given by $a_{\underline{f}^{\underline{m}}} = \underline{A}^{\underline{m}} \prod_{j=1}^{N} \big(\gamma_j(\underline{m})^{\gamma_j(\underline{m})}\big)^{\mu_j}$.

Next we define the b-functions of several variables and give their properties.

LEMMA 1.3.4 ([22, Proposition 14]). *For any* l*-tuple* $\underline{m} = (m_1, \ldots, m_l) \in (\mathbb{Z}_{\geqslant 0})^l$, *we have*

$$\underline{f}^{\vee \underline{m}}(\mathrm{grad})\underline{f}^{\underline{s}+\underline{m}} = b_{\underline{m}}(\underline{s})\underline{f}^{\underline{s}}$$

with some nonzero polynomial $b_{\underline{m}}(\underline{s})$.

These polynomials $b_{\underline{m}}(\underline{s})$ are called the b*-functions* of $\underline{f}$. We write $b_i(\underline{s})$ instead of $b_{\varepsilon_i}(\underline{s})$ for an abbreviation. Let a-functions $a_{\underline{m}}(\underline{s})$ be as in Theorem 1.3.2. The following theorem about the structures of $b_{\underline{m}}(\underline{s})$ is given in [22, §3, 4] and [11].

THEOREM 1.3.5 ([22, Theorem 2], [11]). *The* b*-function* $b_{\underline{m}}(\underline{s})$ *is expressed as*

$$b_{\underline{m}}(\underline{s}) = \underline{A}^{\underline{m}} \prod_{j=1}^{N} \prod_{v=0}^{\gamma_j(\underline{m})-1} \prod_{r=1}^{\mu_j} \big(\gamma_j(\underline{s}) + \alpha_{j,r} + v\big)$$

with some $\alpha_{j,r} \in \mathbb{Q}_{>0}$.

It is clear that $b_{\underline{f}^{\underline{m}}}(s) = b_{\underline{m}}(\underline{m}s)$ by definition. Hence, we have the following corollary by Theorem 1.3.5.

COROLLARY 1.3.6. *The* b*-function* $b_{\underline{f}^{\underline{m}}}(s)$ *of the relative invariant* $\underline{f}^{\underline{m}}$ $(\underline{m} \in (\mathbb{Z}_{\geqslant 0})^l)$ *is given by*

$$b_{\underline{f}^{\underline{m}}}(s) = \underline{A}^{\underline{m}} \prod_{j=1}^{N} \prod_{v=0}^{\gamma_j(\underline{m})-1} \prod_{r=1}^{\mu_j} \big(\gamma_j(\underline{m})s + \alpha_{j,r} + v\big).$$

For any regular prehomogeneous vector space, the b-function of several variables satisfies a certain functional equation, which is due to M. Sato [23]. We note that it is known that a prehomogeneous vector space of the Dynkin–Kostant type is regular.

LEMMA 1.3.7. *If (G, V) is a regular prehomogeneous vector space, there exists a relative invariant whose character is* $\det(g|_V)^2$. *Let* $\underline{f}^{2\underline{\kappa}}$ *be such a relative invariant.*

THEOREM 1.3.8 ([23, Theorem 4]). *Put* $\beta_{\gamma_j}(u) := \prod_{r=1}^{\mu_j}(u + \alpha_{j,r})$, *and let* $\underline{\kappa}$ *be in Lemma 1.3.7. Then for each j, we have the functional equation*

$$\beta_{\gamma_j}(u) = (-1)^{\mu_j}\beta_{\gamma_j}(-u - \gamma_j(\underline{\kappa}) - 1)$$

holds for any u.

Comparing zeros of both sides of the equation in Theorem 1.3.8, we get the following corollary.

COROLLARY 1.3.9. *For each j, we have*

$$\{\alpha_{j,r}\ (1 \leqslant r \leqslant \mu_j)\} = \{\gamma_j(\underline{\kappa}) + 1 - \alpha_{j,r}\ (1 \leqslant r \leqslant \mu_j)\}.$$

2. Method of Calculating *b*-Functions

In this section, we shall exhibit how we actually calculate the *b*-functions using the facts in Section 1.

Consider the weighted Dynkin diagram $\overset{\displaystyle 020020}{0}$ as an example. The prehomogeneous vector space arising from this diagram is given as follows. Let

$$G = GL_4 \times GL_2 \times SL_2 \quad \text{and} \quad V = \bigwedge^2 \mathbb{C}^4 + \bigwedge^2 \mathbb{C}^4 + M_{2,4}(\mathbb{C}),$$

where $\bigwedge^2 \mathbb{C}^4 = \{X \in M_4(\mathbb{C});\ {}^tX = -X\}$. Define the action of G on V by

$$gv = \left((g_1 X {}^t g_1, g_1 Y {}^t g_1) {}^t g_2, g_3 W g_1^{-1}\right)$$

for $g = (g_1, g_2, g_3) \in G$ and $v = (X, Y, W) \in V$.

The kernel of this action is $\{(\varepsilon, 1, \varepsilon);\ \varepsilon = \pm 1\}$. Let

$$v_0 = \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right).$$

Then the isotropy subgroup at v_0 is given by

$$G_{v_0} = \left\{\left(\begin{pmatrix} A & 0 \\ 0 & \varepsilon A \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}, A\right);\ A \in SL_2,\ \varepsilon = \pm 1\right\}.$$

This prehomogeneous vector space has two irreducible relative invariants f_1 and f_2

given by $f_1(v) = \det({}^t(WX), {}^t(WY))$ and

$$f_2(v) = \text{discriminant of the binary quadratic form } \mathrm{Pf}(X\xi + Y\eta)$$

in the variables $(\xi, \eta) \in \mathbb{C}$. Their characters are

$$\phi_1(g) = (\det g_1)(\det g_2)^2 \quad \text{and} \quad \phi_2(g) = (\det g_1)^2(\det g_2)^2.$$

Let $d\phi_i\colon \mathrm{Lie}(G) \to \mathbb{C}$ be the character of $\mathrm{Lie}(G)$ induced by ϕ_i and let $\langle \cdot, \cdot \rangle\colon V^\vee \times V \to \mathbb{C}$ be the pairing between $V^\vee$ and V. Then we have the equation $\langle \mathrm{grad}\log f_i(v), Av \rangle = d\phi_i(A)$ for any $v \in V_{f_i}$ and $A \in \mathrm{Lie}(G)$. See [4, Lemma 1.9 (1)]. Solving these equations, we have that

$$\mathrm{grad}\log f_1(v_0) = \left(\begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \right),$$

$$\mathrm{grad}\log f_2(v_0) = \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right).$$

We can calculate a-functions $a_1(\underline{s})$ and $a_2(\underline{s})$ along the definition in Lemma 1.3.1 as

$$a_1(\underline{s}) = s_1^4(s_1 + s_2)^2(2s_1 + s_2)^2,$$
$$a_2(\underline{s}) = s_2(s_1 + s_2)^2(2s_1 + s_2).$$

Hence we have that

$$N = 4,$$
$$\gamma_1(\underline{s}) = s_1, \qquad \mu_1 = 4,$$
$$\gamma_2(\underline{s}) = s_2, \qquad \mu_2 = 1,$$
$$\gamma_3(\underline{s}) = s_1 + s_2, \qquad \mu_3 = 2,$$
$$\gamma_4(\underline{s}) = 2s_1 + s_2, \qquad \mu_4 = 1,$$

in Theorem 1.3.2. The leading coefficient $a_{\underline{f}^{\underline{m}}}$ of the b-function $b_{\underline{f}^{\underline{m}}}(s)$ of the relative invariant $\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ $(m_1, m_2 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$a_{\underline{f}^{\underline{m}}} = \{(m_1)^{m_1}\}^4 (m_2)^{m_2} \{(m_1 + m_2)^{m_1 + m_2}\}^2 (2m_1 + m_2)^{2m_1 + m_2}.$$

Since the a-functions are as above, the b-function of the relative invariant

$\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ $(m_1, m_2 \in \mathbb{Z}_{\geqslant 0})$ is expressed as

$$\begin{aligned} b_{\underline{f}^{\underline{m}}}(s) = & \left\{ \prod_{v=0}^{m_1-1} \prod_{r=1}^{4} (m_1 s + \alpha_{1,r} + v) \right\} \left\{ \prod_{v=0}^{m_2-1} (m_2 s + \alpha_{2,1} + v) \right\} \times \\ & \times \left\{ \prod_{v=0}^{m_1+m_2-1} \prod_{r=1}^{2} ((m_1 + m_2)s + \alpha_{3,r} + v) \right\} \times \\ & \times \left\{ \prod_{v=0}^{2m_1+m_2-1} ((2m_1 + m_2)s + \alpha_{4,1} + v) \right\} \end{aligned} \tag{1}$$

with some $\alpha_{j,r} \in \mathbb{Q}_{>0}$ by Corollary 1.3.6.

For the purpose to determine positive rational numbers $\alpha_{j,r}$, we use Corollary 1.3.9. Since $\det(g|_V)^2 = (\det g_1)^8 (\det g_2)^{12}$, $\underline{\kappa}$ in Lemma 1.3.7 is given by $\underline{\kappa} = (2, 1)$. By Corollary 1.3.9, we get that

$$\{\alpha_{1,r}(1 \leqslant r \leqslant 4)\} = \{3 - \alpha_{1,r}(1 \leqslant r \leqslant 4)\}, \tag{2}$$

$$\{\alpha_{2,1}\} = \{2 - \alpha_{2,1}\}, \tag{3}$$

$$\{\alpha_{3,r}(1 \leqslant r \leqslant 2)\} = \{4 - \alpha_{3,r}(1 \leqslant r \leqslant 2)\}, \tag{4}$$

$$\{\alpha_{4,1}\} = \{6 - \alpha_{4,1}\}. \tag{5}$$

Hence, we get that $\alpha_{2,1} = 1$ from (3) and $\alpha_{4,1} = 3$ from (5). Now, we show that $(s_1 + 1)$ divides $b_1(\underline{s})$, that is, $\{\alpha_{1,r}\ (1 \leqslant r \leqslant 4)\}$ contains 1. We have that

$$\begin{aligned} b_1(\underline{s}) \underline{f}^{\underline{s}}(x) &= b_1(\underline{s})(f_1^{s_1} f_2^{s_2})(x) \\ &= f_1^{\vee}(\mathrm{grad})\left(f_1^{s_1+1} f_2^{s_2}\right)(x) \\ &= (s_1 + 1) Q(\underline{s}, x)(f_1^{s_1} f_2^{s_2})(x) + f_1^{s_1+1}(x)\{f_1^{\vee}(\mathrm{grad}) f_2^{s_2}(x)\} \end{aligned}$$

with a differential operator $Q(\underline{s}, x)$ whose coefficients belong to $\mathbb{C}[\underline{s}, x]$. Since $f_1^{\vee}(\mathrm{grad}) f_2^{s_2}(x)$ is a relative invariant whose character is $\phi_1^{-1} \phi_2^{s_2}$, we get that $f_1^{\vee}(\mathrm{grad}) f_2^{s_2}(x) = c f_1^{-1}(x) f_2^{s_2}(x)$ with $c \in \mathbb{C}$ by Proposition 1.1.1. On the other hand, $f_1^{\vee}(\mathrm{grad}) f_2^{s_2}(x)$ is a polynimial. Hence we have that $f_1^{\vee}(\mathrm{grad}) f_2^{s_2}(x) = 0$, and that $(s_1 + 1)$ divides $b_1(\underline{s})$. Thus we may put $\alpha_{1,3} = 1$ and $\alpha_{1,4} = 2$ from (2). Then (1) yields that

$$
\begin{aligned}
b_{\underline{f}^{\underline{m}}}(s) = & \left\{ \prod_{\nu=0}^{m_1-1} (m_1 s + 1 + \nu)(m_1 s + 2 + \nu) \times \right. \\
& \left. \times \prod_{r=1}^{2} \big(m_1 s + \alpha_{1,r} + \nu\big) \right\} \left\{ \prod_{\nu=0}^{m_2-1} (m_2 s + 1 + \nu) \right\} \times \\
& \times \left\{ \prod_{\nu=0}^{m_1+m_2-1} \prod_{r=1}^{2} \big((m_1 + m_2)s + \alpha_{3,r} + \nu\big) \right\} \times \\
& \times \left\{ \prod_{\nu=0}^{2m_1+m_2-1} ((2m_1 + m_2)s + 3 + \nu) \right\}
\end{aligned}
\tag{6}
$$

with $\alpha_{j,r} \in \mathbb{Q}_{>0}$ such that $\alpha_{1,1} + \alpha_{1,2} = 3$ and $\alpha_{3,1} + \alpha_{3,2} = 4$.

If we take $\underline{m} = (0, 1)$, then $b_{\underline{f}^{\underline{m}}}(s)$ is the b-function of f_2, which is given by

$$b_{f_2}(s) = (s + 1)(s + 3)\big(s + \tfrac{3}{2}\big)\big(s + \tfrac{5}{2}\big).$$

See [13, (15)]. Hence we get that $\{\alpha_{3,1}, \alpha_{3,2}\} = \{\tfrac{3}{2}, \tfrac{5}{2}\}$.

In order to determine the remaining $\alpha_{j,r}$'s, we consider the localization of $\underline{f}^{\underline{m}}$.

In general, the local b-function of a regular function f in a neighborhood of a point $v' \in V$ is defined by the polynomial $b_{f,v'}(s)$ of minimal degree such that $P(s, x, \partial) f^{s+1} = b_{f,v'}(s) f^s$ for some $P(s, x, \partial)$ whose coefficients are regular in the neighborhood of v'.

For any $v' \in V$, take a linear subspace $W' \subset V$ such that $T_{v'}(Gv') \oplus W' = V$, and define a polynomial $f_{v'}(w)$ in the variable $w \in W'$ by $f_{v'}(w) := f(v' + w)$. Then $f_{v'}(w)$ is a relative invariant with respect to the action of $G_{v'}$ on W'. Define a map $\mu\colon G \times (v' + W') \to V$ by $\mu(g, v' + w) := gv' + gw$. Then we have that

$$(\mu^* f)(g, v' + w) := f(\mu(g, v' + w)) = f(gv' + gw) = \phi(g) f_{v'}(w). \tag{7}$$

Let e be the unit of G. Since the map $\mu\colon G \times (v' + W') \to V$ is smooth in the neighborhood of (e, v'), we have $b_{f,v'} = b_{\mu^* f,(e,v')}$. Hence, the local b-function $b_{f,v'}$ coincides with the b-function of $f_{v'}$. For any v'' in the G-orbit containing v', the local b-function $b_{f,v''}$ coincides with $b_{f,v'}$ from (7). Consider the equation $P(s, x, \partial) f^{s+1} = b_{f,0}(s) f^s$ in the neighborhood of 0. Since the coefficients of $P(s, x, \partial)$ is regular in the neighborhood of 0 and since the above v'' can be taken in this neighborhood, this equation also holds in the neighborhood of v''. Hence $b_{f,v'} (= b_{f,v''})$ divides $b_{f,0}$, which is the b-function of f.

Now recall our situation. Let

$$
v' = \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right),
$$

and take $W' = \{(0, 0, W)\} \subset V$. Then the isotropy algebra $\mathfrak{g}_{v'}$ is isomorphic to $\mathfrak{gl}_2 \times \mathfrak{gl}_2 \times \mathfrak{sl}_2$, and its action on W' is $\Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1$. The relative invariant $(f_1)_{v'}$

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

is the irreducible relative invariant on W', whose b-function is

$$b_{(f_1)_{V'}}(s) = (s+1)(s+2)\left(s+\tfrac{3}{2}\right)^2.$$

The relative invariant $(f_2)_{V'}$ is a constant. Hence, the local b-function $b_{\underline{f}^{\underline{m}},V'}$ coincides the b-function of the relative invariant $\left(f_1^{m_1}\right)_{V'}$. Thus we have that

$$b_{\underline{f}^{\underline{m}},V'}(s) = \prod_{v=0}^{m_1-1}(m_1 s+1+v)(m_1 s+2+v)\left(m_1 s+\tfrac{3}{2}+v\right)^2.$$

Since it devides the b-function of $\underline{f}^{\underline{m}}$, we get that $\alpha_{1,1} = \alpha_{1,2} = \frac{3}{2}$.

Therefore the b-function of the relative invariant $\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ $(m_1, m_2 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$\begin{aligned} b_{\underline{f}}^{\underline{m}}(s) = {} & \left\{\prod_{v=0}^{m_1-1}(m_1 s+1+v)(m_1 s+2+v)\left(m_1 s+\tfrac{3}{2}+v\right)^2\right\}\times \\ & \times \left\{\prod_{v=0}^{m_2-1}(m_2 s+1+v)\right\}\times \\ & \times \left\{\prod_{v=0}^{m_1+m_2-1}\left((m_1+m_2)s+\tfrac{3}{2}+v\right)\left((m_1+m_2)s+\tfrac{5}{2}+v\right)\right\}\times \\ & \times \left\{\prod_{v=0}^{2m_1+m_2-1}((2m_1+m_2)s+3+v)\right\}. \end{aligned}$$

3. *b*-Functions of Prehomogeneous Vector Spaces of the Dynkin–Kostant Type

The list of all the weighted Dykin diagrams associated to nilpotent orbits via the Dynkin–Kostant theory is given in [1] and [3] for exceptional groups. Some of such diagrams are not necessary as far as we are exclusively interested in the prehomogeneous vector spaces associaed to them.

(1) Suppose that a diagram contains 1. For any root $\alpha = \sum_i c_i \alpha_i$, put $h(\alpha) := \sum_i c_i m_i$ with the same m_i as in Section 1.2. Let $\mathfrak{g}' = \mathfrak{h} \oplus (\bigoplus_{\alpha; h(\alpha)\in 2\mathbb{Z}} \mathfrak{g}_\alpha)$. Then the prehomogeneous vector space arising from $(\mathfrak{g}', N)$ is the same as the one arising from $(\mathfrak{g}, N)$.

(2) If a diagram contains two 2's connected by an edge, consider two diagrams Γ_1 and Γ_2 which can be obtained from Γ by removing the edge. Then the prehomogeneous vector space arising from Γ is the direct sum of the two prehomogeneous vector spaces arising from Γ_1 and Γ_2 in the sense of Definition 1.1.7.

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

Hence, we do not need to consider such diagrams. The remaining diagrams appear in Subsection 3.1 for type G_2; in Subsection 3.2 for type F_4; in Subsection 3.3 for type E_6; in Subsection 3.4 for type E_7; in Subsection 3.5 for type E_8. We determine the b-functions of prehomogeneous vector spaces of Dynkin–Kostant type arising from these diagrams.

Some prehomogeneous vector spaces are irreducible or simple. Their b-functions have been determined. The summaries appear in [13] and [21] for iirreducible cases and in [10] and [14] for simple cases. For other prehomogeneous vector spaces whose b-functions have been already determined, the literatures are given there. The remaining task is to determine the b-functions of the other prehomogeneous vector spaces. The method of calculating b-fuctions is the same as that for $\underset{0}{020020}$ in Section 2.

3.1. TYPE G_2

(1–1) Weighted Dynkin diagram $2 \Rrightarrow 0$
This is the irreducible prehomogeneous vector space $(GL_2, 3\Lambda_1, \mathbb{C}^4)$. This prehomogeneous vector space has the irreducible relative invariant f of degree 4. Its b-function is determined by T. Shintani [24]:

$$b_f(s) = (s+1)^2\left(s+\tfrac{5}{6}\right)\left(s+\tfrac{7}{6}\right).$$

We note that $\underline{\kappa} = (1)$ in Lemma 1.3.7, and that Corollary 1.3.9 holds in this case.

3.2. TYPE F_4

(2–1) Weighted Dynkin diagram $20 \Rightarrow 00$
This is the irreducible prehomogeneous vector space $(Sp_6 \times GL_1, \Lambda_3 \otimes \Lambda_1, \mathbb{C}^{14} \otimes \mathbb{C})$. This prehomogeneous vector space has the irreducible relative invariant f of degree 4. Its b-function is determined in [13, (14)]:

$$b_f(s) = (s+1)(s+2)\left(s+\tfrac{5}{2}\right)\left(s+\tfrac{7}{2}\right).$$

We note that $\underline{\kappa} = (\tfrac{7}{2})$.

(2–2) Weighted Dynkin diagram $00 \Rightarrow 02$
This is the irreducible prehomogeneous vector space $(Spin_7 \times GL_1$, spin rep. $\otimes \Lambda_1$, $\mathbb{C}^8 \otimes \mathbb{C})$. This prehomogeneous vector space has the irreducible relative invariant f of degree 2. Its b-function is determined in [13, (16)]: $b_f(s) = (s+1)(s+4)$. We note that $\underline{\kappa} = (4)$.

(2–3) Weighted Dynkin diagram $02 \Rightarrow 00$
This is the irreducible prehomogeneous vector space $(SL_3 \times GL_2,\ 2\Lambda_1 \otimes \Lambda_1,\ \mathbb{C}^6 \otimes \mathbb{C}^2)$. This prehomogeneous vector space has the irreducible relative invariant

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

f of degree 12. Its b-function is determined by T. Kimura and M. Muro [16]:

$$b_f(s) = (s+1)^4\left(s+\tfrac{3}{4}\right)^2\left(s+\tfrac{5}{4}\right)^2\left(s+\tfrac{5}{6}\right)^2\left(s+\tfrac{7}{6}\right)^2.$$

We note that $\underline{\kappa} = (1)$.

(2–4) Weighted Dynkin diagram $02 \Rightarrow 02$

This prehomogeneous vector space is given as follows. Let

$$G = GL_2 \times GL_2 \quad \text{and} \quad V = S^2\mathbb{C}^2 + S^2\mathbb{C}^2 + \mathbb{C}^2,$$

where $S^2\mathbb{C}^2 = \{X \in M_2(\mathbb{C});\, {}^{\mathrm{t}}X = X\}$. Define the action by

$$gv = \left((g_1 X {}^{\mathrm{t}}g_1, g_1 Y {}^{\mathrm{t}}g_1){}^{\mathrm{t}}g_2, {}^{\mathrm{t}}g_1^{-1}p\right)$$

for $g = (g_1, g_2) \in G$ and $v = (X, Y, p) \in V$.

Let

$$v_0 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

Then the isotropy group at v_0 is given by

$$G_{v_0} = \left\{\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\right\}.$$

We can see that this prehomogeneous vector space has two irreducible relative invariants f_1 and f_2 given by

$$f_1(v) = \text{the discriminant of the binary quadratic form } \det(X\xi + Y\eta)$$

in the variables $(\xi, \eta) \in \mathbb{C}^2$ and

$$\begin{aligned} f_2(v) &= \det(Xp, Yp) \\ &= (x_{11}y_{12} - x_{12}y_{11})p_1^2 + (x_{11}y_{22} - x_{22}y_{11})p_1p_2 + (x_{12}y_{22} - x_{22}y_{12})p_2^2. \end{aligned}$$

Their characters are

$$\phi_1(g) = (\det g_1)^4(\det g_2)^2, \qquad \phi_2(g) = (\det g_1)(\det g_2).$$

Define the pairing between $V^\vee$ and V by

$$\langle v^\vee, v\rangle := P(X^\vee, X) + P(Y^\vee, Y) + {}^{\mathrm{t}}p^\vee p$$

for $v^\vee = (X^\vee, Y^\vee, p^\vee) \in V^\vee$ and $v = (X, Y, p) \in V$. Here

$$P\left(\begin{pmatrix} x_{11}^\vee & x_{12}^\vee \\ x_{12}^\vee & x_{22}^\vee \end{pmatrix}, \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix}\right) := x_{11}^\vee x_{11} + x_{12}^\vee x_{12} + x_{22}^\vee x_{22}.$$

Then the relative invariants $f_1^\vee$ and $f_2^\vee$ are given by

$$f_1(v^\vee) = \text{the discriminant of the binary quadratic form } h^\vee(X^\vee\xi + Y^\vee\eta)$$

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

in the variables $(\xi, \eta) \in \mathbb{C}^2$, where $h^\vee(X^\vee) = 4x_{11}^\vee x_{22}^\vee - x_{12}^{\vee 2}$, and

$$f_2(v^\vee) = \left(x_{11}^\vee y_{12}^\vee - x_{12}^\vee y_{11}^\vee\right)p_1^{\vee 2} + 2\left(x_{11}^\vee y_{22}^\vee - x_{22}^\vee y_{11}^\vee\right)p_1^\vee p_2^\vee + \\ + \left(x_{12}^\vee y_{22}^\vee - x_{22}^\vee y_{12}^\vee\right)p_2^{\vee 2}.$$

We have that

$$\operatorname{grad}\log f_1(v_0) = \left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0\\ 0\end{pmatrix}\right),$$
$$\operatorname{grad}\log f_2(v_0) = \left(\begin{pmatrix}\frac{1}{2} & 1\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}\right).$$

Hence we get that

$$a_1(\underline{s}) = s_1(s_1+s_2)(2s_1+s_2)^2, \qquad a_2(\underline{s}) = s_2^2(s_1+s_2)(2s_1+s_2).$$

We have $\underline{\kappa} = (1, 1)$.

Therefore the b-function of the relative invariant $\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ $(m_1, m_2 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$b_{\underline{f}}^{\underline{m}}(s) = \left\{\prod_{v=0}^{m_1-1}(m_1 s + 1 + v)\right\}\left\{\prod_{v=0}^{m_2-1}(m_2 s + 1 + v)^2\right\} \times \\ \times \left\{\prod_{v=0}^{m_1+m_2-1}\left((m_1+m_2)s + \tfrac{3}{2} + v\right)\right\} \times \\ \times \left\{\prod_{v=0}^{2m_1+m_2-1}((2m_1+m_2)s + 2 + v)\right\}.$$

3.3. TYPE E_6

(3–1) Weighted Dynkin diagram $\overset{}{00\underset{0}{2}00}$

This is the irreducible prehomogeneous vector space $(GL_6, \Lambda_3, \mathbb{C}^{20})$. This prehomogeneous vector space has the irreducible relative invariant f of degree 4. Its b-function is determined in [13, (5)]:

$$b_f(s) = (s+1)(s+5)\left(s+\tfrac{5}{2}\right)\left(s+\tfrac{7}{2}\right).$$

We note that $\underline{\kappa} = (5)$.

(3–2) Weighted Dynkin diagram $20\underset{0}{0}02$

This prehomogeneous vector space is the direct sum of two irreducible prehomogeneous vector space $(Spin_8 \times GL_1, \text{vector rep.} \otimes \Lambda_1, \mathbb{C}^8 \otimes \mathbb{C})$ and $(Spin_8 \times GL_1, \text{half spin rep.} \otimes \Lambda_1, \mathbb{C}^8 \otimes \mathbb{C})$ in the sense of Definition 1.1.7. The former prehomogeneous vector space has the irreducible relative invariant f_1 of degree 2, whose b-function is $b_{f_1}(s) = (s+1)(s+4)$. The latter prehomogeneous

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

vector space has the irreducible relative invariant f_2 of degree 2, whose b-function is $b_{f_2}(s) = (s+1)(s+4)$. Hence the b-function of the relative invariant $\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ $(m_1, m_2 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$b_{\underline{f}}^{\underline{m}}(s) = \left\{ \prod_{v=0}^{m_1-1} (m_1 s + 1 + v)(m_1 s + 4 + v) \right\} \times \\ \times \left\{ \prod_{v=0}^{m_2-1} (m_2 s + 1 + v)(m_2 s + 4 + v) \right\}.$$

We note that $\underline{\kappa} = (4, 4)$.

(3–3) Weighted Dynkin diagram $\overset{\displaystyle 00200}{0}$

This is the irreducible prehomogeneous vector space $(SL_3 \times SL_3 \times GL_2, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2)$. This prehomogeneous vector space has the irreducible relative invariant f of degree 12. Its b-function is determined by T. Kimura and M. Muro [16]:

$$b_f(s) = (s+1)^4 \left(s + \tfrac{3}{2}\right)^4 \left(s + \tfrac{4}{3}\right)\left(s + \tfrac{5}{3}\right)\left(s + \tfrac{5}{6}\right)\left(s + \tfrac{7}{6}\right).$$

We note that $\underline{\kappa} = (\tfrac{3}{2})$.

(3–4) Weighted Dynkin diagram $\overset{\displaystyle 20002}{2}$

This prehomogeneous vector space is the direct sum of the irreducible prehomogeneous vector space $(GL_4, \Lambda_2, \bigwedge^2 \mathbb{C}^4)$ and the simple prehomogeneous vector space $(SL_4 \times GL_1 \times GL_1, \Lambda_1 + \Lambda_1^*, \mathbb{C}^4 + \mathbb{C}^4)$ in the sense of Definition 1.1.7. The former prehomogeneous vector space has the irreducible relative invariant f_1 of degree 2, whose b-function is $b_{f_1}(s) = (s+1)(s+3)$. The latter prehomogeneous vector space has the irreducible relative invariant f_2 of degree 2, whose b-function is $b_{f_2}(s) = (s+1)(s+4)$. Hence the b-function of the relative invariant $\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ $(m_1, m_2 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$b_{\underline{f}}^{\underline{m}}(s) = \left\{ \prod_{v=0}^{m_1-1} (m_1 s + 1 + v)(m_1 s + 3 + v) \right\} \times \\ \times \left\{ \prod_{v=0}^{m_2-1} (m_2 s + 1 + v)(m_2 s + 4 + v) \right\}.$$

We note that $\underline{\kappa} = (3, 4)$.

(3–5) Weighted Dynkin diagram $\overset{\displaystyle 20202}{0}$

This prehomogeneous vector space is given as follows. Let

$$G = GL_2 \times GL_2 \times GL_2 \quad \text{and} \quad V = M_2(\mathbb{C}) + M_2(\mathbb{C}) + \mathbb{C}^2 + \mathbb{C}^2.$$

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

Define the action by

$$gv = \big((g_1 X g_2^{-1}, g_1 Y g_2^{-1})^{\mathrm{t}} g_3, {}^{\mathrm{t}}g_1^{-1} p, g_2 q\big)$$

for $g = (g_1, g_2, g_3) \in G$ and $v = (X, Y, p, q) \in V$.

The b-functions of this prehomogeneous vector space are determined by A. Gyoja [8, §8]. This prehomogeneous vector space has three irreducible relative invariants f_1, f_2, and f_3 given by

$$f_1(v) = \text{the discriminant of the binary quadratic form } \det(X\xi + Y\eta)$$

in the variables $(\xi, \eta) \in \mathbb{C}$,

$$f_2(v) = \det\big({}^{\mathrm{t}}Xp, {}^{\mathrm{t}}Yp\big), \qquad f_3(v) = \det(Xq, Yq).$$

We have that

$$\begin{aligned} a_1(\underline{s}) &= s_1(s_1+s_2)(s_1+s_3)(s_1+s_2+s_3),\\ a_2(\underline{s}) &= s_2^2(s_1+s_2)(s_1+s_2+s_3),\\ a_3(\underline{s}) &= s_3^2(s_1+s_3)(s_1+s_2+s_3). \end{aligned}$$

We have $\underline{\kappa} = (1, 1, 1)$.

The b-function of the relative invariant

$$\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2} f_3^{m_3} \ (m_1, m_2, m_3 \in \mathbb{Z}_{\geqslant 0})$$

is given by

$$\begin{aligned} b_{\underline{f}}^{\underline{m}}(s) = &\left\{\prod_{v=0}^{m_1-1}(m_1 s + 1 + v)\right\}\left\{\prod_{v=0}^{m_2-1}(m_2 s + 1 + v)^2\right\}\times\\ &\times\left\{\prod_{v=0}^{m_3-1}(m_3 s + 1 + v)^2\right\}\left\{\prod_{v=0}^{m_1+m_2-1}\big((m_1+m_2)s + \tfrac{3}{2} + v\big)\right\}\times\\ &\times\left\{\prod_{v=0}^{m_1+m_3-1}\big((m_1+m_3)s + \tfrac{3}{2} + v\big)\right\}\times\\ &\times\left\{\prod_{v=0}^{m_1+m_2+m_3-1}((m_1+m_2+m_3)s + 2 + v)\right\}. \end{aligned}$$

3.4. TYPE E_7

(4–1) Weighted Dynkin diagram $\begin{smallmatrix}200000\\ 0\end{smallmatrix}$

This is the irreducible prehomogeneous vector space $(E_6 \times GL_1, \Lambda_1 \otimes \Lambda_1, \mathbb{C}^{27} \otimes \mathbb{C})$. This prehomogeneous vector space has the irreducible relative invariant f of degree 3. Its b-function is determined in [13, (27)]: $b_f(s) = (s+1)(s+5)(s+9)$. We note that $\underline{\kappa} = (9)$.

(4–2) Weighted Dynkin diagram $\overset{\displaystyle 000002}{\scriptstyle 0}$

This is the irreducible prehomogeneous vector space ($Spin_{12} \times GL_1$, half spin rep. $\otimes \Lambda_1, \mathbb{C}^{32} \otimes \mathbb{C}$). This prehomogeneous vector space has the irreducible relative invariant f of degree 4. Its b-function is determined in [13, (23)]: $b_f(s) = (s+1)(s+8)\left(s+\frac{7}{2}\right)\left(s+\frac{11}{2}\right)$. We note that $\underline{\kappa} = (8)$.

(4–3) Weighted Dynkin diagram $\overset{\displaystyle 000000}{\scriptstyle 2}$

This is the irreducible prehomogeneous vector space $(GL_7, \Lambda_3, \mathbb{C}^{35})$. This prehomogeneous vector space has the irreducible relative invariant f of degree 7. See [15]. Its b-function is determined in [13, (6)]:

$$b_f(s) = (s+1)(s+2)(s+3)(s+4)(s+5)\left(s+\tfrac{5}{2}\right)\left(s+\tfrac{7}{2}\right).$$

We note that $\underline{\kappa} = (5)$.

(4–4) Weighted Dynkin diagram $\overset{\displaystyle 020000}{\scriptstyle 0}$

This is the irreducible prehomogeneous vector space ($Spin_{10} \times GL_2$, half spin rep. $\otimes \Lambda_1$, $\mathbb{C}^{16} \otimes \mathbb{C}^2$). This prehomogeneous vector space has the irreducible relative invariant f of degree 4. See [12]. Its b-function is determined in [13, (20)]: $b_f(s) = (s+1)(s+4)(s+5)(s+8)$. We note that $\underline{\kappa} = (8)$.

(4–5) Weighted Dynkin diagram $\overset{\displaystyle 200002}{\scriptstyle 0}$

This is the simple prehomogeneous vector space ($Spin_{10} \times GL_1 \times GL_1$, vector rep. $\otimes \Lambda_1 \otimes 1 +$ half spin rep. $\otimes 1 \otimes \Lambda_1, \mathbb{C}^{10} + \mathbb{C}^{16}$). This prehomogeneous vector space has two irreducible relative invariants f_1 such that $\deg_{\mathbb{C}^{10}} f_1 = 1$ and $\deg_{\mathbb{C}^{16}} f_1 = 2$, and f_2 of degree 2 on $\mathbb{C}^{10}$.

The b-function of the relative invariant $\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ $(m_1, m_2 \in \mathbb{Z}_{\geqslant 0})$ is determined by S. Kasai [10, (13)]:

$$b_{\underline{f}}^{\underline{m}}(s) = \left\{\prod_{v=0}^{m_1-1}(m_1 s + 1 + v)(m_1 s + 8 + v)\right\} \times \\ \times \left\{\prod_{v=0}^{m_2-1}(m_2 s + 1 + v)\right\}\left\{\prod_{v=0}^{m_1+m_2-1}((m_1+m_2)s + 5 + v)\right\}.$$

We note that $\underline{\kappa} = (8, 1)$.

(4–6) Weighted Dynkin diagram $\overset{\displaystyle 000020}{\scriptstyle 0}$

This is the irreducible prehomogeneous vector space ($SL_6 \times GL_2$, $\Lambda_2 \otimes \Lambda_1$, $\mathbb{C}^{15} \otimes \mathbb{C}^2$). This prehomogeneous vector space has the irreducible relative invariant

f of degree 12. Its b-function is determined by T. Kimura and M. Muro [16]:

$$b_f(s) = (s+1)^2(s+2)^2\left(s+\tfrac{3}{2}\right)^2\left(s+\tfrac{5}{2}\right)^2 \times \\ \times \left(s+\tfrac{7}{3}\right)\left(s+\tfrac{8}{3}\right)\left(s+\tfrac{5}{6}\right)\left(s+\tfrac{7}{6}\right).$$

We note that $\underline{\kappa} = (\frac{5}{2})$.

(4–7) Weighted Dynkin diagram $\overset{}{\underset{0}{002000}}$

This is the irreducible prehomogeneous vector space $(SL_5 \times GL_3,\ \Lambda_2 \otimes \Lambda_1,\ \mathbb{C}^{10} \otimes \mathbb{C}^3)$. This prehomogeneous vector space has the irreducible relative invariant f of degree 15. See [7]. Its b-function is determined in [13, (10)]:

$$b_f(s) = (s+1)^3(s+2)^3\left(s+\tfrac{3}{2}\right)^3\left(s+\tfrac{4}{3}\right)^2\left(s+\tfrac{5}{3}\right)^2\left(s+\tfrac{5}{4}\right)\left(s+\tfrac{7}{4}\right).$$

We note that $\underline{\kappa} = (2)$.

(4–8) Weighted Dynkin diagram $\underset{0}{020002}$

This prehomogeneous vector space is the direct sum of two irreducible prehomogeneous vector space $(Spin_8 \times GL_2,\ \text{vector rep.} \otimes \Lambda_1,\ \mathbb{C}^8 \otimes \mathbb{C}^2)$ and $(Spin_8 \times GL_1,\ \text{half spin rep.} \otimes \Lambda_1,\ \mathbb{C}^8 \otimes \mathbb{C})$ in the sense of Definition 1.1.7. The former prehomogeneous vector space has the irreducible relative invariant f_1 of degree 4, whose b-function is

$$b_{f_1}(s) = (s+1)(s+4)\left(s+\tfrac{3}{2}\right)\left(s+\tfrac{7}{2}\right).$$

The latter prehomogeneous vector space has the irreducible relative invariant f_2 of degree 2, whose b-function is $b_{f_2}(s) = (s+1)(s+4)$. Hence the b-function of the relative invariant $\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ $(m_1, m_2 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$b_{\underline{f}^{\underline{m}}}(s) = \left\{\prod_{v=0}^{m_1-1}(m_1 s+1+v)(m_1 s+4+v)\left(m_1 s+\tfrac{3}{2}+v\right)\left(m_1 s+\tfrac{7}{2}+v\right)\right\} \times \\ \times \left\{\prod_{v=0}^{m_2-1}(m_2 s+1+v)(m_2 s+4+v)\right\}.$$

We note that $\underline{\kappa} = (4, 4)$.

(4–9) Weighted Dynkin diagram $\underset{0}{000200}$

This is the irreducible prehomogeneous vector space $(SL_4 \times GL_3 \times SL_2,\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ \mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2)$. Denote it by (G, V) and let f be its irreducible relative invariant.

Consider the irreducible prehomogeneous vector space $(SL_2 \times GL_3 \times SL_2,\ \Lambda_1 \otimes \Lambda_1^* \otimes \Lambda_1^*,\ \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2)$, which is isomorphic to the irreducible prehomogeneous vector space $(SO_4 \times GL_3, \Lambda_1 \otimes \Lambda_1, \mathbb{C}^4 \otimes \mathbb{C}^3)$, and denote it by (G^c, V^c). Then (G, V) is the castling transform of (G^c, V^c). The irreducible pre-

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

homogeneous vector space (G^c, V^c) has the irreducible relative invariant f^c of degree 6, whose b-function is given by

$$b_{f^c}(s) = (s+1)^2(s+2)^2\left(s+\tfrac{3}{2}\right)^2.$$

See [13, (15)]. We see that f is of degree 12, and that b-functions of relative invariants f and f^c satisfy the relation

$$b_f(s) = b_{f^c}(s)\prod_{i=0}^{2}\prod_{j=3}^{4}(3s+i+j)$$

by the transformation formula for b-functions given in [25]. Hence the b-function of the irreducible relative invariant f is given by

$$b_f(s) = (s+1)^3(s+2)^3\left(s+\tfrac{3}{2}\right)^2\left(s+\tfrac{4}{3}\right)^2\left(s+\tfrac{5}{3}\right)^2.$$

We note that $\underline{\kappa} = (2)$.

(4–10) Weighted Dynkin diagram $\overset{\displaystyle 002002}{\underset{0}{}}$.
This prehomogeneous vector space is given as follows. Let

$$G = GL_4 \times GL_3 \quad \text{and} \quad V = \bigwedge^2 \mathbb{C}^4 + \bigwedge^2 \mathbb{C}^4 + \bigwedge^2 \mathbb{C}^4 + \mathbb{C}^4,$$

where $\bigwedge^2 \mathbb{C}^4 = \{X \in M_4(\mathbb{C});\, {}^{\mathrm{t}}X = -X\}$. Define the action by

$$gv = \left((g_1 X {}^{\mathrm{t}}g_1, g_1 Y {}^{\mathrm{t}}g_1, g_1 Z {}^{\mathrm{t}}g_1){}^{\mathrm{t}}g_2, g_1 p\right)$$

for $g = (g_1, g_2) \in G$ and $v = (X, Y, Z, p) \in V$.

Let

$$v_0 = \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right).$$

Then the isotropy group at v_0 is given by

$$G_{v_0} = \left\{\left(\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, {}^{\mathrm{t}}A^{-1}\right);\, A \in SL_3,\, A\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}{}^{\mathrm{t}}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}.$$

This prehomogeneous vector space has two irreducible relative invariants. Let

$$f_1(v) = \text{the discriminant of the ternary quadratic form } \mathrm{Pf}(X\xi + Y\eta + Z\zeta)$$

in the variables $(\xi, \eta, \zeta) \in \mathbb{C}^3$. Here the discriminant of the ternary quadratic form

$$x_1\xi^2 + x_2\eta^2 + x_3\zeta^2 + x_4\xi\eta + x_5\eta\zeta + x_6\xi\zeta$$

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

is given by

$$4x_1x_2x_3 + x_4x_5x_6 - x_1x_5^2 - x_2x_6^2 - x_3x_4^2.$$

Then f_1 is an irreducible relative invariant of degree 6, whose character is $\phi_1(g) = (\det g_1)^3(\det g_2)^2$. The other irreducible relative invariant f_2 of degree 5 is explicitly constructed in the same way as in [27]. Let $\mathfrak{S} = \mathfrak{S}_4 \times \mathfrak{S}_3 \subset G$ be the group which permutes the bases. Then $\sigma = (\sigma_1, \sigma_2) \in \mathfrak{S}$ acts on the polynomial ring $\mathbb{C}[V]$ as σ_1 permutes the subscripts of x, y, z, and p, and σ_2 permutes the set $\{X, Y, Z\}$. The irreducible relative invariant f_2 is given by

$$f_2(v) = \sum{}' \pm x_{23}y_{24}z_{34}p_1^2 - \sum{}' \pm x_{13}y_{24}z_{34}p_1p_2.$$

Here $\sum' \pm m$ means the sum of the distinct terms of

$$\sum_{\sigma_1 \in \mathfrak{S}_4, \sigma_2 \in \mathfrak{S}_3} \operatorname{sgn}(\sigma_2)(\sigma_1, \sigma_2)m.$$

Its character is

$$\phi_2(g) = (\det g_1)^2(\det g_2).$$

We have that

$$\operatorname{grad}\log f_1(v_0) = \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right),$$

$$\operatorname{grad}\log f_2(v_0) = \left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right).$$

Hence we get that

$$a_1(\underline{s}) = s_1^3(s_1 + s_2)^3, \qquad a_2(\underline{s}) = s_2^2(s_1 + s_2)^3.$$

We have $\underline{\kappa} = (2, 2)$. Hence, the b-function of the relative invariant $f^{\underline{m}} = f_1^{m_1}f_2^{m_2}$ is

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

given by

$$b_{\underline{f}^{\underline{m}}}(s) = \left\{ \prod_{\nu=0}^{m_1-1} (m_1 s + 1 + \nu)(m_1 s + 2 + \nu)\left(m_1 s + \tfrac{3}{2} + \nu\right) \right\} \times$$
$$\times \left\{ \prod_{\nu=0}^{m_2-1} (m_2 s + 1 + \nu)(m_2 s + 2 + \nu) \right\} \times$$
$$\times \left\{ \prod_{\nu=0}^{m_1+m_2-1} \left((m_1+m_2)s + \tfrac{5}{2} + \nu\right)\left((m_1+m_2)s + \alpha_{3,1} + \nu\right) \times \right.$$
$$\left. \times \left((m_1+m_2)s + \alpha_{3,2} + \nu\right) \right\}$$

with some $\alpha_{3,r} \in \mathbb{Q}_{>0}$ such that $\alpha_{3,1} + \alpha_{3,2} = 5$. If we take $\underline{m} = (1, 0)$, then $b_{\underline{f}^{\underline{m}}}(s)$ is the b-function of f_1, which is

$$b_{f_1}(s) = (s+1)(s+2)^2(s+3)\left(s+\tfrac{3}{2}\right)\left(s+\tfrac{5}{2}\right).$$

See [13, (15)]. By an easy consideration, we get that $\{\alpha_{3,1}, \alpha_{3,2}\} = \{2, 3\}$.

Therefore the b-function of the relative invariant $\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ $(m_1, m_2 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$b_{\underline{f}^{\underline{m}}}(s) = \left\{ \prod_{\nu=0}^{m_1-1} (m_1 s + 1 + \nu)(m_1 s + 2 + \nu)\left(m_1 s + \tfrac{3}{2} + \nu\right) \right\} \times$$
$$\times \left\{ \prod_{\nu=0}^{m_2-1} (m_2 s + 1 + \nu)(m_2 s + 2 + \nu) \right\} \times$$
$$\times \left\{ \prod_{\nu=0}^{m_1+m_2-1} ((m_1+m_2)s + 2 + \nu)((m_1+m_2)s + 3 + \nu) \times \right.$$
$$\left. \times \left((m_1+m_2)s + \tfrac{5}{2} + \nu\right) \right\}.$$

(4–11) Weighted Dynkin diagram $\overset{}{\underset{0}{020020}}$
See Section 2.

(4–12) Weighted Dynkin diagram $\underset{0}{200200}$
This prehomogeneous vector space is given as follows. Let

$$G = GL_3 \times GL_3 \times SL_2 \quad \text{and} \quad V = M_3(\mathbb{C}) + M_3(\mathbb{C}) + \mathbb{C}^3.$$

Define the action by

$$gv = \left((g_1 X g_2^{-1}, g_1 Y g_2^{-1})^{t} g_3, g_2 p\right)$$

for $g = (g_1, g_2, g_3) \in G$ and $v = (X, Y, p) \in V$.

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

The b-functions of this prehomogeneous vector space are determined in [27]. This prehomogeneous vector space has two irreducible relative invariants f_1 of degree 12 on $M_3(\mathbb{C}) + M_3(\mathbb{C})$, and f_2 such that $\deg_{(X,Y)} f_2 = 6$ and $\deg_p f_2 = 3$.

The b-function of the relative invariant $\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ $(m_1, m_2 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$\begin{aligned} b_{\underline{f}^{\underline{m}}}(s) = &\left\{ \prod_{\nu=0}^{m_1-1} (m_1 s + 1 + \nu)^2 \big(m_1 s + \tfrac{5}{6} + \nu\big)\big(m_1 s + \tfrac{7}{6} + \nu\big) \right\} \times \\ &\times \left\{ \prod_{\nu=0}^{m_2-1} (m_2 s + 1 + \nu)^3 \right\} \times \\ &\times \left\{ \prod_{\nu=0}^{m_1+m_2-1} \big((m_1 + m_2)s + \tfrac{3}{2} + \nu\big)^2 \times \right. \\ &\left. \times \big((m_1 + m_2)s + \tfrac{4}{3} + \nu\big)\big((m_1 + m_2)s + \tfrac{5}{3} + \nu\big) \right\} \times \\ &\times \left\{ \prod_{\nu=0}^{2m_1+m_2-1} ((2m_1 + m_2)s + 2 + \nu)^2 \right\}. \end{aligned}$$

We note that $\underline{\kappa} = (1, 1)$.

(4–13) Weighted Dynkin diagram $\underset{0}{020200}$.
This prehomogeneous vector space is given as follows. Let

$$G = GL_2 \times GL_2 \times SL_3 \times SL_2 \quad \text{and} \quad V = M_2(\mathbb{C}) + M_2(\mathbb{C}) + M_2(\mathbb{C}) + M_2(\mathbb{C}).$$

Define the action by

$$gv = \big((g_1 X g_2^{-1}, g_1 Y g_2^{-1}, g_1 Z g_2^{-1})^t g_3, g_2 W g_4^{-1}\big)$$

for

$$g = (g_1, g_2, g_3, g_4) \in G \quad \text{and} \quad v = (X, Y, Z, W) \in V.$$

This prehomogeneous vector space is the direct sum of two irreducible prehomogeneous vector spaces, namely, the one consisting of the first three matrices and the one consisting of the last matrix in the sense of Definition 1.1.7. The former prehomogeneous vector space is isomophic to the irreducible prehomogeneous vector space $(SO_4 \times GL_3, \Lambda_1 \otimes \Lambda_1, \mathbb{C}^4 \otimes \mathbb{C}^3)$. Hence it has the irreducible relative invariant f_1 of degree 6, whose b-function is

$$b_{f_1}(s) = (s+1)^2 (s+2)^2 \big(s + \tfrac{3}{2}\big)^2.$$

See [13, (15)]. The latter prehomogeneous vector space has the irreducible relative invariant f_2 of degree 2, whose b-function is

$$b_{f_2}(s) = (s+1)(s+2).$$

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

Hence the b-function of the relative invariant $f^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ $(m_1, m_2 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$b_{f^{\underline{m}}}(s) = \left\{ \prod_{\nu=0}^{m_1-1} (m_1 s + 1 + \nu)^2 (m_1 s + 2 + \nu)^2 \left(m_1 s + \tfrac{3}{2} + \nu\right)^2 \right\} \times \\ \times \left\{ \prod_{\nu=0}^{m_2-1} (m_2 s + 1 + \nu)(m_2 s + 2 + \nu) \right\}.$$

We note that $\underline{\kappa} = (2, 2)$.

(4–14) Weighted Dynkin diagram $\underset{0}{200202}$

This prehomogeneous vector space is given as follows. Let

$$G = GL_3 \times GL_2 \times GL_2 \quad \text{and} \quad V = M_{3,2}(\mathbb{C}) + M_{3,2}(\mathbb{C}) + \mathbb{C}^3 + \mathbb{C}^2.$$

Define the action by

$$gv = \left((g_1 X g_2^{-1}, g_1 Y g_2^{-1})^{\mathrm{t}} g_3, {}^{\mathrm{t}} g_1^{-1} p, g_2 q\right)$$

for $g = (g_1, g_2, g_3) \in G$ and $v = (X, Y, p, q) \in V$.

Let

$$v_0 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right).$$

Then the isotropy group at v_0 is given by

$$G_{v_0} = \left\{ \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \right. \\ \left. \times \left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \right\}.$$

This prehomogeneous vector space has three irreducible relative invariants f_1 of degree 7, f_2 of degree 6, and f_3 of degree 4. The irreducible relative invariant f_1 is explicitly constructed in the same way as in [27]. Let $\mathfrak{S} = \mathfrak{S}_3 \times \mathfrak{S}_2 \times \mathfrak{S}_2 \subset G$ be the group which permutes the bases. We write

$$\begin{pmatrix} i_1 i_2 & k_1 k_2 \\ j_2 j_2 & l_1 l_2 \end{pmatrix}, s, t_1 t_2$$

for a monomial $x_{i_1 j_1} x_{i_2 j_2} y_{k_1 l_1} y_{k_2 l_2} p_s q_{t_1} q_{t_2}$. Then $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathfrak{S}$ acts on the poly-

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

nomial ring $\mathbb{C}[V]$ as

$$(\sigma_1, \sigma_2, 1)\begin{pmatrix} i_1 i_2 \\ j_2 j_2 \end{pmatrix}, \begin{pmatrix} k_1 k_2 \\ l_1 l_2 \end{pmatrix}, s, t_1 t_2 \Big)$$

$$= \left(\begin{matrix} \sigma_1(i_1)\sigma_1(i_2) \\ \sigma_2(j_2)\sigma_2(j_2) \end{matrix}, \begin{matrix} \sigma_1(k_1)\sigma_1(k_2) \\ \sigma_2(l_1)\sigma_2(l_2) \end{matrix}, \sigma_1(s)\sigma_2(t_1)\sigma_2(t_2)\right),$$

$$(1, 1, (12))\left(\begin{matrix} i_1 i_2 \\ j_2 j_2 \end{matrix}, \begin{matrix} k_1 k_2 \\ l_1 l_2 \end{matrix}, s, t_1 t_2\right) = \left(\begin{matrix} k_1 k_2 \\ l_2 l_2 \end{matrix}, \begin{matrix} i_1 i_2 \\ j_1 j_2 \end{matrix}, s, t_1, t_2\right).$$

The irreducible relative invariant f_1 is given by $f_1(v) = \sum_i \big(\alpha_i \sum' \pm m_i\big)$. Here monomials m_i and coefficients α_i are listed in Table I, and $\sum' \pm m_i$ is the sum of the distinct terms of

$$\sum_{\sigma_1 \in \mathfrak{S}_3, \sigma_2, \sigma_3 \in \mathfrak{S}_2} \operatorname{sgn}(\sigma_1\sigma_2)(\sigma_1, \sigma_2, \sigma_3)m_i.$$

Its character is $\phi_1(g) = (\det g_1)(\det g_2)^{-1}(\det g_3)^2$. The second relative invariant is given by

$$f_2(v) = \text{the discriminant of the ternary quadratic form}$$
$$\det\left(\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}\xi + \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}\eta + \begin{pmatrix} X_3 \\ Y_3 \end{pmatrix}\zeta\right)$$

in the variables $(\xi, \eta, \zeta) \in \mathbb{C}^3$ for

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}.$$

For the discriminant of ternary quadratic forms, see (4–10). Its character is

$$\phi_2(g) = (\det g_1)^2(\det g_2)^{-3}(\det g_3)^3.$$

Table I

Representative	Coefficient	Isotropy group
$m_1 = \left(\begin{smallmatrix} 11 \\ 11 \end{smallmatrix}, \begin{smallmatrix} 23 \\ 12 \end{smallmatrix}, 1, 11\right)$	$\alpha_1 = 1$	1
$m_2 = \left(\begin{smallmatrix} 11 \\ 12 \end{smallmatrix}, \begin{smallmatrix} 23 \\ 11 \end{smallmatrix}, 1, 11\right)$	$\alpha_2 = 0$	$\mathfrak{S}_2 \cong \langle (23), 1, 1 \rangle$
$m_3 = \left(\begin{smallmatrix} 12 \\ 11 \end{smallmatrix}, \begin{smallmatrix} 13 \\ 12 \end{smallmatrix}, 1, 11\right)$	$\alpha_3 = -1$	1
$m_4 = \left(\begin{smallmatrix} 12 \\ 11 \end{smallmatrix}, \begin{smallmatrix} 13 \\ 21 \end{smallmatrix}, 1, 11\right)$	$\alpha_4 = 1$	1
$m_5 = \left(\begin{smallmatrix} 11 \\ 11 \end{smallmatrix}, \begin{smallmatrix} 23 \\ 22 \end{smallmatrix}, 1, 12\right)$	$\alpha_5 = 0$	$\mathfrak{S}_2 \cong \langle (23), 1, 1 \rangle$
$m_6 = \left(\begin{smallmatrix} 11 \\ 12 \end{smallmatrix}, \begin{smallmatrix} 23 \\ 12 \end{smallmatrix}, 1, 12\right)$	$\alpha_6 = 2$	$\mathfrak{S}_2 \cong \langle (23), (12), 1 \rangle$
$m_7 = \left(\begin{smallmatrix} 12 \\ 11 \end{smallmatrix}, \begin{smallmatrix} 13 \\ 22 \end{smallmatrix}, 1, 12\right)$	$\alpha_7 = 0$	$\mathfrak{S}_2 \cong \langle (23), (12), (12) \rangle$
$m_8 = \left(\begin{smallmatrix} 12 \\ 12 \end{smallmatrix}, \begin{smallmatrix} 13 \\ 12 \end{smallmatrix}, 1, 12\right)$	$\alpha_8 = 0$	$\mathfrak{S}_2 \cong \langle (23), 1, (12) \rangle$
$m_9 = \left(\begin{smallmatrix} 12 \\ 12 \end{smallmatrix}, \begin{smallmatrix} 13 \\ 21 \end{smallmatrix}, 1, 12\right)$	$\alpha_9 = 2$	$\mathfrak{S}_2 \cong \langle (23), (12), (12) \rangle$

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

The third relative invariant is given by $f_3(v) = \det({}^tXp, {}^tYp)$, whose character is $\phi_3(g) = (\det g_2)^{-1}(\det g_3)$.We have that

$$\operatorname{grad}\log f_1(v_0) = \left(\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 0 \\ 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right),$$
$$\operatorname{grad}\log f_2(v_0) = \left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right),$$
$$\operatorname{grad}\log f_3(v_0) = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right).$$

Hence we get that

$$\begin{aligned} a_1(\underline{s}) &= s_1^2(s_1+s_2)(s_1+s_3)(s_1+2s_2)(s_1+s_2+s_3)^2, \\ a_2(\underline{s}) &= s_2(s_1+s_2)(s_1+2s_2)^2(s_1+s_2+s_3)^2, \\ a_3(\underline{s}) &= s_3(s_1+s_3)(s_1+s_2+s_3)^2. \end{aligned}$$

We have $\underline{\kappa} = (1, 1, 1)$. Hence the b-function of the relative invariant $\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2} f_3^{m_3}$ is given by

$$\begin{aligned} b_{\underline{f}^{\underline{m}}}(s) = {} & \left\{ \prod_{v=0}^{m_1-1} (m_1 s + 1 + v)^2 \right\} \left\{ \prod_{v=0}^{m_2-1} (m_2 s + 1 + v) \right\} \times \\ & \times \left\{ \prod_{v=0}^{m_3-1} (m_3 s + 1 + v) \right\} \left\{ \prod_{v=0}^{m_1+m_2-1} \left((m_1+m_2)s + \tfrac{3}{2} + v \right) \right\} \times \\ & \times \left\{ \prod_{v=0}^{m_1+m_3-1} \left((m_1+m_3)s + \tfrac{3}{2} + v \right) \right\} \times \\ & \times \left\{ \prod_{v=0}^{m_1+2m_2-1} \left((m_1+2m_2)s + 2 + v \right) \right\} \times \\ & \times \left\{ \prod_{v=0}^{m_1+m_2+m_3-1} \prod_{r=1}^{2} \left((m_1+m_2+m_3)s + \alpha_{7,r} + v \right) \right\} \end{aligned}$$

with some $\alpha_{7,r} \in \mathbb{Q}_{>0}$ such that $\alpha_{7,1} + \alpha_{7,2} = 4$. If we take $\underline{m} = (0, 1, 0)$, then $b_{\underline{f}^{\underline{m}}}(s)$ is the b-function of f_2, which is

$$b_{f_2}(s) = (s+1)^2(s+2)^2\left(s+\tfrac{3}{2}\right)^2.$$

By an easy consideration, we get that $\{\alpha_{7,1}, \alpha_{7,2}\} = \{2, 2\}$.

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

Therefore the b-function of the relative invariant $f^{\underline{m}} = f_1^{m_1} f_2^{m_2} f_3^{m_3}$ $(m_1, m_2, m_3 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$b_{f^{\underline{m}}}(s) = \left\{\prod_{v=0}^{m_1-1}(m_1 s + 1 + v)^2\right\}\left\{\prod_{v=0}^{m_2-1}(m_2 s + 1 + v)\right\}\times$$
$$\times\left\{\prod_{v=0}^{m_3-1}(m_3 s + 1 + v)\right\}\left\{\prod_{v=0}^{m_1+m_2-1}\big((m_1+m_2)s + \tfrac{3}{2} + v\big)\right\}\times$$
$$\times\left\{\prod_{v=0}^{m_1+m_3-1}\big((m_1+m_3)s + \tfrac{3}{2} + v\big)\right\}\times$$
$$\times\left\{\prod_{v=0}^{m_1+2m_2-1}((m_1+2m_2)s + 2 + v)\right\}\times$$
$$\times\left\{\prod_{v=0}^{m_1+m_2+m_3-1}\big((m_1+m_2+m_3)s + \alpha_{7,r} + v\big)\right\}$$

(4–15) Weighted Dynkin diagram $\overset{}{0202}\!\!\underset{0}{}02$

This prehomogeneous vector space is given as follows. Let $G = GL_2 \times GL_2 \times GL_2 \times SL_2$ and $V = M_2(\mathbb{C}) + M_2(\mathbb{C}) + M_2(\mathbb{C}) + \mathbb{C}^2$. Define the action by

$$gv = \big((g_1 X g_2^{-1}, g_1 Y g_2^{-1})^{t} g_3, g_4 W g_1^{-1}, g_2 p\big)$$

for $g = (g_1, g_2, g_3, g_4) \in G$ and $v = (X, Y, W, p) \in V$.

This prehomogeneous vector space is the direct sum of two prehomogeneous vector spaces $\{(X, Y, p)\}$ and $\{(W)\}$ in the sense of Definition 1.1.7. The former prehomogeneous vector space is the submodule of (3–5), which has two irreducible relative invariants f_1 and f_2 given by

$$f_1(v) = \text{the discriminant of the binary quadratic form } \det(X\xi + Y\eta)$$

in the variables $(\xi, \eta) \in \mathbb{C}^2$ and $f_2(v) = \det(Xp, Yp)$. The latter prehomogeneous vector space has the irreducible relative invariant f_3 given by $f_3(v) = \det(W)$, whose b-function is $b_{f_3}(s) = (s+1)(s+2)$. Hence the b-function of the relative invariant $f^{\underline{m}} = f_1^{m_1} f_2^{m_2} f_3^{m_3} (m_1, m_2, m_3 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$b_{f^{\underline{m}}}(s) = \left\{\prod_{v=0}^{m_1-1}(m_1 s + 1 + v)\big(m_1 s + \tfrac{3}{2} + v\big)\right\}\left\{\prod_{v=0}^{m_2-1}(m_2 s + 1 + v)^2\right\}\times$$
$$\times\left\{\prod_{v=0}^{m_3-1}(m_3 s + 1 + v)(m_3 s + 2 + v)\right\}\times$$
$$\times\left\{\prod_{v=0}^{m_1+m_2-1}\big((m_1+m_2)s + \tfrac{3}{2} + v\big)((m_1+m_2)s + 2 + v)\right\}.$$

We note that $\underline{\kappa} = (\frac{3}{2}, 1, 2)$.

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

3.5. TYPE E_8

(5–1) Weighted Dynkin diagram $\overset{\displaystyle 2000000}{0}$

This is the irreducible prehomogeneous vector space $(E_7 \times GL_1, \Lambda_1 \otimes \Lambda_1, \mathbb{C}^{56} \otimes \mathbb{C})$. This prehomogeneous vector space has the irreducible relative invariant f of degree 4. Its b-function is determined in [13, (29)]:

$$b_f(s) = (s+1)(s+14)\left(s+\frac{11}{2}\right)\left(s+\frac{19}{2}\right).$$

We note that $\underline{\kappa} = (14)$.

(5–2) Weighted Dynkin diagram $\overset{\displaystyle 0000002}{0}$

This is the irreducible prehomogeneous vector space $(Spin_{14} \times GL_1$, half spin rep. $\otimes \Lambda_1, \mathbb{C}^{64} \otimes \mathbb{C})$. This prehomogeneous vector space has the irreducible relative invariant f of degree 8. See [7]. Its b-function is determined in the appendix of [13] due to I. Ozeki:

$$b_f(s) = (s+1)(s+4)(s+5)(s+8)\left(s+\frac{5}{2}\right)\left(s+\frac{7}{2}\right)\left(s+\frac{11}{2}\right)\left(s+\frac{13}{2}\right).$$

We note that $\underline{\kappa} = (8)$.

(5–3) Weighted Dynkin diagram $\overset{\displaystyle 0200000}{0}$

This is the irreducible prehomogeneous vector space $(E_6 \times GL_2, \Lambda_1 \otimes \Lambda_1, \mathbb{C}^{27} \otimes \mathbb{C}^2)$. This prehomogeneous vector space has the irreducible relative invariant f of degree 12. Its b-function is determined by T. Kimura and M. Muro [16]:

$$b_f(s) = (s+1)^2(s+3)^2\left(s+\tfrac{5}{2}\right)^2\left(s+\tfrac{9}{2}\right)^2\left(s+\tfrac{13}{3}\right)\left(s+\tfrac{14}{3}\right)\left(s+\tfrac{5}{6}\right)\left(s+\tfrac{7}{6}\right).$$

We note that $\underline{\kappa} = \left(\tfrac{9}{2}\right)$.

(5–4) Weighted Dynkin diagram $\overset{\displaystyle 2000002}{0}$

This prehomogeneous vector space is the direct sum of two irreducible prehomogeneous vector space $(Spin_{12} \times GL_1$, vector rep. $\otimes \Lambda_1, \mathbb{C}^{12} \otimes \mathbb{C})$ and $(Spin_{12} \times GL_1$, half spin rep. $\otimes \Lambda_1, \mathbb{C}^{32} \otimes \mathbb{C})$ in the sense of Definition 1.1.7. The latter prehomogeneous vector space has the irreducible relative invariant f_1 of degree 4, whose b-function is

$$b_{f_1}(s) = (s+1)(s+8)\left(s+\tfrac{7}{2}\right)\left(s+\tfrac{11}{2}\right).$$

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

See [13, (23)]. The former prehomogeneous vector space has the irreducible relative invariant f_2 of degree 2, whose b-function is

$$b_{f_2}(s) = (s+1)(s+6).$$

Hence the b-function of the relative invariant $\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ $(m_1, m_2 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$b_{\underline{f}^{\underline{m}}}(s) = \left\{ \prod_{v=0}^{m_1-1} (m_1 s + 1 + v)(m_1 s + 8 + v)\left(m_1 s + \tfrac{7}{2} + v\right)\left(m_1 s + \tfrac{11}{2} + v\right) \right\} \times \\ \times \left\{ \prod_{v=0}^{m_2-1} (m_2 s + 1 + v)(m_2 s + 6 + v) \right\}.$$

We note that $\underline{\kappa} = (8, 6)$.

(5–5) Weighted Dynkin diagram $\overset{}{000}\underset{2}{0}\overset{}{000}$

This is the irreducible prehomogeneous vector space $(GL_8, \Lambda_3, \mathbb{C}^{56})$. This prehomogeneous vector space has the irreducible relative invariant f of degree 16. See [15]. Its b-function is determined by I. Ozeki [18]:

$$b_f(s) = (s+1)(s+2)^3(s+3)^2\left(s+\tfrac{3}{2}\right)^2\left(s+\tfrac{5}{2}\right)^3\left(s+\tfrac{7}{2}\right)\left(s+\tfrac{7}{3}\right)\left(s+\tfrac{8}{3}\right)\left(s+\tfrac{11}{6}\right)\left(s+\tfrac{13}{6}\right).$$

We note that $\underline{\kappa} = \left(\tfrac{7}{2}\right)$.

(5–6) Weighted Dynkin diagram $002\underset{0}{0}000$

This is the irreducible prehomogeneous vector space $(Spin_{10} \times GL_3$, half spin rep. $\otimes \Lambda_1, \mathbb{C}^{16} \otimes \mathbb{C}^3)$. This prehomogeneous vector space has the irreducible relative invariant f of degree 12. See [7]. Its b-function is determined by T. Kimura and I. Ozeki [17]:

$$b_f(s) = (s+1)(s+2)^2(s+3)^2(s+4)\left(s+\tfrac{3}{2}\right)\left(s+\tfrac{7}{2}\right)\left(s+\tfrac{5}{3}\right)\left(s+\tfrac{7}{3}\right)\left(s+\tfrac{8}{3}\right)\left(s+\tfrac{10}{3}\right).$$

We note that $\underline{\kappa} = (4)$.

(5–7) Weighted Dynkin diagram $200\underset{2}{0}000$

This is the simple prehomogeneous vector space $(GL_7 \times GL_1, \Lambda_3 + \Lambda_1^*, \mathbb{C}^{35} + \mathbb{C}^7)$. This prehomogeneous vector space has two irreducible relative invariants f_1 of degree 7 on $\mathbb{C}^{35}$ and f_2 such that $\deg_{\mathbb{C}^{35}} f_2 = 3$ and $\deg_{\mathbb{C}^7} f_2 = 2$.

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

The b-function of the relative invariant $f^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ ($m_1, m_2 \in \mathbb{Z}_{\geqslant 0}$) is determined by S. Kasai [10, (11)]:

$$b_{f^{\underline{m}}}(s) = \left\{ \prod_{v=0}^{m_1-1} (m_1 s + 1 + v)(m_1 s + 2 + v)\left(m_1 s + \tfrac{5}{2} + v\right)\left(m_1 s + \tfrac{7}{2} + v\right) \right\} \times$$
$$\times \left\{ \prod_{v=0}^{m_2-1} (m_2 s + 1 + v)\left(m_2 s + \tfrac{7}{2} + v\right) \right\} \times$$
$$\times \left\{ \prod_{v=0}^{m_1+m_2-1} ((m_1+m_2)s + 3 + v)((m_1+m_2)s + 4 + v) \times \right.$$
$$\left. \times\, ((m_1+m_2)s + 5 + v) \right\}.$$

We note that $\underline{\kappa} = (\frac{7}{2}, \frac{7}{2})$.

(5–8) Weighted Dynkin diagram $\overset{}{0200}\underset{0}{0}02$

This prehomogeneous vector space is given as follows. Let $G = Spin_{10} \times GL_2 \times GL_1$ and $V = \mathbb{C}^{16} + M_{10,2}(\mathbb{C})$. Define the action by

$$\rho(g)v = \left(\lambda(g_1)p\,{}^{\mathrm{t}}g_3,\ \chi(g_1)X\,{}^{\mathrm{t}}g_2\right)$$

for $g = (g_1, g_2, g_3) \in G$ and $v = (p, X) \in V$. Here λ denotes the half spin representation and χ denotes the vector representation. We use the same notation as in [21, 110–114] about the half spin representation.

The kernel of ρ is generated by

$$\left(\prod_{i=1}^{5} \left(\sqrt{-1} e_i f_i + \frac{1}{\sqrt{-1}} f_i e_i \right), -1, \sqrt{-1} \right).$$

We take a point

$$v_0 = \left(1 + e_1 e_2 e_3 e_4,\ \varepsilon_{4,1} + \varepsilon_{5,2} + \varepsilon_{9,1} + \varepsilon_{10,2}\right) \in O_0,$$

where $\varepsilon_{i,j}$ is the matrix whose (i,j)-element is 1 and other elements are 0. The isotropy algebra at v_0 is isomorphic to the algebra of type G_2. Let $G' = G/\mathrm{Ker}\rho$. Then we see that $G'/(G')_0 \cong \mathbb{Z}_2$ by [1]. Let

$$g' = \left(\sqrt{-1} e_5 f_5 + \frac{1}{\sqrt{-1}} f_5 e_5,\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \sqrt{-1} \right).$$

Then g' belongs to G_{v_0} and not to the group generated by $(G_{v_0})_0$ and $\mathrm{Ker}\rho$. Hence we see that the isotropy group G_{v_0} is generated by $(G_{v_0})_0$, $\mathrm{Ker}\rho$, and g'. We see that $X^*(G, V)$ in Proposition 1.1.3 is generated by $(\det g_2)^2$ and $(\det g_3)^4$. Hence this prehomogeneous vector space has two irreducible relative invariants. Let $f_2(v) = \det({}^{\mathrm{t}}XJX)$, where $J = \left(\begin{smallmatrix} 0 & I_5 \\ I_5 & 0 \end{smallmatrix}\right)$ and I_5 is the unit matrix of size 5. Then it

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

is an irreducible relative invariant of degree 4, whose character is $\phi_2(g) = (\det g_2)^2$. Hence, the character of the other irreducible relative invariant f_1 is expressed as

$$\phi_1(g) = \left\{(\det g_3)^4\right\}^{\varepsilon} \times \left(\phi_2(g)\right)^k$$

with some $\varepsilon = \pm 1$ and $k \in \mathbb{Z}$.

Let

$$v_1 = \left(1 + e_1e_2e_3e_4, \varepsilon_{4,1} + \varepsilon_{5,2} + \varepsilon_{9,1}\right).$$

Since v_1 belongs to an open G-orbit in $S_2 = \{v \in V; f_2(v) = 0\}$, we see that $f_1(v_1) \neq 0$. We have that $f_1(v_1) = f_1(gv_1) = \phi_1(g)f_1(v_1)$ for $g \in G_{v_1}$. Hence we have that

$$\langle \phi_1 \rangle \subset \left\{\phi \in \mathrm{Hom}(G, \mathbb{C}^\times);\ \phi|_{G_{v_1}} \equiv 1\right\}.$$

By calculating the isotropy subalgebra $\mathfrak{g}_{v_1}$, we see that $(\det g_2)(\det g_3)^2$ divides $\phi_1(g)$. Hence the character of f_1 is determined as $\phi_1(g) = (\det g_2)^2(\det g_3)^4$. We have that $\deg_{\mathbb{C}^{16}} f_1 = \deg_{M_{10,2}(\mathbb{C})} f_1 = 4$.

We have that

$$\begin{aligned}
\operatorname{grad}\log f_1(v_0) &= \left(2 + 2e_1e_2e_3e_4, \varepsilon_{4,1} + 2\varepsilon_{5,2} + \varepsilon_{9,1}\right),\\
\operatorname{grad}\log f_2(v_0) &= \left(0, \varepsilon_{4,1} + \varepsilon_{5,2} + \varepsilon_{9,1} + \varepsilon_{10,2}\right).
\end{aligned}$$

Since the relative invariant f_2 is given explicitly, the a-function $a_2(\underline{s})$ can be calculated. Then using the structure theorem for a-functions (i.e., Theorem 1.3.2), we can derive the explicit form of $a_1(\underline{s})$ from that of $a_2(\underline{s})$, and hence we get that

$$\begin{aligned}
a_1(\underline{s}) &= s_1^4(s_1 + s_2)^2(2s_1 + s_2)^2,\\
a_2(\underline{s}) &= s_2(s_1 + s_2)^2(2s_1 + s_2).
\end{aligned}$$

We have $\underline{\kappa} = (4, 1)$. Hence the b-function of the relative invariant $\underline{f}^{\underline{m}} = f_1^{m_1}f_2^{m_2}$ is given by

$$\begin{aligned}
b_{\underline{f}^{\underline{m}}}(s) = &\left\{\prod_{v=0}^{m_1-1}(m_1s + 1 + v)(m_1s + 4 + v)\left(m_1s + \alpha_{1,1} + v\right)\left(m_1s + \alpha_{1,2} + v\right)\right\}\times\\
&\times\left\{\prod_{v=0}^{m_2-1}(m_2s + 1 + v)\right\}\times\\
&\times\left\{\prod_{v=0}^{m_1+m_2-1}\left((m_1 + m_2)s + \alpha_{3,1} + v\right)\left((m_1 + m_2)s + \alpha_{3,2} + v\right)\right\}\times\\
&\times\left\{\prod_{v=0}^{2m_1+m_2-1}((2m_1 + m_2)s + 5 + v)\right\}
\end{aligned}\tag{1}$$

with some $\alpha_{j,r} \in \mathbb{Q}_{>0}$ such that $\alpha_{1,1} + \alpha_{1,2} = 5$ and $\alpha_{3,1} + \alpha_{3,2} = 6$. If we take

$\underline{m} = (0, 1)$, then $b_{f^{\underline{m}}}(s)$ is the b-function of f_2, which is

$$b_{f_2}(s) = (s+1)(s+5)\left(s+\tfrac{3}{2}\right)\left(s+\tfrac{9}{2}\right).$$

See [13, (15)]. By an easy consideration, we get $\{\alpha_{3,1}, \alpha_{3,2}\} = \{\tfrac{3}{2}, \tfrac{9}{2}\}$.

Let

$$v' = \left(0, \varepsilon_{4,1} + \varepsilon_{5,2} + \varepsilon_{9,1} + \varepsilon_{10,2}\right),$$

and take $W' = \{(p, 0)\} \subset V$. The isotropy algebra $\mathfrak{g}_{v'}$ is isomorphic to $\mathfrak{so}_8 + \mathfrak{gl}_1 + \mathfrak{gl}_1$ and its action on W' is even half spin rep. + odd half spin rep.. Let h_0 and h_1 be irreducible relative invariants on W'. Their b-functions are $b_{h_i}(s) = (s+1)(s+4)$. Considering the degree of $(f_1)_{v'}$, we see that $(f_1)_{v'}$ is one of $\{h_0^2, h_0h_1, h_1^2\}$. On the other hand, the relative invariant $(f_2)_{v'}$ is a constant. Hence the local b-function $b_{f^{\underline{m}},v'}$ coincides with the b-function of $(f_1^{m_1})_{v'}$. Since it divides $b_{f^{\underline{m}}}$, we have

$$b_{f^{\underline{m}},v'}(s) = \left\{\prod_{v=0}^{m_1-1} (m_1s+1+v)(m_1s+4+v)\left(m_1s+\alpha_{1,1}+v\right)\left(m_1s+\alpha_{1,2}+v\right)\right\}$$

with the same $\alpha_{1,r}$ as in Equation (1). Hence, the b-function of $(f_1)_{v'}$ is given by

$$b_{(f_1)_{v'}}(s) = (s+1)(s+4)\left(s+\alpha_{1,1}\right)\left(s+\alpha_{1,2}\right).$$

Since

$$b_{h_0^2}(s) = b_{h_1^2}(s) = (s+1)(s+2)\left(s+\tfrac{1}{2}\right)\left(s+\tfrac{5}{2}\right),$$

we see that $(f_1)_{v'} = h_0h_1$ and that its b-function is

$$b_{(f_1)_{v'}}(s) = (s+1)^2(s+4)^2.$$

Hence we get that $\{\alpha_{1,1}, \alpha_{1,2}\} = \{1, 4\}$.

Therefore the b-function of the relative invariant $f^{\underline{m}} = f_1^{m_1}f_2^{m_2}$ $(m_1, m_2 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$\begin{aligned} b_{f^{\underline{m}}}(s) = & \left\{\prod_{v=0}^{m_1-1} (m_1s+1+v)^2(m_1s+4+v)^2\right\} \times \\ & \times \left\{\prod_{v=0}^{m_2-1} (m_2s+1+v)\right\} \times \\ & \times \left\{\prod_{v=0}^{m_1+m_2-1} \left((m_1+m_2)s+\tfrac{3}{2}+v\right)\left((m_1+m_2)s+\tfrac{9}{2}+v\right)\right\} \times \\ & \times \left\{\prod_{v=0}^{2m_1+m_2-1} ((2m_1+m_2)s+5+v)\right\}. \end{aligned}$$

(5–9) Weighted Dynkin diagram $\overset{}{000}\underset{0}{2}000$

This is the irreducible prehomogeneous vector space $(SL_5 \times GL_4, \ \Lambda_2 \otimes \Lambda_1, \mathbb{C}^{10} \otimes \mathbb{C}^4)$. This prehomogeneous vector space has the irreducible relative invariant f of degree 40. Its b-function is determined by T. Yano and I. Ozeki [28]:

$$b_f(s) = (s+1)^8\left(s+\tfrac{2}{3}\right)^4\left(s+\tfrac{4}{3}\right)^4\left(s+\tfrac{3}{4}\right)^4\left(s+\tfrac{5}{4}\right)^4\left(s+\tfrac{5}{6}\right)^4\left(s+\tfrac{7}{6}\right)^4 \times \left(s+\tfrac{7}{10}\right)^2\left(s+\tfrac{9}{10}\right)^2\left(s+\tfrac{11}{10}\right)^2\left(s+\tfrac{13}{10}\right)^2.$$

We note that $\underline{\kappa} = (1)$.

(5–10) Weighted Dynkin diagram $002\underset{0}{0}002$

This prehomogeneous vector space is the direct sum of two irreducible prehomogeneous vector spaces $(Spin_8 \times GL_3, \text{vector rep.} \otimes \Lambda_1, \mathbb{C}^8 \otimes \mathbb{C}^3)$ and $(Spin_8 \times GL_1, \text{half spin rep.} \otimes \Lambda_1, \mathbb{C}^8 \otimes \mathbb{C})$ in the sense of Definition 1.1.7. The former prehomogeneous vector space has the irreducible relative invariant f_1 of degree 6, whose b-function is

$$b_{f_1}(s) = (s+1)(s+2)(s+3)(s+4)\left(s+\tfrac{3}{2}\right)\left(s+\tfrac{7}{2}\right).$$

See [13, (15)]. The latter prehomogeneous vector space has the irreducible relative invariant f_2 of degree 2, whose b-function is $b_{f_2}(s) = (s+1)(s+4)$. Hence the b-function of the relative invariant $\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ $(m_1, m_2 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$b_{\underline{f}^{\underline{m}}}(s) = \left\{\prod_{\nu=0}^{m_1-1}(m_1 s + 1 + \nu)(m_1 s + 2 + \nu)(m_1 s + 3 + \nu) \times \times (m_1 s + 4 + \nu)\left(m_1 s + \tfrac{3}{2} + \nu\right)\left(m_1 s + \tfrac{7}{2} + \nu\right)\right\} \times \times \left\{\prod_{\nu=0}^{m_2-1}(m_2 s + 1 + \nu)(m_2 s + 4 + \nu)\right\}.$$

We note that $\underline{\kappa} = (4, 4)$.

(5–11) Weighted Dynkin diagram $200\underset{0}{2}000$

This prehomogeneous vector space is given as follows. Let

$$G = GL_5 \times GL_3 \quad \text{and} \quad V = \bigwedge^2 \mathbb{C}^5 + \bigwedge^2 \mathbb{C}^5 + \bigwedge^2 \mathbb{C}^5 + \mathbb{C}^3,$$

where $\bigwedge^2 \mathbb{C}^5 = \{X \in M_5(\mathbb{C});\, {}^{\mathrm{t}}X = -X\}$. Define the action by

$$gv = \left((g_1 X {}^{\mathrm{t}}g_1, g_1 Y {}^{\mathrm{t}}g_1, g_1 Z {}^{\mathrm{t}}g_1){}^{\mathrm{t}}g_2, {}^{\mathrm{t}}g_2^{-1} p\right)$$

for $g = (g_1, g_2) \in G$ and $v = (X, Y, Z, p) \in V$.

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

Let

$$v_0 = \left(\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right).$$

Then the isotropy group at v_0 is given by

$$G_{v_0} = \left\{ \begin{array}{c} \left(\begin{pmatrix} a^2 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & 0 & a^{-2} \end{pmatrix}, \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix} \right); a \in \mathbb{C}^\times, \\ \left(\sqrt{-1} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right) \end{array} \right\}.$$

Hence, $X^*(G, V)$ in Proposition 1.1.3 is generated by $(\det g_1)^2(\det g_2)$ and $(\det g_2)^2$, and this prehomogeneous vector space has two irreducible relative invariants. One of them, say f_1, is the same as in the case (4–7), whose degree is 15, and its character is $\phi_1(g) = (\det g_1)^6(\det g_2)^5$. Hence the character of the other irreducible relative invariant f_2 is expressed as

$$\phi_2(g) = \{(\det g_1)^2(\det g_2)\}^{\varepsilon} \times (\phi_1(g))^k.$$

with some $\varepsilon = \pm 1$ and $k \in \mathbb{Z}$.

Let

$$v_1 = \left(\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right).$$

Since v_1 belongs to an open G-orbit in $S_1 = \{v \in V; f_1(v) = 0\}$, we see that $f_2(v_1) \neq 0$. By calculating the isotropy subalgebra $\mathfrak{g}_{v_1}$, we see that $(\det g_1)^2(\det g_2)$ divides $\phi_2(g)$ in the same reason as in the case (5–8). Hence the character of f_2 is determined as $\phi_2(g) = (\det g_1)^2(\det g_2)$. We have that $\deg_{(X,Y,Z)} f_2 = 5$ and $\deg_p f_2 = 2$.

Take another generic point v_0' as

$$v_0' = \left(\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right).$$

Then we have that

$$\operatorname{grad}\log f_1(v_0') = \left(\begin{pmatrix} 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right),$$

$$\operatorname{grad}\log f_2(v_0') = \left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right).$$

Since f_1 is explicitly constructed in [7], we get that

$$a_1(\underline{s}) = s_1^4(s_1+s_2)(2s_1+s_2)^4(3s_1+s_2)^6,$$
$$a_2(\underline{s}) = s_2^2(s_1+s_2)(2s_1+s_2)^2(3s_1+s_2)^2,$$

in the same reason as in the case (5–8). We have $\underline{\kappa} = (\frac{3}{2}, \frac{3}{2})$. Hence, the *b*-function of the relative invariant $f^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ is given by

$$b_{f^{\underline{m}}}(s) = \left\{\prod_{v=0}^{m_1-1}(m_1 s+1+v)\left(m_1 s+\tfrac{3}{2}+v\right)\left(m_1 s+\alpha_{1,1}+v\right)\left(m_1 s+\alpha_{1,2}+v\right)\right\}\times$$
$$\times\left\{\prod_{v=0}^{m_2-1}(m_2 s+1+v)\left(m_2 s+\tfrac{3}{2}+v\right)\right\}\times$$
$$\times\left\{\prod_{v=0}^{m_1+m_2-1}((m_1+m_2)s+2+v)\right\}\times$$
$$\times\left\{\prod_{v=0}^{2m_1+m_2-1}\left((2m_1+m_2)s+\alpha_{3,1}+v\right)\left((2m_1+m_2)s+\alpha_{3,2}+v\right)\right\}$$
$$\times\left\{\prod_{v=0}^{3m_1+m_2-1}\left((3m_1+m_2)s+\alpha_{4,1}+v\right)\left((3m_1+m_2)s+\alpha_{4,2}+v\right)\right\}$$

with some $\alpha_{j,r} \in \mathbb{Q}_{>0}$ such that

$$\alpha_{1,1}+\alpha_{1,2}=\tfrac{5}{2}, \qquad \alpha_{3,1}+\alpha_{3,2}=\tfrac{11}{2}, \quad \text{and} \quad \alpha_{4,1}+\alpha_{4,2}=7.$$

If we take $\underline{m}=(1,0)$, then $b_{f^{\underline{m}}}(s)$ is the b-function of f_1, which is determined as

$$b_{f_1}(s)=(s+1)^3(s+2)^3\left(s+\tfrac{3}{2}\right)^3\left(s+\tfrac{4}{3}\right)^2\left(s+\tfrac{5}{3}\right)^2\left(s+\tfrac{5}{4}\right)\left(s+\tfrac{7}{4}\right).$$

See [13, (10)]. By an easy consideration, we get that

$$\{\alpha_{1,1},\alpha_{1,2}\}=\{1,\tfrac{3}{2}\},\ \{\alpha_{3,1},\alpha_{3,2}\}=\{3,\tfrac{5}{2}\}, \quad \text{and} \quad \{\alpha_{4,1},\alpha_{4,2}\}=\{3,4\}.$$

Therefore the b-function of the relative invariant $f^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ $(m_1, m_2 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$b_{f^{\underline{m}}}(s) = \left\{\prod_{v=0}^{m_1-1}(m_1 s+1+v)^2\left(m_1 s+\tfrac{3}{2}+v\right)^2\right\}\times$$
$$\times\left\{\prod_{v=0}^{m_2-1}(m_2 s+1+v)\left(m_2 s+\tfrac{3}{2}+v\right)\right\}\times$$
$$\times\left\{\prod_{v=0}^{m_1+m_2-1}((m_1+m_2)s+2+v)\right\}\times$$
$$\times\left\{\prod_{v=0}^{2m_1+m_2-1}((2m_1+m_2)s+3+v)\left((2m_1+m_2)s+\tfrac{5}{2}+v\right)\right\}\times$$
$$\times\left\{\prod_{v=0}^{3m_1+m_2-1}((3m_1+m_2)s+3+v)((3m_1+m_2)s+4+v)\right\}.$$

(5–12) Weighted Dynkin diagram $\overset{}{\underset{0}{2020002}}$

This prehomogeneous vector space is the direct sum of two prehomogeneous vector spaces $(Spin_8 \times GL_2 \times GL_1$, vector rep. $\otimes \Lambda_1^* \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^*$, $\mathbb{C}^8 \otimes \mathbb{C}^2 + \mathbb{C}^2 \otimes \mathbb{C})$ and $(Spin_8 \times GL_1$, half spin rep. $\otimes \Lambda_1$, $\mathbb{C}^8)$ in the sense of Definition 1.1.7.

When we identify the representation space of the former prehomogeneous vector space with $M_{1,2}(\mathbb{C}) + M_{2,8}(\mathbb{C})$, the action is given by $gv = \big(g_3 X g_2^{-1}, g_2 Y {}^{t}\chi(g_1)\big)$ for

$$g = (g_1, g_2, g_3) \in Spin_8 \times GL_2 \times GL_1 \quad \text{and} \quad v = (X, Y) \in M_{1,2}(\mathbb{C}) + M_{2,8}(\mathbb{C}).$$

Here χ denotes the vector representation of $Spin_8$. It has two irreducible relative invariants f_1 of degree 4 on $M_{2,8}(\mathbb{C})$ and f_2 such that $\deg_X f_2 = \deg_Y f_2 = 2$. Their b-functions are determined by F. Sato [19].

The latter prehomogeneous vector space has the irreducible relative invariant f_3 of degree 2, whose b-function is $b_{f_3}(s) = (s+1)(s+4)$. Hence the b-function of the relative invariant $\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2} f_3^{m_3}$ $(m_1, m_2, m_3 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$\begin{aligned} b_{\underline{f}^{\underline{m}}}(s) = &\left\{\prod_{v=0}^{m_1-1} (m_1 s + 1 + v)\big(m_1 s + \tfrac{7}{2} + v\big)\right\}\left\{\prod_{v=0}^{m_2-1} (m_2 s + 1 + v)^2\right\} \times \\ &\times \left\{\prod_{v=0}^{m_3-1} (m_3 s + 1 + v)(m_3 s + 4 + v)\right\} \times \\ &\times \left\{\prod_{v=0}^{m_1+m_2-1} ((m_1+m_2)s + 4 + v)\big((m_1+m_2)s + \tfrac{3}{2} + v\big)\right\}. \end{aligned}$$

We note that $\underline{\kappa} = (\frac{7}{2}, 1, 4)$.

(5–13) Weighted Dynkin diagram $\underset{0}{2000200}$

This prehomogeneous vector space is given as follows. Let $G = GL_4 \times GL_3 \times SL_2$ and $V = M_{4,3}(\mathbb{C}) + M_{4,3}(\mathbb{C}) + \mathbb{C}^4$. Define the action by

$$gv = \big((g_1 X g_2^{-1}, g_1 Y g_2^{-1})\,{}^{t}g_3, {}^{t}g_1^{-1} p\big)$$

for $g = (g_1, g_2, g_3) \in G$ and $v = (X, Y, p) \in V$.

Let

$$v_0 = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

Then the isotropy group at v_0 is given by

$$G_{v_0} = \left\{ \begin{array}{l} \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 \\ 0 & 0 & \xi^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \xi^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi \end{pmatrix}, \begin{pmatrix} \xi^2 & 0 \\ 0 & \xi \end{pmatrix} \right); \xi^3 = 1, \\ \left(\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \sqrt{-1}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \sqrt{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \end{array} \right\}.$$

Hence $X^*(G, V)$ in Proposition 1.1.3 is generated by $(\det g_1)$ and $(\det g_2)^4$, and this prehomogeneous vector space has two irreducible relative invariants f_1 and f_2. The irreducible relative invariant f_2 is the same as in the case (4–9), whose degree is 12, and its character is $\phi_2(g) = (\det g_1)^3(\det g_2)^{-4}$. Hence the character of the other irreducible relative invariant f_1 is expressed as $\phi_1(g) = \{(\det g_1)\}^{\varepsilon} \times (\phi_2(g))^k$ with some $\varepsilon = \pm 1$ and $k \in \mathbb{Z}$.

Since $\dim G = \dim V$, this prehomogeneous vector space has the following relative invariant f_0. Fix linear bases of $\mathfrak{g} := \mathrm{Lie}(G)$ and of V. Put $f_0(v) := \det(\mathfrak{g} \to V; A \mapsto Av)$ for $v \in V$ and $\phi_0(g) := \det(V \to V; v \mapsto gv)$ for $g \in G$. Then f_0 is a relative invariant whose character is ϕ_0, and $V \backslash f_0^{-1}(0) = O_0$ by [2, Remark 6.3.1].

We have that $f_0 = f_1^{m_1} f_2^{m_2}$ with $m_1, m_2 > 0$ and that $\phi_0(g) = (\det g_1)^5(\det g_2)^{-8}$. Comparing the characters of f_i, we have that

$$(\varepsilon + 3k)m_1 + 3m_2 = 5, \qquad -4km_1 - 4m_2 = -8.$$

Since f_1 is a polymonial, we can easily see that $k > 0$. Hence we get that $\varepsilon = -1$ and $k = 1$. Thus the character of f_1 is determined as $\phi_1(g) = (\det g_1)^2(\det g_2)^{-4}$. We have that $\deg_{(X,Y)} f_1 = 12$ and $\deg_p f_1 = 4$.

We have that

$$\operatorname{grad} \log f_1(v_0) = \left(\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix} \right),$$

$$\operatorname{grad} \log f_2(v_0) = \left(\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right).$$

Since f_0 is determined, we can calculate as

$$\begin{aligned} a_1(\underline{s})a_2(\underline{s}) &= f_0^{\vee}(\operatorname{grad} \log \underline{f}^{\underline{s}}(v_0)) f_0(v_0) \\ &= s_1^4 s_2^2 (s_1 + s_2)^8 (2s_1 + s_2)^9 (2s_1 + 3s_2)^5. \end{aligned}$$

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

Hence we get that

$$a_1(\underline{s}) = s_1^4(s_1+s_2)^4(2s_1+s_2)^6(2s_1+3s_2)^2,$$
$$a_2(\underline{s}) = s_2^2(s_1+s_2)^4(2s_1+s_2)^3(2s_1+3s_2)^3,$$

by Theorem 1.3.2. We have $\underline{\kappa} = (1,1)$. Hence the b-function of the relative invariant $f^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ is given by

$$\begin{aligned} b_{f^{\underline{m}}}(s) = & \left\{ \prod_{v=0}^{m_1-1} (m_1 s + 1 + v)^2 \big(m_1 s + \alpha_{1,1} + v\big)\big(m_1 s + \alpha_{1,2} + v\big) \right\} \times \\ & \times \left\{ \prod_{v=0}^{m_2-1} (m_2 s + 1 + v)^2 \right\} \times \\ & \times \left\{ \prod_{v=0}^{m_1+m_2-1} \prod_{r=1}^{4} \big((m_1+m_2)s + \alpha_{3,r} + v\big) \right\} \times \\ & \times \left\{ \prod_{v=0}^{2m_1+m_2-1} ((2m_1+m_2)s + 2 + v)\big((2m_1+m_2)s + \alpha_{4,1} + v\big) \times \right. \\ & \left. \times \big((2m_1+m_2)s + \alpha_{4,2} + v\big) \right\} \times \\ & \times \left\{ \prod_{v=0}^{2m_1+3m_2-1} ((2m_1+3m_2)s + 3 + v) \right\} \end{aligned}$$

with some $\alpha_{j,r} \in \mathbb{Q}_{>0}$ such that

$$\alpha_{1,1} + \alpha_{1,2} = 2, \qquad \alpha_{3,1} + \alpha_{3,2} = 3, \qquad \alpha_{3,3} + \alpha_{3,4} = 3, \quad \text{and} \quad \alpha_{4,1} + \alpha_{4,2} = 4.$$

If we take $\underline{m} = (0,1)$, then $b_{f^{\underline{m}}}(s)$ is the b-function of f_2, which is determined as

$$b_{f_2}(s) = (s+1)^3(s+2)^3\big(s+\tfrac{3}{2}\big)^2\big(s+\tfrac{4}{3}\big)^2\big(s+\tfrac{5}{3}\big)^2.$$

See the case (4–9). By an easy consideration, we get that

$$\{\alpha_{3,1}, \alpha_{3,2}, \alpha_{3,3}, \alpha_{3,4}\} = \{\tfrac{3}{2}, \tfrac{3}{2}, \tfrac{4}{3}, \tfrac{5}{3}\} \text{and} \{\alpha_{4,1}, \alpha_{4,2}\} = \{2, 2\}.$$

Let

$$v' = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right),$$

and take $W' = \{(0,0,p)\} \subset V$. Then the isotropy algebra $\mathfrak{g}_{v'}$ is isomorphic to $\mathfrak{gl}_2$, and its action on W' is $3\Lambda_1$. The relative invariant $(f_1)_{v'}$ is the irreducible relative invariant on W'. The relative invariant $(f_2)_{v'}$ is a constant. Hence the local b-function $b_{f^{\underline{m}},v'}$ is

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

given by

$$b_{\underline{f}^{\underline{m}},v'}(s) = \prod_{v=0}^{m_1-1}(m_1 s+1+v)^2\left(m_1 s+\tfrac{5}{6}+v\right)\left(m_1 s+\tfrac{7}{6}+v\right).$$

Since it devides the b-function of $\underline{f}^{\underline{m}}$, we get that $\{\alpha_{1,1}, \alpha_{1,2}\} = \{\frac{5}{6}, \frac{7}{6}\}$.

Therefore the b-function of the relative invariant $\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ $(m_1, m_2 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$\begin{aligned} b_{\underline{f}^{\underline{m}}}(s) = {} & \left\{\prod_{v=0}^{m_1-1}(m_1 s+1+v)^2\left(m_1 s+\tfrac{5}{6}+v\right)\left(m_1 s+\tfrac{7}{6}+v\right)\right\}\times \\ & \times\left\{\prod_{v=0}^{m_2-1}(m_2 s+1+v)^2\right\}\times \\ & \times\left\{\prod_{v=0}^{m_1+m_2-1}\left((m_1+m_2)s+\tfrac{3}{2}+v\right)^2\left((m_1+m_2)s+\tfrac{4}{3}+v\right)\times\right. \\ & \left.\times\left((m_1+m_2)s+\tfrac{5}{3}+v\right)\right\}\left\{\prod_{v=0}^{2m_1+m_2-1}((2m_1+m_2)s+2+v)^3\right\}\times \\ & \times\left\{\prod_{v=0}^{2m_1+3m_2-1}((2m_1+3m_2)s+3+v)\right\}. \end{aligned}$$

(5–14) Weighted Dynkin diagram $\underset{0}{200002}$

This prehomogeneous vector space is given as follows. Let

$$G = GL_4\times GL_3\times GL_1 \quad\text{and}\quad V = \bigwedge^2\mathbb{C}^4+\bigwedge^2\mathbb{C}^4+\bigwedge^2\mathbb{C}^4+\mathbb{C}^4+\mathbb{C}^3,$$

where $\bigwedge^2\mathbb{C}^4 = \{X\in M_4(\mathbb{C});\ {}^{\mathrm{t}}X = -X\}$. Define the action by

$$gv = \left((g_1 X {}^{\mathrm{t}}g_1, g_1 Y {}^{\mathrm{t}}g_1, g_1 Z {}^{\mathrm{t}}g_1){}^{\mathrm{t}}g_2, g_1 p, {}^{\mathrm{t}}g_2^{-1} q {}^{\mathrm{t}}g_3\right)$$

for $g = (g_1, g_2, g_3)\in G$ and $v = (X, Y, Z, p, q)\in V$.

Let

$$v_0 = \left(\begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1\\ 0\\ 0\\ 0 \end{pmatrix}, \begin{pmatrix} 1\\ 0\\ 0 \end{pmatrix}\right).$$

Then the isotropy group at v_0 is given by

$$G_{v_0} = \left\{ \begin{array}{c} \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & A \end{pmatrix}, \begin{pmatrix} \varepsilon & 0 \\ 0 & {}^{\mathrm{t}}A^{-1} \end{pmatrix}, \varepsilon \right); \\ A \in GL_2, A\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}{}^{\mathrm{t}}A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \det A = \varepsilon, \varepsilon = \pm 1 \end{array} \right\}.$$

This prehomogeneous vector space has three irreducible relative invariants. Note that this prehomogeneous vector space has (4–10) as a submodule. Hence, two of the irrducible relative invariants are the same as in the case (4–10). Let f_1 be the irreducible relative invariant of degree 6, whose character is $\phi_1(g) = (\det g_1)^3(\det g_2)^2$,and let f_2 be the irreducible relative invariant of degree 5, whose character is $\phi_2(g) = (\det g_1)^2(\det g_2)$. Let $f_3(v) = Pf((X, Y, Z)^{\mathrm{t}}q)$. Then f_3 is the remaining irreducible relative invariant of degree 4, whose character is $\phi_3(g) = (\det g_1)(\det g_3)^2$.

We have that

$$\operatorname{grad}\log f_1(v_0) = \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right),$$

$$\operatorname{grad}\log f_2(v_0) = \left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right),$$

$$\operatorname{grad}\log f_3(v_0) = \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right).$$

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

Hence we get that

$$a_1(\underline{s}) = s_1^2(s_1+s_2)^2(s_1+s_3)(s_1+s_2+s_3),$$
$$a_2(\underline{s}) = s_2^2(s_1+s_2)^2(s_1+s_2+s_3),$$
$$a_3(\underline{s}) = s_3^2(s_1+s_3)(s_1+s_2+s_3).$$

We have $\underline{\kappa} = (\frac{3}{2}, 2, \frac{3}{2})$. Hence the b-function of the relative invariant $f^{\underline{m}} = f_1^{m_1} f_2^{m_2} f_3^{m_3}$ is given by

$$\begin{aligned} b_{f^{\underline{m}}}(s) = &\left\{\prod_{\nu=0}^{m_1-1}(m_1 s+1+\nu)\big(m_1 s+\tfrac{3}{2}+\nu\big)\right\}\times \\ &\times\left\{\prod_{\nu=0}^{m_2-1}(m_2 s+1+\nu)(m_2 s+2+\nu)\right\}\left\{\prod_{\nu=0}^{m_3-1}(m_3 s+1+\nu)\big(m_3 s+\tfrac{3}{2}+\nu\big)\right\}\times \\ &\times\left\{\prod_{\nu=0}^{m_1+m_2-1}\big((m_1+m_2)s+\alpha_{4,1}+\nu\big)\big((m_1+m_2)s+\alpha_{4,2}+\nu\big)\right\}\times \\ &\times\left\{\prod_{\nu=0}^{m_1+m_3-1}((m_1+m_3)s+2+\nu)\right\}\times \\ &\times\left\{\prod_{\nu=0}^{m_1+m_2+m_3-1}((m_1+m_2+m_3)s+3+\nu)\right\} \end{aligned}$$

with some $\alpha_{4,r} \in \mathbb{Q}_{>0}$ such that $\alpha_{4,1}+\alpha_{4,2} = \frac{9}{2}$. If we take $\underline{m} = (m_1, m_2, 0)$, then $b_{f^{\underline{m}}}(s)$ is the b-function of (4–10). By an easy consideration, we get that $\{\alpha_{4,1}, \alpha_{4,2}\} = \{2, \frac{5}{2}\}$.

Therefore the b-function of the relative invariant $f^{\underline{m}} = f_1^{m_1} f_2^{m_2} f_3^{m_3}$ $(m_1, m_2, m_3 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$\begin{aligned} b_{f^{\underline{m}}}(s) = &\left\{\prod_{\nu=0}^{m_1-1}(m_1 s+1+\nu)\big(m_1 s+\tfrac{3}{2}+\nu\big)\right\}\times \\ &\times\left\{\prod_{\nu=0}^{m_2-1}(m_2 s+1+\nu)(m_2 s+2+\nu)\right\}\times \\ &\times\left\{\prod_{\nu=0}^{m_3-1}(m_3 s+1+\nu)\big(m_3 s+\tfrac{3}{2}+\nu\big)\right\}\times \\ &\times\left\{\prod_{\nu=0}^{m_1+m_2-1}((m_1+m_2)s+2+\nu)\big((m_1+m_2)s+\tfrac{5}{2}+\nu\big)\right\}\times \\ &\times\left\{\prod_{\nu=0}^{m_1+m_3-1}((m_1+m_3)s+2+\nu)\right\}\times \\ &\times\left\{\prod_{\nu=0}^{m_1+m_2+m_3-1}((m_1+m_2+m_3)s+3+\nu)\right\}. \end{aligned}$$

(5–15) Weighted Dynkin diagram $\overset{}{0200}\underset{0}{2}00$
This prehomogeneous vector space is given as follows. Let

$$G = GL_3 \times GL_3 \times SL_2 \times SL_2 \quad \text{and} \quad V = M_3(\mathbb{C}) + M_3(\mathbb{C}) + M_{3,2}(\mathbb{C}).$$

Define the action by

$$gv = \left((g_1 X g_2^{-1}, g_1 Y g_2^{-1})^{\mathrm{t}} g_3, g_2 W g_4^{-1}\right)$$

for $g = (g_1, g_2, g_3, g_4) \in G$ and $v = (X, Y, W) \in V$.
Let

$$v_0 = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}\right).$$

Then the isotropy group at v_0 is given by

$$G_{v_0} = \left\{ \begin{array}{c} (\varepsilon, 1, \varepsilon, 1);\ \varepsilon = \pm 1, \\ \left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sqrt{-1}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sqrt{-1}\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \sqrt{-1}\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}\right), \\ \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \sqrt{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \sqrt{-1}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \sqrt{-1}\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}\right) \end{array} \right\}.$$

This prehomogeneous vector space has two irreducible relative invariants f_1 and f_2. The irreducible relative invariant f_1 is of degree 12, which is given by

$$f_1(v) = \text{the discriminant of the binary cubic form } \det(X\xi + Y\eta)$$

in the variables $(\xi, \eta) \in \mathbb{C}^2$, and its character is $\phi_1(g) = (\det g_1)^4 (\det g_2)^{-4}$. The irreducible relative invariant f_2 is of degree 12, which is given by

$$f_2(v) = \text{the discriminant of the ternary quadratic form}$$
$$\det\left(\begin{pmatrix} (XW)_1 \\ (YW)_1 \end{pmatrix}\xi + \begin{pmatrix} (XW)_2 \\ (YW)_2 \end{pmatrix}\eta + \begin{pmatrix} (XW)_3 \\ (YW)_3 \end{pmatrix}\zeta\right)$$

in the variables $(\xi, \eta, \zeta) \in \mathbb{C}^3$ for

$$XW = \begin{pmatrix} (XW)_1 \\ (XW)_2 \\ (XW)_3 \end{pmatrix} \quad \text{and} \quad YW = \begin{pmatrix} (YW)_1 \\ (YW)_2 \\ (YW)_3 \end{pmatrix}.$$

For the discriminant of ternary quadratic forms, see (4–10). Its character is

$\phi_2(g) = (\det g_1)^2$. We have that

$$\operatorname{grad}\log f_1(v_0) = \left(\begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right),$$
$$\operatorname{grad}\log f_2(v_0) = \left(\begin{pmatrix} 2 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -1 & -1 \\ -1 & 2 \end{pmatrix} \right).$$

Hence we get that

$$a_1(\underline{s}) = s_1^4(s_1+s_2)^4(2s_1+s_2)^4,$$
$$a_2(\underline{s}) = s_2^6(s_1+s_2)^4(2s_1+s_2)^2.$$

We have $\underline{\kappa} = (1,1)$. Hence, the b-function of the relative invariant $\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ is given by

$$\begin{aligned} b_{\underline{f}^{\underline{m}}}(s) = {} & \left\{ \prod_{v=0}^{m_1-1} (m_1 s + 1 + v)^2 (m_1 s + \alpha_{1,1} + v)(m_1 s + \alpha_{1,2} + v) \right\} \times \\ & \times \left\{ \prod_{v=0}^{m_2-1} (m_2 s + 1 + v)^2 \prod_{r=1}^{4} (m_2 s + \alpha_{2,r} + v) \right\} \times \\ & \times \left\{ \prod_{v=0}^{m_1+m_2-1} \prod_{r=1}^{4} ((m_1+m_2)s + \alpha_{3,r} + v) \right\} \times \\ & \times \left\{ \prod_{v=0}^{2m_1+m_2-1} \prod_{r=1}^{2} ((2m_1+m_2)s + \alpha_{4,r} + v) \right\} \end{aligned}$$

with some $\alpha_{j,r} \in \mathbb{Q}_{>0}$ such that

$$\alpha_{1,1} + \alpha_{1,2} = 2, \qquad \alpha_{2,1} + \alpha_{2,2} = \alpha_{2,3} + \alpha_{2,4} = 2,$$
$$\alpha_{3,1} + \alpha_{3,2} = \alpha_{3,3} + \alpha_{3,4} = 3, \quad \text{and} \quad \alpha_{4,1} + \alpha_{4,2} = 4.$$

If we take $\underline{m} = (1,0)$, then $b_{\underline{f}^{\underline{m}}}(s)$ is the b-function of f_1, which is determined as

$$b_{f_1}(s) = (s+1)^4 \left(s+\tfrac{3}{2}\right)^4 \left(s+\tfrac{4}{3}\right)\left(s+\tfrac{5}{3}\right)\left(s+\tfrac{5}{6}\right)\left(s+\tfrac{7}{6}\right).$$

See [16]. By an easy consideration, we get that

$$\{\alpha_{1,1}, \alpha_{1,2}\} = \left\{\tfrac{5}{6}, \tfrac{7}{6}\right\},$$
$$\{\alpha_{3,1}, \alpha_{3,2}, \alpha_{3,3}, \alpha_{3,4}\} = \left\{\tfrac{3}{2}, \tfrac{3}{2}, \tfrac{4}{3}, \tfrac{5}{3}\right\},$$
$$\{\alpha_{4,1}, \alpha_{4,2}\} = \{2, 2\}.$$

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

Let

$$v' = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right),$$

and take $W' = \{(0, 0, W)\} \subset V$. Then the isotropy algebra $\mathfrak{g}_{v'}$ is isomorphic to $\mathfrak{gl}_1 + \mathfrak{gl}_1 + \mathfrak{gl}_1 + \mathfrak{sl}_2$, and its action on W' is $\Lambda_1 \otimes 1 \otimes 1 \otimes \Lambda_1^* + 1 \otimes \Lambda_1 \otimes 1 \otimes \Lambda_1^* + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^*$. The relative invariant $(f_1)_{v'}$ is a constant. The relative invariant $(f_2)_{v'}$ is the product of irreducible relative invariants on W'. We see that

$$(f_2)_{v'}(w') = \det\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \det\begin{pmatrix} w_{21} & w_{22} \\ w_{31} & w_{32} \end{pmatrix} \det\begin{pmatrix} w_{31} & w_{32} \\ w_{11} & w_{12} \end{pmatrix}$$

for

$$w' = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{pmatrix} \in W'$$

by the direct calculation. Hence, the local b-function $b_{\underline{f}^{\underline{m}}, v'}$ is given by

$$b_{\underline{f}^{\underline{m}}, v'}(s) = \prod_{v=0}^{m_2-1} (m_2 s + 1 + v)^4 \left(m_2 s + \tfrac{2}{3} + v\right)\left(m_2 s + \tfrac{4}{3} + v\right).$$

Since it devides $\underline{f}^{\underline{m}}$, we get that $\{\alpha_{2,1}, \alpha_{2,2}, \alpha_{2,3}, \alpha_{2,4}\} = \{1, 1, \frac{2}{3}, \frac{4}{3}\}$.

Therefore the b-function of the relative invariant $\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2}$ $(m_1, m_2 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$\begin{aligned} b_{\underline{f}^{\underline{m}}}(s) = &\left\{ \prod_{v=0}^{m_1-1} (m_1 s + 1 + v)^2 \left(m_1 s + \tfrac{5}{6} + v\right)\left(m_1 s + \tfrac{7}{6} + v\right) \right\} \times \\ &\times \left\{ \prod_{v=0}^{m_2-1} (m_2 s + 1 + v)^4 \left(m_2 s + \tfrac{2}{3} + v\right)\left(m_2 s + \tfrac{4}{3} + v\right) \right\} \times \\ &\times \left\{ \prod_{v=0}^{m_1+m_2-1} \left((m_1 + m_2)s + \tfrac{3}{2} + v\right)^2 \left((m_1 + m_2)s + \tfrac{4}{3} + v\right) \times \right. \\ &\left. \times \left((m_1 + m_2)s + \tfrac{5}{3} + v\right) \right\} \times \\ &\times \left\{ \prod_{v=0}^{2m_1+m_2-1} ((2m_1 + m_2)s + 2 + v)^2 \right\}. \end{aligned}$$

(5–16) Weighted Dynkin diagram $\underset{0}{0200202}$

This prehomogeneous vector space is given as follows. Let

$$G = GL_3 \times GL_2 \times GL_2 \times SL_2 \quad \text{and} \quad V = M_{3,2}(\mathbb{C}) + M_{3,2}(\mathbb{C}) + M_{2,3}(\mathbb{C}) + \mathbb{C}^2.$$

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

The action is given by

$$gv = \left((g_1 X g_2^{-1}, g_1 Y g_2^{-1})^{\mathrm{t}} g_3, g_4 W g_1^{-1}, g_2 p\right)$$

for $g = (g_1, g_2, g_3, g_4) \in G$ and $v = (X, Y, W, p) \in V$.

Let

$$v_0 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right).$$

Then the isotropy group at v_0 is given by

$$G_{v_0} = \left\{ \begin{matrix} \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \\ \left(\sqrt{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sqrt{-1}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \sqrt{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \end{matrix} \right\}.$$

This prehomogeneous vector space has three irreducible relative invariants f_1, f_2, and f_3 as follows. The first relative invariant is given by

$$f_1(v) = \text{the discriminant of the binary quadratic form } \det(WX\xi + WY\eta)$$

in the variables $(\xi, \eta) \in \mathbb{C}^2$. Its character is $\phi_1(g) = (\det g_2)^{-2}(\det g_3)^2$. The second relative invariant is given by

$$f_2(v) = \text{the discriminant of the ternary quadratic form}$$
$$\det\left(\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}\xi + \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}\eta + \begin{pmatrix} X_3 \\ Y_3 \end{pmatrix}\zeta\right)$$

in the variables $(\xi, \eta, \zeta) \in \mathbb{C}^3$ for

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}.$$

For the discriminant of ternary quadratic forms, see (4–10). Its character is

$$\phi_2(g) = (\det g_1)^2(\det g_2)^{-3}(\det g_3)^3.$$

The third relative invariant is given by $f_3(v) = \det(WXp, WYp)$, whose character is $\phi_3(g) = (\det g_3)$.

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

We have that

$$\operatorname{grad}\log f_1(v_0) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right),$$
$$\operatorname{grad}\log f_2(v_0) = \left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right),$$
$$\operatorname{grad}\log f_3(v_0) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right).$$

Hence we get that

$$a_1(\underline{s}) = s_1(s_1+s_2)(s_1+s_3)(2s_1+s_3)^2(s_1+s_2+s_3)(2s_1+2s_2+s_3)^2,$$
$$a_2(\underline{s}) = s_2^2(s_1+s_2)(s_1+s_2+s_3)(2s_1+2s_2+s_3)^2,$$
$$a_3(\underline{s}) = s_3^2(s_1+s_3)(2s_1+s_3)(s_1+s_2+s_3)(2s_1+2s_2+s_3).$$

We have $\underline{\kappa} = (1, 1, 1)$.

Therefore the b-function of the relative invariant $\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2} f_3^{m_3}$ $(m_1, m_2, m_3 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$\begin{aligned} b_{\underline{f}^{\underline{m}}}(s) = {} & \left\{ \prod_{v=0}^{m_1-1} (m_1 s + 1 + v) \right\} \left\{ \prod_{v=0}^{m_2-1} (m_2 s + 1 + v)^2 \right\} \times \\ & \times \left\{ \prod_{v=0}^{m_3-1} (m_3 s + 1 + v)^2 \right\} \left\{ \prod_{v=0}^{m_1+m_2-1} \left((m_1+m_2)s + \tfrac{3}{2} + v\right) \right\} \times \\ & \times \left\{ \prod_{v=0}^{m_1+m_3-1} \left((m_1+m_3)s + \tfrac{3}{2} + v\right) \right\} \times \\ & \times \left\{ \prod_{v=0}^{2m_1+m_3-1} ((2m_1+m_3)s + 2 + v) \right\} \times \\ & \times \left\{ \prod_{v=0}^{m_1+m_2+m_3-1} ((m_1+m_2+m_3)s + 2 + v) \right\} \times \\ & \times \left\{ \prod_{v=0}^{2m_1+2m_2+m_3-1} ((2m_1+2m_2+m_3)s + 3 + v) \right\}. \end{aligned}$$

(5–15) Weighted Dynkin diagram $\overset{\textstyle 2020202}{0}$

This prehomogeneous vector space is given as follows. Let

$$G = GL_2 \times GL_2 \times GL_2 \times GL_2 \ \text{ and } \ V = M_2(\mathbb{C}) + M_2(\mathbb{C}) + M_2(\mathbb{C}) + \mathbb{C}^2 + \mathbb{C}^2.$$

Define the action by

$$gv = \left((g_1 X g_2^{-1}, g_1 Y g_2^{-1})^{\mathrm{t}} g_3, g_2 W g_4^{-1}, {}^{\mathrm{t}}g_1^{-1} p, g_4 q\right)$$

for $g = (g_1, g_2, g_3, g_4) \in G$ and $v = (X, Y, W, p, q) \in V$.

Let

$$v_0 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

Then the isotropy group at v_0 is given by

$$G_{v_0} = \left\{ \begin{matrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \\ \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \end{matrix} \right\}.$$

This prehomogeneous vector space has four irreducible relative invariants f_1, f_2, f_3 and f_4 given by

$$\begin{aligned} &f_1(v) = \det(XWq, YWq), \\ &f_2(v) = \text{the discriminant of the binary quadratic form } \det(X\xi + Y\eta) \\ &\text{in the variables } (\xi, \eta) \in \mathbb{C}^2, \\ &f_3(v) = \det\left({}^{\mathrm{t}}Xp, {}^{\mathrm{t}}Yp\right), \qquad f_4(v) = \det(W). \end{aligned}$$

Their characters are

$$\begin{aligned} &\phi_1(g) = (\det g_1)(\det g_3), \qquad \phi_2(g) = (\det g_1)^2(\det g_2)^{-2}(\det g_3)^2, \\ &\phi_3(g) = (\det g_2)^{-1}(\det g_3), \qquad \phi_4(g) = (\det g_2)(\det g_4)^{-1}. \end{aligned}$$

We have that

$$\begin{aligned} \operatorname{grad}\log f_1(v_0) &= \left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \\ \operatorname{grad}\log f_2(v_0) &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right), \\ \operatorname{grad}\log f_3(v_0) &= \left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right), \\ \operatorname{grad}\log f_4(v_0) &= \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right). \end{aligned}$$

Hence we get that

$$a_1(\underline{s}) = s_1^2(s_1+s_2)(2s_1+s_4)^2(s_1+s_2+s_3),$$
$$a_2(\underline{s}) = s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3),$$
$$a_3(\underline{s}) = s_3^2(s_2+s_3)(s_1+s_2+s_3),$$
$$a_4(\underline{s}) = s_4(2s_1+s_4).$$

We have $\underline{\kappa} = (1,1,1,1)$.

Therefore the *b*-function of the relative invariant $\underline{f}^{\underline{m}} = f_1^{m_1} f_2^{m_2} f_3^{m_3} f_4^{m_4}$ $(m_1, m_2, m_3, m_4 \in \mathbb{Z}_{\geqslant 0})$ is given by

$$\begin{aligned} b_{\underline{f}^{\underline{m}}}(s) = & \left\{\prod_{v=0}^{m_1-1}(m_1 s+1+v)^2\right\}\left\{\prod_{v=0}^{m_2-1}(m_2 s+1+v)\right\}\times \\ & \times\left\{\prod_{v=0}^{m_3-1}(m_3 s+1+v)^2\right\}\left\{\prod_{v=0}^{m_4-1}(m_4 s+1+v)\right\}\times \\ & \times\left\{\prod_{v=0}^{m_1+m_2-1}\left((m_1+m_2)s+\tfrac{3}{2}+v\right)\right\}\times \\ & \times\left\{\prod_{v=0}^{m_2+m_3-1}\left((m_2+m_3)s+\tfrac{3}{2}+v\right)\right\}\times \\ & \times\left\{\prod_{v=0}^{2m_1+m_4-1}((2m_1+m_4)s+2+v)\right\}\times \\ & \times\left\{\prod_{v=0}^{m_1+m_2+m_3-1}((m_1+m_2+m_3)s+2+v)\right\}. \end{aligned}$$

4. Tables

We give the summary of the results obtained in Section 3. Notations in the tables are the same as in Section 1.

The prehomogeneous vector space arising from the diagram in the table has l irreducible relative invariants $f_1, , f_l$. The *b*-function of the relative invariant $\underline{f}^{\underline{m}} = f_1^{m_1} \cdots f_l^{m_l}$ $(m_1, \ldots, m_l \in \mathbb{Z}_{\geqslant 0})$ is given by

$$b_{\underline{f}^{\underline{m}}}(s) = \prod_{j=1}^{N} \prod_{v=0}^{\gamma_j(\underline{m})-1} \prod_{r=1}^{\mu_j} \left(\gamma_j(\underline{m})s + \alpha_{j,r} + v\right).$$

The ordering of irreducible relative invariants is the same as in Section 3. The symbol $^{\times *}$ about $\alpha_{j,r}$ means multiplicity. For example, $1^{\times 2}, \frac{5}{6}, \frac{7}{6}$ means that $\{\alpha_{j,r}\ (1 \leqslant r \leqslant 4)\} = \{1, 1, \frac{5}{6}, \frac{7}{6}\}$.

The degrees $d_i = (d_{i1}, \ldots, d_{ik})$ of irreducible relative invariants in the fourth entries of the row of s_i in the table mean the following. Let $V_1, \ldots, V_k$ be the set of subspaces

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

corresponding to the 2 of the diagram, whose ordering is from left to right. Then we define $d_i = (d_{i1}, \ldots, d_{ik}) := (\deg_{V_1} f_i, \ldots, \deg_{V_k} f_i)$.

4.1. TYPE G_2

	Diagram	l	d_i	γ_j	$\alpha_{j,r}$
(1)	$2 \Rrightarrow 0$	1	4	s_1	$1^{\times 2}, \frac{5}{6}, \frac{7}{6}$

4.2. TYPE F_4

	Diagram	l	d_i	γ_j	$\alpha_{j,r}(1)$
(1)	$20 \Rightarrow 00$	1	4	s_1	$1, 2, \frac{5}{2}, \frac{7}{2}$
(2)	$00 \Rightarrow 02$	1	2	s_1	1, 4
(3)	$02 \Rightarrow 00$	1	12	s_1	$1^{\times 4}, \frac{3}{4}^{\times 2}, \frac{5}{4}^{\times 2}, \frac{5}{6}^{\times 2}, \frac{7}{6}^{\times 2}$
(4)	$02 \Rightarrow 02$	2	(4, 0) (2, 2)	s_1 s_2 $s_1 + s_2$ $2s_1 + s_2$	1 $1^{\times 2}$ $\frac{3}{2}$ 2

4.3. TYPE E_6

	Diagram	l	d_i	γ_j	$\alpha_{j,r}$
(1)	00000 2	1	4	s_1	$1, 5, \frac{5}{2}, \frac{7}{2}$
(2)	20002 0	2	(2, 0) (0, 2)	s_1 s_2	1, 4 1, 4
(3)	00200 0	1	12	s_1	$1^{\times 4}, \frac{3}{2}^{\times 4}, \frac{4}{3}, \frac{5}{3}, \frac{5}{6}, \frac{7}{6}$
(4)	20002 2	2	(0, 2, 0) (1, 0, 1)	s_1 s_2	1, 3 1, 4
(5)	20202 0	3	(0, 4, 0) (2, 2, 0) (0, 2, 2)	s_1 s_2 s_3 $s_1 + s_2$ $s_1 + s_3$ $s_1 + s_2 + s_3$	1 $1^{\times 2}$ $1^{\times 2}$ $\frac{3}{2}$ $\frac{3}{2}$ 2

4.4. TYPE E_7

	Diagram	l	d_i	γ_j	$\alpha_{j,r}$
(1)	200000 0	1	3	s_1	$1,\ 5,\ 9$
(2)	000002 0	1	4	s_1	$1,\ 8,\ \frac{7}{2},\ \frac{11}{2}$
(3)	000000 2	1	7	s_1	$1,\ 2,\ 3,\ 4,\ 5,\ \frac{5}{2},\ \frac{7}{2}$
(4)	020000 0	1	4	s_1	$1,\ 4,\ 5,\ 8$
(5)	200002 0	2	(1, 2) (2, 0)	s_1 s_2	$1,\ 8$ 1
(6)	000020 0	1	12	s_1+s_2 s_1	5 $1^{\times 2},\ 2^{\times 2},\ \frac{3}{2}^{\times 2},\ \frac{5}{2}^{\times 2},\ \frac{7}{3},\ \frac{8}{3},\ \frac{5}{6},\ \frac{7}{6}$
(7)	002000 0	1	15	s_1	$1^{\times 3},\ 2^{\times 3},\ \frac{3}{2}^{\times 3},\ \frac{4}{3}^{\times 2},\ \frac{5}{3}^{\times 2},\ \frac{5}{4},\ \frac{7}{4}$
(8)	020002 0	2	(4, 0) (0, 2)	s_1 s_2	$1,\ 4,\ \frac{3}{2},\ \frac{7}{2}$ $1,\ 4$
(9)	000200 0	1	12	s_1	$1^{\times 3},\ 2^{\times 3},\ \frac{3}{2}^{\times 2},\ \frac{4}{3}^{\times 2},\ \frac{5}{3}^{\times 2}$
(10)	002002 0	2	(6, 0) (3, 2)	s_1 s_2 s_1+s_2	$1,\ 2,\ \frac{3}{2}$ $1,\ 2$ $2,\ 3,\ \frac{5}{2}$
(11)	020020 0	2	(4, 4) (0, 4)	s_1 s_2 s_1+s_2 $2s_1+s_2$	$1,\ 2,\ \frac{3}{2}^{\times 2}$ 1 $\frac{3}{2},\ \frac{5}{2}$ 3
(12)	200200 0	2	(0, 12) (3, 6)	s_1 s_2 s_1+s_2 $2s_1+s_2$	$1^{\times 2},\ \frac{5}{6},\ \frac{7}{6}$ $1^{\times 3}$ $\frac{3}{2}^{\times 2},\ \frac{4}{3},\ \frac{5}{3}$ $2^{\times 2}$
(13)	020200 0	2	(0, 6) (2, 0)	s_1 s_2	$1^{\times 2},\ 2^{\times 2},\ \frac{3}{2}^{\times 2}$ $1,\ 2$

4.4. TYPE E_7 *Continued*

	Diagram	l	d_i	γ_j	$\alpha_{j,r}$
(14)	200202 0	3	(1, 4, 2) (0, 6, 0) (2, 2, 0)	s_1 s_2 s_3 s_1+s_2 s_1+s_3 s_1+2s_2 $s_1+s_2+s_3$	$1^{\times 2}$ 1 1 $\frac{3}{2}$ $\frac{3}{2}$ 2 $2^{\times 2}$
(15)	020202 0	3	(0, 4, 0) (0, 2, 2) (2, 0, 0)	s_1 s_2 s_3 s_1+s_2	$1, \frac{3}{2}$ $1^{\times 2}$ 1, 2 $2, \frac{3}{2}$

4.5. TYPE E_8

	Diagram	l	d_i	γ_j	$\alpha_{j,r}$
(1)	2000000 0	1	4	s_1	$1, 14, \frac{11}{2}, \frac{19}{2}$
(2)	0000002 0	1	8	s_1	$1, 4, 5, 8, \frac{5}{2}, \frac{7}{2}, \frac{11}{2}, \frac{13}{2}$
(3)	0200000 0	1	12	s_1	$1^{\times 2}, 3^{\times 2}, \frac{5}{2}^{\times 2}, \frac{9}{2}^{\times 2}, \frac{13}{3}, \frac{14}{3}, \frac{5}{6}, \frac{7}{6}$
(4)	2000002 0	2	(0, 4) (2, 0)	s_1 s_2	$1, 8, \frac{7}{2}, \frac{11}{2}$ 1, 6
(5)	0000000 2	1	16	s_1	$1, 2^{\times 3}, 3^{\times 2}, \frac{3}{2}^{\times 2}, \frac{5}{2}^{\times 3}$ $\frac{7}{2}, \frac{7}{3}, \frac{8}{3}, \frac{11}{6}, \frac{13}{6}$
(6)	0020000 0	1	12	s_1	$1, 2^{\times 2}, 3^{\times 2}, 4, \frac{3}{2}, \frac{7}{2}, \frac{5}{3}, \frac{7}{3}, \frac{8}{3}, \frac{10}{3}$
(7)	2000000 2	2	(0, 7) (2, 3)	s_1 s_2 s_1+s_2	$1, 2, \frac{5}{2}, \frac{7}{2}$ $1, \frac{7}{2}$ 3, 4, 5
(8)	0200002 0	2	(4, 4) (4, 0)	s_1 s_2 s_1+s_2 $2s_1+s_2$	$1^{\times 2}, 4^{\times 2}$ 1 $\frac{3}{2}, \frac{9}{2}$ 5
(9)	0002000 0	1	40	s_1	$1^{\times 8}, \frac{2}{3}^{\times 4}, \frac{4}{3}^{\times 4}, \frac{3}{4}^{\times 4}, \frac{5}{4}^{\times 4}, \frac{5}{6}^{\times 4},$ $\frac{7}{6}^{\times 4}, \frac{7}{10}^{\times 2}, \frac{9}{10}^{\times 2}, \frac{11}{10}^{\times 2}, \frac{13}{10}^{\times 2}$

4.5. TYPE E_8 *Continued*

	Diagram	l	d_i	γ_j	$\alpha_{j,r}$
(10)	0020002 0	2	(6, 0) (0, 2)	s_1 s_2	$1, 2, 3, 4, \frac{3}{2}, \frac{7}{2}$ $1, 4$
(11)	2002000 0	2	(0, 15) (2, 5)	s_1 s_2 s_1+s_2 $2s_1+s_2$ $3s_1+s_2$	$1^{\times 2}, \frac{3}{2}^{\times 2}$ $1, \frac{3}{2}$ 2 $3, \frac{5}{2}$ $3, 4$
(12)	2020002 0	3	(0, 4, 0) (2, 2, 0) (0, 0, 2)	s_1 s_2 s_3 s_1+s_2	$1, \frac{7}{2}$ $1^{\times 2}$ $1, 4$ $4, \frac{3}{2}$
(13)	2000200 0	2	(4, 12) (0, 12)	s_1 s_2 s_1+s_2 $2s_1+s_2$ $2s_1+3s_2$	$1^{\times 2}, \frac{5}{6}, \frac{7}{6}$ $1^{\times 2}$ $\frac{3}{2}^{\times 2}, \frac{4}{3}, \frac{5}{3}$ $2^{\times 3}$ 3
(14)	2002002 0	3	(0, 6, 0) (0, 3, 2) (2, 2, 0)	s_1 s_2 s_3 s_1+s_2 s_1+s_3 $s_1+s_2+s_3$	$1, \frac{3}{2}$ $1, 2$ $1, \frac{3}{2}$ $2, \frac{5}{2}$ 2 3
(15)	0200200 0	2	(0, 12) (6, 6)	s_1 s_2 s_1+s_2 $2s_1+s_2$	$1^{\times 2}, \frac{5}{6}, \frac{7}{6}$ $1^{\times 4}, \frac{2}{3}, \frac{4}{3}$ $\frac{3}{2}^{\times 2}, \frac{4}{3}, \frac{5}{3}$ $2^{\times 2}$
(16)	0200202 0	3	(4, 4, 0) (0, 6, 0) (2, 2, 2)	s_1 s_2 s_3 s_1+s_2 s_1+s_3 $2s_1+s_3$ $s_1+s_2+s_3$ $2s_1+2s_2+2s_3$	1 $1^{\times 2}$ $1^{\times 2}$ $\frac{3}{2}$ $\frac{3}{2}$ 2 2 3
(17)	2020202 0	4	(2, 2, 2, 0) (0, 0, 4, 0) (0, 0, 2, 2) (0, 2, 0, 0)	s_1 s_2 s_3 s_4 s_1+s_2 s_2+s_3 $2s_1+s_4$ $s_1+s_2+s_3$	$1^{\times 2}$ 1 $1^{\times 2}$ 1 $\frac{3}{2}$ $\frac{3}{2}$ 2 2

https://doi.org/10.1023/A:1021768026536 Published online by Cambridge University Press

References

1. Alekseevsky, A. J.: Component groups of centralizer for unipotent elements in semi-simple algebraic groups, *Trudy Tbiliss. Mat. Inst.* **62** (1979), 2–27.
2. Denef, J. and Gyoja, A.: Character sums associated to prehomogeneous vector spaces, *Compositio Math.* **113** (1998), 273–346.
3. Elašvili, A. G.: The centralizers of nilpotent elements in the semi-simple Lie algebras, *Trudy Tbiliss. Mat. Inst.* **46** (1975), 109–132.
4. Gyoja, A.: Theory of prehomogeneous vector spaces without regularity condition, *Publ. RIMS Kyoto Univ.* **27** (1991), 861–922.
5. Gyoja, A.: A theorem of Chevalley type for prehomogeneous vector spaces, *J. Math. Soc. Japan* **48** (1996), 161–167.
6. Gyoja, A.: Lefschetz principle in the theory of prehomogeneous vector spaces, *Adv. Stud. Pure Math.* **21** (1992), 87–99.
7. Gyoja, A.: Construction of invariants, *Tsukuba J. Math.* **14** (1990), 437–457.
8. Gyoja, A.: Lecture notes, Nagoya University (1999).
9. Kaneko, Y.: On the *b*-functions of prehomogeneous vector spaces associated to nilpotent orbits with classical Lie algebra actions (in Japanese), Master Thesis, Nagoya University, 2001.
10. Kasai, S.: On the *b*-functions of regular prehomogeneous vector spaces with simple group action (in Japanese), *RIMS Kokyuroku, Kyoto Univ.* **629** (1987), 31–81.
11. Kashiwara, M.: B-functions and holonomic systems, *Invent. Math.* **38** (1976), 33–58.
12. Kawahara, H.: Prehomogeneous vector spaces related with the spin group, Master Thesis in Japanese, University of Tokyo (1974).
13. Kimura, T.: The *b*-functions and holonomy diagrams of irreducible regular prehomogeneous vector spaces, *Nagoya Math. J.* **85** (1982), 1–80.
14. Kimura, T.: A classification of prehomogeneous vector spaces of simple algebraic groups with scalar multiplications, *J. Algebra* **83** (1983), 72–100.
15. Kimura, T.: Remark on some combinatorial construction of relative invariants, *Tsukuba J. Math.* **5** (1981), 101–115.
16. Kimura, T. and Muro, M.: On some series of regular irreducible prehomogeneous vector spaces, *Proc. Japan Acad.* **55** (1979), 384–389.
17. Kimura, T. and Ozeki, I.: On the microlocal structure of a regular prehomogeneous vector space associated with $Spin(10) \times GL(3)$, *Proc. Japan Acad. Ser. A* **58** (1982), 239–242.
18. Ozeki, I.: On the micro-local structure of a regular prehomogeneous vector associated with $GL(8)$, *Proc. Japan Acad. A* **56** (1) (1980), 18–21.
19. Sato, F.: *b*-Functions of prehomogeneous vector spaces attached to flag manifolds of the general linear group, *Comment. Math. Univ. St. Pauli* **48** (1999), 129–136.
20. Sato, M., Kashiwara, M., Kimura, T. and Oshima, T.: Micro-local analysis of prehomogeneous vector spaces, *Invent. Math.* **62** (1980), 117–179.
21. Sato, M. and Kimura, T.: A classification of irreducible prehomogeneous vector spaces and their relative invariants, *Nagoya Math. J.* **65** (1977), 1–55.
22. Sato, M., Shintani, T. and Muro, M.: Theory of prehomogeneous vector spaces (algebraic part), *Nagoya Math. J.* **120** (1990), 1–34.
23. Sato, M.: Theory of prehomogeneous vector spaces (notes by T. Shintani in Japanese), *Sugaku no Ayumi* **15-1** (1970), 75–157.
24. Shintani, T.: On Dirichlet series whose coefficients are class numbers of integral binary cubic forms, *J. Math. Soc. Japan* **24** (1972), 132–188.

25. Shintani, T.: On transformation formula for *b*-functions with respect to the castling transformation (notes by T.Kimura in Japanese), *RIMS Kokyuroku, Kyoto Univ.* **497** (1982), 73–76.
26. Springer, T. A. and Steinberg, R.: Conjugacy classes, In: Lecture Notes in Math. 131, Springer, New York, 1970, pp. 167–266.
27. Ukai, K.: *b*-Functions of the prehomogeneous vector space arising from a cuspidal character sheaf of E_7, *J. Algebra* **237** (2001), 358–381.
28. Yano, T. and Ozeki, I.: Microlocal structure of the regular prehomogeneous vector space associated with $SL(5) \times GL(4)$, II, *RIMS Kokyuroku, Kyoto Univ.* **999** (1997), 92–115.