J. Aust. Math. Soc. **106** (2019), 200–234
doi:10.1017/S1446788718000101

THE LOOMIS–SIKORSKI THEOREM FOR *EMV*-ALGEBRAS

ANATOLIJ DVUREČENSKIJ and OMID ZAHIRI✉

(Received 3 July 2017; accepted 19 March 2018; first published online 23 August 2018)

Communicated by J. East

Abstract

An EMV-algebra resembles an MV-algebra in which a top element is not guaranteed. For σ-complete EMV-algebras, we prove an analogue of the Loomis–Sikorski theorem showing that every σ-complete EMV-algebra is a σ-homomorphic image of an EMV-tribe of fuzzy sets where all algebraic operations are defined by points. To prove it, some topological properties of the state-morphism space and the space of maximal ideals are established.

2010 *Mathematics subject classification*: primary 06C15; secondary 06D35.

Keywords and phrases: MV-algebra, idempotent element, EMV-algebra, σ-complete EMV-algebra, EMV-clan, EMV-tribe, state-morphism, ideal, filter, hull-kernel topology, the Loomis–Sikorski theorem.

1. Introduction

Boolean algebras are well-known structures that have been studied over many decades. They describe an algebraic semantics for two-valued logic. In the 1930s, Boolean rings appeared, or equivalently, generalized Boolean algebras, which have almost Boolean features, but a top element is not assumed. For such structures, Stone, see for example [16, Theorem 6.6], developed a representation of Boolean rings by rings of subsets, and also some logical models with such incomplete information were established, see [20, 21].

Our approach in [10] was based on analogous ideas: develop a Łukasiewicz-type algebraic structure with incomplete total information, that is, find an algebraic semantics very similar to MV-algebras with incomplete information, which however in a local sense is complete, meaning the following: conjunctions and disjunctions exist, negation only in a local sense, that is, negation of a in b exists whenever $a \leq b$ but total negation of the event a is not assumed. For such ideas we have introduced in [10] EMV-algebras which are locally close to MV-algebras, however, a top element

The first author is grateful for support from grants APVV-16-0073, VEGA no. 2/0069/16 SAV and GAČR 15-15286S.

is not assumed. Every EMV-algebra with a top element is termwise equivalent to an MV-algebra and vice versa.

The basic representation theorem says, [10, Theorem 5.21], that even in such a case, we can find an EMV-algebra with a top element where the original algebra can be embedded as its maximal ideal, that is, incomplete information hidden in an EMV-algebra is sufficient to find a Łukasiewicz logical system where a top element exists and where all original statements are valid.

EMV-algebras generalize Chang's MV-algebras, [3]. Nowadays, MV-algebras have many important applications in different areas of mathematics and logic. Therefore, MV-algebras have many different generalizations, like BL-algebras, pseudo MV-algebras, [8, 12], GMV-algebras in the realm of residuated lattices, [11], and so on. In recent years, MV-algebras have also been studied in frames of involutive semirings, see [6]. The presented EMV-algebras are another kind of generalization of MV-algebras inspired by Boolean rings.

We note that for σ-complete MV-algebras, a variant of the Loomis–Sikorski theorem was established in [1, 7, 18]. It was shown that, for every σ-complete MV-algebra M, there is a tribe of fuzzy sets, which is a σ-complete MV-algebra of $[0,1]$-valued functions with all MV-operations defined by points, that can be σ-homomorphically embedded onto M.

The aim of the present paper is to formulate and prove a Loomis–Sikorski-type theorem for σ-complete EMV-algebras showing that every σ-complete EMV-algebra is a σ-homomorphic image of an EMV-tribe of fuzzy sets, where all EMV-operations are defined by points.

To show this, we introduce the hull-kernel topology of the maximal ideals of EMV-algebras and the weak topology of state-morphisms which are EMV-homomorphisms from the EMV-algebra into the MV-algebra of the real interval $[0,1]$, or equivalently, a variant of extremal probability measures.

The paper is organized as follows. Section 2 gathers the main notions and results on EMV-algebras showing that every EMV-algebra without a top element can be embedded into an EMV-algebra with a top element as its maximal ideal. Dedekind σ-complete EMV-algebras are studied in Section 3 where some one-to-one relationships among maximal ideals, maximal filters and state-morphisms are also established. In Section 4 we introduce the weak topology of state-morphisms and the hull-kernel topology of maximal ideals. We show that these spaces are always mutually homeomorphic, locally compact Hausdorff spaces which are compact if and only if the EMV-algebra possesses a top element. We prove that if our EMV-algebra has no top element, then the state-morphism space of the representing MV-algebra is the one-point compactification of the state-morphism space of the original EMV-algebra. The Loomis–Sikorski representation theorem will be established in Section 5 together with some topological properties of the state-morphism space and the space of the maximal ideals.

2. Elements of EMV-algebras

An *MV-algebra* is an algebra $(M; \oplus, ^*, 0, 1)$ (henceforth write simply $M = (M; \oplus, ^*, 0, 1)$) of type $(2, 1, 0, 0)$, where $(M; \oplus, 0)$ is a commutative monoid with the neutral element 0 and for all $x, y \in M$:

(i) $x^{**} = x$;
(ii) $x \oplus 1 = 1$;
(iii) $x \oplus (x \oplus y^*)^* = y \oplus (y \oplus x^*)^*$.

In any MV-algebra $(M; \oplus, ^*, 0, 1)$, we can also define the following operations:

$$x \odot y := (x^* \oplus y^*)^*, \quad x \ominus y := (x^* \oplus y)^*.$$

Then M is a distributive lattice where $x \vee y = (x \ominus y) \oplus y$ and $x \wedge y = x \odot (x^* \oplus y)$. Note that, for each $x \in M$, x^* is the least element of the set $\{z \in M \mid x \oplus z = 1\}$, that is,

$$x^* := \min\{z \in M \mid z \oplus x = 1\}. \tag{2.1}$$

For example, if (G, u) is an Abelian unital ℓ-group with strong unit u, then the interval $[0, u]$ can be converted into an MV-algebra as follows: $x \oplus y := (x + y) \wedge u$, $x^* := u - x$ for all $x, y \in [0, u]$. Then $\Gamma(G, u) := ([0, u]; \oplus, ^*, 0, u)$ is an MV-algebra and due to the Mundici result, every MV-algebra is isomorphic to some $\Gamma(G, u)$, see [17]. For more information about MV-algebras, see [4].

An element $a \in M$ is said to be *Boolean* or *idempotent* if $a \oplus a = a$, or equivalently, $a \vee a^* = 1$. The set $B(M)$ of Boolean elements of M forms a Boolean algebra.

Given $a \in B(M)$, we can define a new MV-algebra M_a whose universe is the interval $[0, a]$ and the MV-operations are inherited from the original one as follows: $M_a = ([0, a]; \oplus, ^{*_a}, 0, a)$, where $x^{*_a} = a \odot x^*$ for each $x \in [0, a]$. Then,

$$x^{*_a} = \min\{z \in [0, a] : z \oplus x = a\}, \quad x \in [0, a].$$

In this paper, we will also write $\lambda_a(x) := x^{*_a}$, $x \in [0, a]$.

Inspired by these properties of MV-algebras, in [10], we have introduced EMV-algebras as follows. Let $(M; \oplus, 0)$ be a commutative monoid with a neutral element 0. An element $a \in M$ is said to be an *idempotent* if $a \oplus a = a$. We denote by $\mathcal{I}(M)$ the set of idempotent elements of M; clearly $0 \in \mathcal{I}(M)$, and if $a, b \in I(M)$, then $a \oplus b \in \mathcal{I}(M)$.

According to [10], an *EMV-algebra* is an algebra $(M; \vee, \wedge, \oplus, 0)$ of type $(2, 2, 2, 0)$ such that:

(i) $(M; \oplus, 0)$ is a commutative ordered monoid with a neutral element 0;
(ii) $(M; \vee, \wedge, 0)$ is a distributive lattice with the bottom element 0;
(iii) for each idempotent $a \in \mathcal{I}(M)$, the algebra $([0, a]; \oplus, \lambda_a, 0, a)$ is an MV-algebra, where

$$\lambda_a(x) = \min\{z \in [0, a] \mid z \oplus x = a\}, \quad x \in [0, a];$$

(iv) for each $x \in M$, there is an idempotent a of M such that $x \leq a$.

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

We note that the existence of a top element in an EMV-algebra is not assumed, and if it exists, then $M = (M; \oplus, \lambda_1, 0, 1)$ is an MV-algebra. We note that every MV-algebra $(M; \oplus, ^*, 0, 1)$ forms an EMV-algebra $(M; \vee, \wedge, \oplus, 0)$ with top element 1, every Boolean ring or equivalently a generalized Boolean algebra (= a relatively complemented distributive lattice with a bottom element) is an EMV-algebra.

Besides the operation $\oplus$ we can define an operation $\odot$ as follows: let $x, y \in M$ and let $x, y \leq a \in \mathcal{I}(M)$. Then

$$x \odot y := \lambda_a(\lambda_a(x) \oplus \lambda_a(y)).$$

As shown in [10, Lemma 5.1], the operation $\odot$ does not depend on $a \in \mathcal{I}(M)$. Then, if $x, y \in [0, a]$ for some idempotent $a \in M$, then

$$x \odot \lambda_a(y) = x \odot \lambda_a(x \wedge y) \quad \text{and} \quad x = (x \wedge y) \oplus (x \odot \lambda_a(y)). \tag{2.2}$$

For any integer $n \geq 1$ and any x of an EMV-algebra M, we can define

$$0.x = 0, \quad 1.x = x, \quad (n+1).x = (n.x) \oplus x,$$

and

$$x^1 = x, \quad x^n = x^{n-1} \odot x, \ n \geq 2,$$

and if M has a top element 1, we define also $x^0 = 1$.

We define the classical notions like ideal: an *ideal* of an EMV-algebra is a nonvoid subset I of M such that (i) if $x \leq y \in I$, then $x \in I$, and (ii) if $x, y \in I$, then $x \oplus y \in I$. An ideal is *maximal* if it is a proper ideal of M which is not properly contained in another proper ideal of M. Despite M not necessarily having a top element, every $M \neq \{0\}$ has a maximal ideal, see [10, Theorem 5.6]. We denote by $\mathrm{MaxI}(M)$ the set of maximal ideals of M. The *radical* $\mathrm{Rad}(M)$ of M, is the intersection of all maximal ideals of M, and for it,

$$\mathrm{Rad}(M) = \{x \in M \setminus \{0\} \mid \exists\, a \in \mathcal{I}(M) : x \leq a \ \& \ (n.x \leq \lambda_a(x),\ \forall\, n \in \mathbb{N})\} \cup \{0\}. \tag{2.3}$$

A *filter* is a dual notion to ideals, that is, a nonvoid subset F of M such that (i) $x \geq y \in F$ implies $x \in F$, and (ii) if $x, y \in F$, then $x \odot y \in F$.

A subset $A \subseteq M$ is called an *EMV-subalgebra* of M if A is closed under $\vee, \wedge, \oplus$ and 0 and, for each $b \in \mathcal{I}(M) \cap A$, the set $[0, b]_A := [0, b] \cap A$ is a subalgebra of the MV-algebra $([0, b]; \oplus, \lambda_b, 0, b)$. Clearly, the last condition is equivalent to the following condition:

$$\forall b \in A \cap \mathcal{I}(M),\ \forall x \in [0, b]_A,\ \min\{z \in [0, b]_A \mid x \oplus z = b\} = \min\{z \in [0, b] \mid x \oplus z = b\},$$

or equivalently, $x \in [0, b] \cap A$ implies $\lambda_b(x) \in [0, b] \cap A$ whenever $b \in A \cap \mathcal{I}(M)$. Let $(M_1; \vee, \wedge, \oplus, 0)$ and $(M_2; \vee, \wedge, \oplus, 0)$ be EMV-algebras. A map $f : M_1 \to M_2$ is called an *EMV-homomorphism* if f preserves the operations $\vee$, $\wedge$, $\oplus$ and 0, and for each $b \in \mathcal{I}(M_1)$ and for each $x \in [0, b]$, $f(\lambda_b(x)) = \lambda_{f(b)}(f(x))$.

As it was said, it can happen that an EMV-algebra M has no top element, however, it can be embedded into an EMV-algebra N with a top element as its maximal ideal as it was proved in [10, Theorem 5.21].

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

Theorem 2.1 (Basic representation theorem). *Every EMV-algebra $(M; \vee, \wedge, \oplus, 0)$ is either termwise equivalent to the MV-algebra $(M; \oplus, \lambda_1, 0, 1)$ or M can be embedded into an EMV-algebra N with a top element as a maximal ideal of N such that every element $x \in N$ either belongs to the image of the embedding of M, or it is a complement of some element x_0 belonging to the image of the embedding of M, that is, $x = \lambda_1(x_0)$.*

The EMV-algebra N from the latter theorem is said to be *representing* the EMV-algebra M. A similar result for generalized Boolean algebras was established in [5, Theorem. 2.2].

A mapping $s : M \to [0, 1]$ is called a *state-morphism* if s is an EMV-homomorphism from M into the EMV-algebra of the real interval $[0, 1]$ such that there is an element $x \in M$ with $s(x) = 1$. We denote by $\mathcal{SM}(M)$ the set of state-morphisms on M. In [10, Theorem 4.2] it was shown that if $M \neq \{0\}$, M admits at least one state-morphism. In addition, there is a one-to-one correspondence between state-morphisms and maximal ideals given by a relation: if s is a state-morphism, then $\mathrm{Ker}(s) = \{x \in M \mid s(x) = 0\}$ is a maximal ideal of M, and conversely, for each maximal ideal I there is a unique state-morphism s on M such that $\mathrm{Ker}(s) = I$.

An EMV-algebra M is said to be *semisimple* if $\mathrm{Rad}(M) = \{0\}$. Semisimple EMV-algebras can be characterized by EMV-clans. A system $\mathcal{T} \subseteq [0, 1]^{\Omega}$ of fuzzy sets of a set $\Omega \neq \emptyset$ is said to be an *EMV-clan* if:

(i) $0_{\Omega} \in \mathcal{T}$ where $0_{\Omega}(\omega) = 0$ for each $\omega \in \Omega$;

(ii) if $a \in \mathcal{T}$ is a characteristic function (that is, $Im(a) \subseteq \{0, 1\}$), then (a) $a - f \in \mathcal{T}$ for each $f \in \mathcal{T}$ such that $f(\omega) \leq a(\omega)$ for each $\omega \in \Omega$, (b) if $f, g \in \mathcal{T}$ with $f(\omega), g(\omega) \leq a(\omega)$ for each $\omega \in \Omega$, then $f \oplus g \in \mathcal{T}$, where $(f \oplus g)(\omega) = \min\{f(\omega) + g(\omega), a(\omega)\}$, $\omega \in \Omega$;

(iii) for each $f \in \mathcal{T}$, there is a characteristic function $a \in \mathcal{T}$ such that $f(\omega) \leq a(\omega)$ for each $\omega \in \Omega$;

(iv) given $\omega \in \Omega$, there is $f \in \mathcal{T}$ such that $f(\omega) = 1$.

Then M is semisimple if and only if there is an EMV-clan $\mathcal{T}$ that is isomorphic to M, see [10, Theorem 4.11].

For other unexplained notions and results, please see the paper [10].

3. Dedekind σ-complete EMV-algebras

In the present section, we study Dedekind σ-complete EMV-algebras and we show a one-to-one correspondence between the set of maximal ideals and the set of maximal filters using the notion of state-morphisms.

We say that an EMV-algebra M is *Archimedean in the sense of Belluce* if, for each $x, y \in M$ with $n.x \leq y$ for all $n \geq 0$, we have $x \odot y = x$. This notion was introduced by [2] for MV-algebras, see also [9, page 395].

Proposition 3.1. *Let M be an EMV-algebra. The following statements are equivalent.*

(i) *M is Archimedean in the sense of Belluce.*

(ii) *For each $a \in \mathcal{I}(M)$, the MV-algebra $[0, a]$ is Archimedean in the sense of Belluce.*

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

(iii) *For each* $a \in \mathcal{I}(M)$*, the* MV*-algebra* $[0, a]$ *is semisimple.*
(iv) M *is semisimple.*

PROOF. (i) ⇒ (ii) If $x, y \in [0, a]$, then $x \odot y \in [0, a]$ so that the implication is evident.

(ii) ⇒ (i) Let $x, y \in M$ and let $n.x \le y$ for each $n \ge 0$. There is an idempotent $a \in M$ such that $x, y \le a$. Hence $n.x \le y \le a$, so that $x \odot y = x$.

(ii) ⇔ (iii) It follows from [2, Theorems 31, 33].

(iii) ⇒ (iv) We use equation (2.3). Assume $x \in \mathrm{Rad}(M)$. By [10, Theorem 5.14], there is an idempotent $a \in M$ such that $x \le a$ and $n.x \le \lambda_a(x)$. Using Archimedeanicity in the sense of Belluce holding in the MV-algebra $[0, a]$, we have $0 = x \odot \lambda_a(x) = x$, so that $\mathrm{Rad}(M) = \{0\}$ and M is semisimple.

(iv) ⇒ (iii) Let a be an arbitrary idempotent of M. If I is a maximal ideal of M, then by [10, Proposition 3.23], $[0, a] \cap I$ is either $[0, a]$ or a maximal ideal of $[0, a]$. Since $\{0\} = \mathrm{Rad}(M) = \bigcap\{I \mid I \in \mathrm{MaxI}(M)\}$, we have $\mathrm{Rad}([0, a]) \subseteq [0, a] \cap \mathrm{Rad}(M) = \{0\}$ proving $[0, a]$ is a semisimple MV-algebra. □

According to the basic representation theorem, Theorem 2.1, every EMV-algebra M is either termwise equivalent to an MV-algebra or it can be embedded into an EMV-algebra N with a top element as its maximal ideal, so that we can assume that M is an EMV-subalgebra of N. We define a notion of Dedekind σ-complete EMV-algebras as follows.

We say that an EMV-algebra M is *Dedekind σ-complete* if, for each sequence $\{x_n\}$ of elements of M for which there is an element $x_0 \in M$ such that $x_n \le x_0$ for each n, $\bigvee_n x_n$ exists in M. It is easy to see that M is Dedekind σ-complete if and only if $[0, a]$ is a σ-complete MV-algebra for each idempotent $a \in M$.

LEMMA 3.2.

(i) *If* $x \in M$ *is the least upper bound of a sequence* $\{x_n\}$ *of elements of an* EMV*-algebra* M*, then it is the least upper bound in* N.
(ii) *If* $\{x_n\}$ *has an upper bound* $a \in \mathcal{I}(M)$*, then* $\bigvee_n x_n$ *exists in* M *if and only if it exists in the* MV*-algebra* $[0, a]$*. In either case, the suprema coincide.*
(iii) M *is Dedekind* σ*-complete if and only if, given a sequence* $\{y_n\}$ *of elements of* M*, there is* $y = \bigwedge_n y_n \in M$.

If $x = \bigvee_n x_n \le a \in \mathcal{I}(M)$*, then*

$$\lambda_a(x) = \bigwedge_n \lambda_a(x_n),$$

and if $y = \bigwedge_n y_n$ *and* $y_n \le a \in \mathcal{I}(M)$*, then*

$$\lambda_a(y) = \bigvee_n \lambda_a(y_n).$$

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

PROOF.

(i) If $M = N$, the statement is trivial. So let M be a proper EMV-algebra, that is, $M \subsetneq N$. Assume that for $y \in N \setminus M$, we have $x_n \leq y$ for each n. Then $y = y_0^* := \lambda_1(y_0)$ for some $y_0 \in M$, where 1 is the top element of N. We have $x_n \leq x \wedge y_0^* \leq x, y_0^*$. Since M is a maximal ideal of N, we have $x \wedge y_0^* \in M$ which entails $x \leq x \wedge y_0^* \leq x$, and finally $x \leq y_0^*$ proving x is the least upper bound also in N.

(ii) Let $x = \bigvee_n x_n$, and $x \leq a \in \mathcal{I}(M)$. If $y \in [0, a]$ is an upper bound of $\{x_n\}$, then clearly $x \leq y$, so that x is also its least upper bound taken in $[0, a]$. Conversely, let x be the least upper bound of $\{x_n\}$ taken in the MV-algebra $[0, a]$ and let $y \in M$ be an arbitrary upper bound of $\{x_n\}$. Then $x_n \leq y \wedge a \leq a$ so that $x \leq y \wedge a \leq y$.

(iii) Assume M is Dedekind σ-complete and let $\{y_n\}$ be a sequence of elements of M. Since M is a lattice, we can assume $y_{n+1} \leq y_n \leq y_1$ for each $n \geq 1$. There is an idempotent $a \in M$ such that $y_n \leq a$ for each $n \geq 1$. Then $\lambda_a(y_n) \leq \lambda_a(y_{n+1}) \leq a$, so that there is $y_0 = \bigvee_n \lambda_a(y_n) \in [0, a]$. We assert $\lambda_a(y_0) = \bigwedge_n y_n$. Let $y' \leq y_n$ for each $n \geq 1$, then $\lambda_a(y_n) \leq \lambda_a(y')$ so that $y_0 \leq \lambda_a(y')$, and $y' = \lambda_a^2(y') \leq \lambda_a(y_0)$.

Conversely, let every sequence from M have the infimum in M. Let $\{x_n\}$ be an arbitrary sequence from M with an upper bound $x_0 \in M$; we can assume $x_n \leq x_{n+1}$ for each $n \geq 1$. There is an idempotent $a \in M$ such that $x_n \leq x_0 \leq a$. Then $a \geq \lambda_a(x_n) \geq \lambda_a(x_{n+1}) \geq \lambda_a(x_0)$, and there is $z_0 = \bigwedge_n \lambda_a(x_n)$. As in the previous case, we can show $\lambda_a(z_0) = \bigvee_n x_n$. □

For the next result, we need the following notion. We say that an EMV-algebra M satisfies the *general comparability property* if, given $a \in \mathcal{I}(M)$ and $x, y \in [0, a]$, there is an idempotent e, $e \in [0, a]$ such that $x \wedge e \leq y$ and $y \wedge \lambda_a(e) \leq x$.

PROPOSITION 3.3. *If an EMV-algebra M is Dedekind σ-complete, then M is a semisimple EMV-algebra satisfying the general comparability property, and the set of idempotent elements $\mathcal{I}(M)$ is a Dedekind σ-complete subalgebra of M.*

PROOF. Let $a \in M$ be an idempotent. Since M is Dedekind σ-complete, then $[0, a]$ is a σ-complete MV-algebra, and by [4, Proposition 6.6.2], $[0, a]$ is semisimple. Applying Proposition 3.1, we conclude that M is semisimple. Using [13, Theorem 9.9], we can conclude that every MV-algebra $[0, a]$ satisfies the general comparability property, consequently, so does M.

Now let $\{a_n\}$ be a sequence of idempotent elements of M bounded by some element x. Clearly, $\{a_n\}$ is bounded by some idempotent a_0. Let $a = \bigvee_n a_n$ exist in M. For any n, let $b_n = a_1 \vee \cdots \vee a_n$. Then $a = \bigvee_n b_n$. Using [12, Proposition 1.21], we have $a \oplus a = a \oplus (\bigvee_n b_n) = \bigvee_n (a \oplus b_n) = \bigvee_n \bigvee_m (b_n \oplus b_m) = \bigvee_n (\bigvee_{m \leq n} (b_n \oplus b_m) \vee \bigvee_{m > n} (b_n \oplus b_m)) = \bigvee_n (\bigvee_{m \leq n} b_n \vee \bigvee_{m > n} b_m) = \bigvee_n b_n = a$. That is, a is an idempotent of M. □

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

Proposition 3.4. *Let M be an EMV-algebra. If $\bigvee_t y_t$ exists in M, then for each $x \in M$, $\bigvee_t (x \wedge y_t)$ exists and*

$$x \wedge \bigvee_t y_t = \bigvee_t (x \wedge y_t),$$
$$\Big(\bigvee_t y_t\Big) \odot x = \bigvee_t (y_t \odot x).$$

Proof. Let $y = \bigvee_t y_t$ exist in M. Clearly, $x \wedge y \geq x \wedge y_t$ for each t. Now let $z \geq x \wedge y_t$ for each t. There is an idempotent $a \in M$ such that $x, y, z \leq a$. Then the statement holds in the MV-algebra $[0, a]$, see for example [12, Proposition 1.18], and also does in M.

The second property holds also in the MV-algebra $[0, a]$ as it follows from [12, Proposition 1.16]. □

Let s be a state-morphism on M. We define two sets

$$\mathrm{Ker}(s) := \{x \in M \mid s(x) = 0\}, \quad \mathrm{Ker}_1(s) = \{x \in M \mid s(x) = 1\}.$$

We have the following simple but useful characterization of maximal ideals and maximal filters by state-morphisms.

Lemma 3.5. *Let s be a state-morphism on an EMV-algebra M. Then* $\mathrm{Ker}(s)$ *is a maximal ideal of M and* $\mathrm{Ker}_1(s)$ *is a maximal filter of M. Conversely, for each maximal ideal I and each maximal filter F, there are unique state-morphisms s and s_1 on M such that $I =$* $\mathrm{Ker}(s)$ *and $F =$* $\mathrm{Ker}_1(s_1)$.

Proof. The one-to-one correspondence between $\mathrm{Ker}(s)$ and a maximal ideal I of M was established [10, Theorem 4.2].

Now we show that $\mathrm{Ker}_1(s)$ is a maximal filter of M. It is clear that $\mathrm{Ker}_1(s)$ is a filter. Let $x \notin \mathrm{Ker}_1(s)$. Then $s(x) < 1$ and since $s(x)$ is a real number in the MV-algebra of the real interval $[0, 1]$, we have that there is an integer n such that $s(x^n) = (s(x))^n = 0$ and an idempotent $b \in \mathcal{I}(M)$ such that $x \leq b$ and $s(b) = 1$. Then $x^n \oplus \lambda_b(x^n) = b$, so that $\lambda_b(x^n) \in \mathrm{Ker}_1(s)$ which by criterion (ii) of [10, Proposition 5.4] means $\mathrm{Ker}_1(s)$ is a maximal filter.

Now let F be a maximal filter of M. Define $I_F = \{\lambda_a(x) \mid x \in F, a \in \mathcal{I}(M), x \leq a\}$. By [10, Theorem 5.6], I_F is a maximal ideal of M so that, there is a unique state-morphism s such that $\mathrm{Ker}(s) = I_F$. Now let $x \in F$ and let a be an idempotent of M such that $x \leq a$ and $s(a) = 1$. Then $s(\lambda_a(x)) = 0$, so that $1 = s(a) = s(x \oplus \lambda_a(x)) = s(x)$, and $F \subseteq \mathrm{Ker}_1(s)$. The maximality of F and $\mathrm{Ker}_1(s)$ yields $F = \mathrm{Ker}_1(s)$.

If there is another state-morphism s' such that $\mathrm{Ker}_1(s) = F = \mathrm{Ker}_1(s')$, then $\mathrm{Ker}(s) = I_F = \mathrm{Ker}(s')$, which by [10, Theorem 4.3] means $s = s'$. □

4. Hull-kernel topologies and the weak topology of state-morphisms

The present section is devoted to the hull-kernel topology of the set of maximal ideals and the weak topology of the set of state-morphisms. We show that these spaces

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

are homeomorphic, and more information can be derived for EMV-algebras with the general comparability property. In addition, using the basic representation theorem, we show that if an EMV-algebra M has no top elements, the state-morphism space is only locally compact and not compact, and its one-point compactification is homeomorphic to the state-morphism space of N. A similar property holds for the set of maximal filters of M and N, respectively.

We recall that a topological space $\Omega \neq \emptyset$ is:

(i) *regular* if, for each point $\omega \in \Omega$ and any closed subspace A of Ω not-containing ω, there are two disjoint open sets U and V such that $\omega \in U$ and $A \subseteq V$;
(ii) *completely regular* if, for each nonempty closed set F and each point $a \in \Omega \setminus F$, there is a continuous function $f : \Omega \to [0, 1]$ such that $f(\omega) = 1$ for each $\omega \in F$ and $f(a) = 0$;
(iii) *totally disconnected* if every two different points are separated by a clopen subset of Ω;
(iv) *locally compact* if every point of Ω has a compact neighborhood;
(v) *basically disconnected* if the closure of every open F_σ subset of Ω is open.

Of course, (i) implies (ii). We note that the weak topology of state-morphisms on a σ-complete MV-algebra is basically disconnected, see for example [7, Proposition 4.3].

On the set $\mathrm{MaxI}(M)$ of maximal ideals of M we introduce the following hull-kernel topology $\mathcal{T}_M$.

Proposition 4.1. *Let M be an EMV-algebra. Given an ideal I of M, let*

$$O(I) := \{A \in \mathrm{MaxI}(M) \mid A \not\supseteq I\},$$

and let $\mathcal{T}_M$ be the collection of all subsets of the above form. Then $\mathcal{T}_M$ defines a topology on $\mathrm{MaxI}(M)$ *which is a Hausdorff one.*

Given $a \in M$, we set

$$M(a) = \{I \in \mathrm{MaxI}(M) \mid a \notin I\}.$$

Then $\{M(a) \mid a \in M\}$ is a base for $\mathcal{T}_M$. In addition:

(i) $M(0) = \emptyset$;
(ii) $M(a) \subseteq M(b)$ *whenever* $a \leq b$;
(iii) $M(a \wedge b) = M(a) \cap M(b)$, $M(a \vee b) = M(a) \cup M(b)$.

Moreover, any closed subset of $\mathcal{T}_M$ is of the form

$$C(I) := \{A \in \mathrm{MaxI}(M) : A \supseteq I\}.$$

Proof. We have (i) $O(\{0\}) = \emptyset$, $O(M) = \mathrm{MaxI}(M)$, (ii) if $I \subseteq J$, then $O(I) \subseteq O(J)$, (iii) $\bigcup_\alpha O(I_\alpha) = O(I)$, where $I = \bigvee_\alpha I_\alpha$, and (iv) $\bigcap_{i=1}^n O(I_i) = O(\bigcap_{i=1}^n I_i)$ which implies $\{O(I) \mid I \in \mathrm{Ideal}(M)\}$ defines the topology $\mathcal{T}_M$ on $\mathrm{MaxI}(M)$.

Given $a \in M$, let I_a be the ideal of M generated by a. Then $O(I_a) = M(a)$. Since $O(I) = \bigcup\{M(a) \mid a \in I\}$, we see that $\{M(a) \mid a \in M\}$ is a base for $\mathcal{T}_M$.

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

To see that $M(a) \cap M(b) = M(a \wedge b)$, we have trivially $M(a) \cap M(b) \supseteq M(a \wedge b)$. Let $A \in M(a) \cap M(b)$ and let $A \notin M(a \wedge b)$. Then $a \wedge b \in A$ and since A is prime, either $a \in A$ or $b \in A$ which is impossible. Then $A \in M(y)$ and $B \in M(x)$.

Hausdorffness. Let A and B be two maximal ideals of M, $A \neq B$. There are $x \in A \setminus B$ and $y \in B \setminus A$. Then $x \wedge y \in A \cap B$. Let a be an idempotent of M such that $x, y \leq a$. Then $x \odot \lambda_a(y) \in [0, a]$. Since $x = (x \odot \lambda_a(y)) \oplus (x \wedge y)$, we see that $x \odot \lambda_a(y) \in A \setminus B$. In a similar way, we have $y \odot \lambda_a(x) \in B \setminus A$. Due to $(x \odot \lambda_a(y)) \wedge (y \odot \lambda_a(x)) = 0$, we have also $A \in M(y \odot \lambda_a(x))$ and $B \in M(x \odot \lambda_a(y))$ and $M(y \odot \lambda_a(x)) \cap M(x \odot \lambda_a(y)) = M((x \odot \lambda_a(y)) \wedge (y \odot \lambda_a(x))) = M(0) = \emptyset$. □

Lemma 4.2. *Let M be an EMV-algebra. Then:*

(i) *if $O(I) = O(M)$, then $I = M$;*
(ii) *$M(a) = M(0)$ if and only if $a \in \mathrm{Rad}(M)$;*
(iii) *if, for some $a \in \mathcal{I}(M)$, we have $M(a) = O(M)$, then a is the top element of M and M is an EMV-algebra with a top element;*
(iv) *if, for some $x \in M$, we have $M(x) = O(M)$, then M has a top element;*
(v) *the space $\mathrm{MaxI}(M)$ is compact if and only if M has the top element.*

Proof.

(i) Assume I is a proper ideal of M. There is a maximal ideal A of M containing I, then $A \notin O(I) = O(M)$ which yields a contradiction with $A \in O(M)$.
(ii) It follows from the definition of $\mathrm{Rad}(M)$.
(iii) Let a be an idempotent and let I_a be the ideal of M generated by a. From (i), we conclude $I_a = M$. Hence, if $x \in M$, then $x \in I_a$ and henceforth, there is an integer n such that $x \leq n.a = a$, that is, a is the top element of M.
(iv) Let I_x be the ideal of M generated by x. There is an idempotent a of M such that $x \leq a$. We assert a is the top element of M. Indeed, from (i), we have $I_x = M$, that is, for any $z \in M$, there is an integer n such that $z \leq n.x$. But then $z \leq n.a = a$.
(v) Let $\mathrm{MaxI}(M)$ be a compact space. Since $\{M(x) \mid x \in M\}$ is an open covering of $\mathrm{MaxI}(M)$, there are finitely many elements $x_1, \ldots, x_n \in M$ such that $\bigcup_{i=1}^n M(x_i) = O(M)$, so that if $x_0 = x_1 \vee \cdots \vee x_n$, then $M(x_0) = O(M)$ which by (iv) means that x_0 is the top element of M.

Conversely, if M has the top element, then M is in fact an MV-algebra, and the compactness of $\mathrm{MaxI}(M)$ is well known, see for example [9, Proposition 7.1.3], [13, Corollary 12.19]. □

We say that a net $\{s_\alpha\}_\alpha$ of state-morphisms on M *converges weakly* to a state-morphism s on M, if $\lim_\alpha s_\alpha(a) = s(a)$. Hence, $\mathcal{SM}(M)$ is a subset of $[0,1]^M$ and if we endow $[0,1]^M$ with the product topology which is a compact Hausdorff space, we see that the weak topology, which is in fact a relative topology (or a subspace topology) of the product topology of $[0,1]^M$, yields a nonempty Hausdorff topological space whenever $M \neq \{0\}$; if $M = \{0\}$, the set $\mathcal{SM}(M)$ is empty. In addition, the system

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

of subsets of $\mathcal{SM}(M)$ of the form $S(x)_{\alpha,\beta} = \{s \in \mathcal{SM}(M) \mid \alpha < s(x) < \beta\}$, where $x \in M$ and $\alpha < \beta$ are real numbers, forms a subbase of the weak topology of state-morphisms.

We note that $\mathcal{SM}(M)$ is closed in the product topology whenever M has a top element. In general, it is not closed because if, for a net $\{s_\alpha\}_\alpha$ of state-morphisms, there exists $s(a) = \lim_\alpha s_\alpha(a)$ for each $a \in M$, then s preserves $\oplus, \vee, \wedge$, but there is no guarantee that there is $x \in M$ such that $s(x) = 1$ as the following example shows.

Example 4.3. Let $\mathcal{T}$ be the set of all finite subsets of the set $\mathbb{N}$ of natural numbers. Then $\mathcal{T}$ is a generalized Boolean algebra having no top element, and $\mathcal{SM}(\mathcal{T}) = \{s_n \mid n \in \mathbb{N}\}$, where $s_n(A) = \chi_A(n)$, $A \in \mathcal{T}$. However, $s(A) = \lim_n s_n(A) = 0$ for each $A \in \mathcal{T}$, so that s is not a state-morphism.

Therefore, a nonempty set X of state-morphisms is closed if and only if, for each net of states $\{s_\alpha\}_\alpha$ of state-morphisms from X, such that there exists $s(x) = \lim_\alpha s_\alpha(x)$ for each $x \in M$, then s is a state-morphism on M and s belongs to X.

We note that if $x \in M$, then the function $\hat{x} : \mathcal{SM}(M) \to [0, 1]$ defined by

$$\hat{x}(s) := s(x), \quad s \in \mathcal{SM}(M),$$

is a continuous function on $\mathcal{SM}(M)$. We denote by $\widehat{M} = \{\hat{x} \mid x \in M\}$.

According to basic representation theorem 2.1, every EMV-algebra M is either termwise equivalent to the MV-algebra $(M; \oplus, \lambda_1, 0, 1)$ or it can be embedded into an EMV-algebra N with a top element as its maximal ideal, so that we can assume that M is an EMV-subalgebra of N and $N = \{x \in N \mid \text{ either } x \in M \text{ or } \lambda_1(x) \in M\}$. If M is a proper EMV-algebra, that is, it does not contain any top element, the state-morphism space $\mathcal{SM}(N)$ can be characterized as follows.

Proposition 4.4. *Let M be a proper EMV-algebra and, for each $x \in M$, we put $x^* = \lambda_1(x)$. Given a state-morphism s on M, the mapping $\tilde{s} : N \to [0, 1]$, defined by*

$$\tilde{s}(x) = \begin{cases} s(x) & \text{if } x \in M, \\ 1 - s(x_0) & \text{if } x = x_0^*,\ x_0 \in M, \end{cases} \qquad x \in N, \tag{4.1}$$

is a state-morphism on N, and the mapping $s_\infty : N \to [0, 1]$ defined by $s_\infty(x) = 0$ if $x \in M$ and $s_\infty(x) = 1$ if $x \notin M$, is a state-morphism on N. Moreover, $\mathcal{SM}(N) = \{\tilde{s} \mid s \in \mathcal{SM}(M)\} \cup \{s_\infty\}$ and $\mathrm{Ker}(\tilde{s}) = \mathrm{Ker}(s) \cup \mathrm{Ker}_1^*(s)$, $s \in \mathcal{SM}(M)$, where $\mathrm{Ker}_1^*(s) = \{\lambda_1(x) \mid x \in \mathrm{Ker}_1(s)\}$.

A net $\{s_\alpha\}_\alpha$ of state-morphisms on M converges weakly to a state-morphism s on M if and only if $\{\tilde{s}_\alpha\}_\alpha$ converges weakly on N to $\tilde{s}$.

Proof. Assume that $N = \Gamma(G, u)$ for some unital Abelian ℓ-group (G, u). Then $1 = u$ and $x^* = \lambda_1(x) = u - x$, where $-$ is the subtraction taken from the ℓ-group G.

Take $s \in \mathcal{SM}(M)$. We have $\tilde{s}(1) = 1$. If $x, y \in M$, then $\tilde{s}(x \oplus y) = \tilde{s}(x) \oplus \tilde{s}(y)$. If $x = x_0^*$, $y = y_0^*$ for $x_0, y_0 \in M$, then $x \oplus y = (x_0 \odot y_0)^*$, so that $\tilde{s}(x \oplus y) = 1 - \tilde{s}(x_0 \odot y_0) = (1 - s(x_0)) \oplus (1 - s(y_0)) = \tilde{s}(x) \oplus \tilde{s}(y)$. Finally, if $x = x_0$, $y = y_0^*$ for $x_0, y_0 \in M$, there exists an idempotent $b \in \mathcal{I}(M)$ such that $x_0, y_0 \leq b$ and $s(b) = 1$. Since $x \oplus y = x_0 \oplus y_0^* =$

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

$(y_0 \odot x_0^*)^* = (y_0 \odot \lambda_b(x_0))^*$ which yields $\tilde{s}(x \oplus y) = 1 - s(y_0 \odot \lambda_b(x_0)) = 1 - (s(y_0) \odot (s(b) - s(x_0)) = (1 - s(y_0)) \oplus s(x_0) = \tilde{s}(x) \oplus s(y)$. Whence, $\tilde{s}$ is a state-morphism on N.

It is easy to verify that s_∞ is a state-morphism on N. We note that the restriction of s_∞ onto M is not a state-morphism on M because it is the zero function on M.

We note that

$$\mathcal{I}(N) = \{x \in N \mid \text{ either } x \in \mathcal{I}(M) \text{ or } x^* \in \mathcal{I}(M)\}.$$

Let s be a state-morphism on N. We have two cases: (i) there is an idempotent $a \in M$ such that $s(a) = 1$, then the restriction s_0 of s onto M is a state-morphism on M, so that $s = \widetilde{s_0} \in \mathcal{SM}(N)$. (ii) For each idempotent $a \in M$, $s(a) = 0$. Since given $x \in M$, there is an idempotent $a \in \mathcal{I}(M)$ with $x \leq a$, we have $s(x) = 0$ for each $x \in M$ which says $s = s_\infty$.

The last assertions are evident. □

The latter proposition can be illustrated by the following example.

Example 4.5. Let $\mathcal{T}$ be the system of all finite subsets of the set $\mathbb{N}$ of integers. Then $\mathcal{T}$ is an EMV-algebra that is a generalized Boolean algebra of subsets, $\mathcal{T}$ has no top element, $\mathcal{SM}(\mathcal{T}) = \{s_n \mid n \in \mathbb{N}\}$ where $s_n = \chi_A(n)$, $A \in \mathcal{T}$. If we define $\mathcal{N}$ as the set of all finite or co-finite subsets of $\mathbb{N}$, $\mathcal{N}$ is an EMV-algebra with the top element such that $\mathcal{N} = \{A \subseteq \mathbb{N} \mid \text{ either } A \in \mathcal{T} \text{ or } A^c \in \mathcal{T}\}$, and $\mathcal{N}$ is representing $\mathcal{T}$. Then $\mathcal{SM}(\mathcal{N}) = \{\tilde{s}_n \mid n \in \mathbb{N}\} \cup \{s_\infty\}$, where $\tilde{s}_n = \chi_A(n)$, $A \in \mathcal{N}$, and $s_\infty(A) = 0$ if A is finite and $s_\infty(A) = 1$ if A is co-finite. In addition, $\lim_n s_n(A) = 0$ for each $A \in \mathcal{T}$ and $\lim_n \tilde{s}_n(A) = s_\infty(A)$, $A \in \mathcal{N}$.

Remark 4.6. Since a net $\{s_\alpha\}_\alpha$ of state-morphisms of M converges weakly to a state-morphism $s \in \mathcal{SM}(M)$ if and only if $\{\tilde{s}_\alpha\}_\alpha$ converges weakly on N to $\tilde{s}$, the mapping $\phi : \mathcal{SM}(M) \to \mathcal{SM}(N)$, defined by $\phi(s) = \tilde{s}$, $s \in \mathcal{SM}(M)$, is injective and continuous, $\phi(\mathcal{SM}(M))$ is open, but ϕ is not necessarily closed, see Example 4.5. We have that ϕ is closed if and only if M possesses a top element.

Proof. If $x \in M$, then $S_N(x) = \{s \in \mathcal{SM}(N) \mid s(x) > 0\} = \widetilde{S(x)} := \{\tilde{s} \mid s \in S(x)\}$, where $S(x) = \{s \in \mathcal{SM}(M) \mid s(x) > 0\}$. Clearly $s_\infty \notin S_N(x)$ and $S_N(x)$ is an open set of $\mathcal{SM}(N)$. Therefore, for each $\tilde{s}$, there is an open set of $\mathcal{SM}(N)$, namely $S_N(x)$, which contains $\tilde{s}$ and $\tilde{s} \in S_N(x) \subseteq \phi(X)$. Whence $\phi(X)$ is open in $\mathcal{SM}(N)$.

If M has a top element, then $N = M$ and ϕ is the identity, so it is closed and open as well. Conversely, let ϕ be closed, then $\phi(X)$ is closed and compact, where $X = \mathcal{SM}(M)$.

Hence, for each open subset O of $\mathcal{SM}(M)$, we have $\phi(O) = \phi(X \setminus C) = \phi(X) \setminus \phi(C)$, where C is a closed subset of $\mathcal{SM}(M)$, so that ϕ is an open mapping. Now let $\{O_\alpha \mid \alpha \in A\}$ be an open covering of X, then $\phi(X) = \phi(\bigcup_\alpha O_\alpha) = \bigcup_\alpha \phi(O_\alpha)$, and the compactness of $\phi(X)$ yields $\phi(X) = \bigcup_{i=1}^n \phi(O_{\alpha_i})$, so that $X = \bigcup_{i=1}^n O_{\alpha_i}$ which says $\mathcal{SM}(M)$ is compact. Since $X = \bigcup\{S(x) \mid x \in M\}$, there are finitely many elements $x_1, \ldots, x_k \in M$ such that $X = \bigcup_{i=1}^k S(x_i) = S(x_0)$, where $x_0 = x_1 \vee \cdots \vee x_k$. If I_{x_0} is the ideal of M generated by x_0, then $S(x_0) = \{s \in \mathcal{SM}(M) \mid \mathrm{Ker}(s) \not\supseteq I_{x_0}\}$, so that $O(I_{x_0}) = O(M) = M(x_0)$ which, by Lemma 4.2(iv), gives M as having a top element. □

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

PROPOSITION 4.7. *Let M be an EMV-algebra and X be a nonempty subspace of state-morphisms on M that is closed in the weak topology of state-morphisms. Let t be a state-morphism such that $t \notin X$. There exists an $a \in M$ such that $t(a) > 1/2$ while $s(a) < 1/2$ for all $s \in X$. Moreover, the element $a \in M$ can be chosen such that $t(a) = 1$ and $s(a) = 0$ for each $s \in X$.*

In particular, the space $\mathcal{SM}(M)$ is completely regular.

PROOF. (1) Let t be a state-morphism such that $t \notin X$. We assert that there exists an $a \in M$ such that $t(a) > 1/2$ while $s(a) < 1/2$ for all $s \in X$.

Indeed, set $A = \{a \in M : t(a) > 1/2\}$, and for all $a \in A$, let

$$W(a) := \{s \in \mathcal{SM}(M)) \mid s(a) < 1/2\},$$

which is an open subset of $\mathcal{SM}(M)$. We note that $A \neq \emptyset$ and A is downward directed and closed under $\oplus$.

We assert that these open subsets cover X. Consider any $s \in X$. Since $\mathrm{Ker}(s)$ and $\mathrm{Ker}(t)$ are noncomparable subsets of M, there exists $x \in \mathrm{Ker}(t) \setminus \mathrm{Ker}(s)$. Hence $t(x) = 0$ and $s(x) > 0$. Choose an idempotent $b \in M$ such that $x \leq b$ and $t(b) = 1$. There exists an integer $n \geq 1$ such that $s(n.x) > 1/2$. Since there is also an integer k such that $s(k.x) = k.s(x) = 1$ and $k.x \leq b$, we conclude $s(b) = 1$. Due to t being a state-morphism, we have $t(n.x) = 0$. Putting $a = \lambda_b(n.x)$, we have $t(a) = 1 > 1/2$ and $s(a) < 1/2$. Therefore, $\{W(a) \mid a \in A\}$ is an open covering of X.

(i) If M has a top element, the state-morphism space $\mathcal{SM}(M)$ is compact and Hausdorff, so that X is compact, and $X \subseteq W(a_1) \cup \cdots \cup W(a_n)$ for some $a_1, \ldots, a_n \in A$.

(ii) If M has no top element, embed M into the EMV-algebra N with a top element as its maximal ideal. Since $s(1) = 1$ for each state-morphism s on N, we see that $\mathcal{SM}(N)$ is a compact set in the product topology, consequently, it is compact in the weak topology of state-morphisms on N. The mapping $\phi : \mathcal{SM}(M) \to \mathcal{SM}(N)$ defined by $\phi(s) = \tilde{s}$, where $\tilde{s}$ is defined through (4.1), is by Proposition 4.4 injective and continuous.

We assert the set $\phi(X) \cup \{s_\infty\}$ is a compact subset of $\mathcal{SM}(N)$. Indeed, let $\{s_\alpha\}_\alpha$ be a net of state-morphisms from $\phi(X) \cup \{s_\infty\}$. Since $\mathcal{SM}(N)$ is compact, there is a subnet $\{s_{\alpha_\beta}\}_\beta$ of the net $\{s_\alpha\}_\alpha$ converging weakly to a state-morphism s on N. If $s = s_\infty$, $s \in \phi(X) \cup \{s_\infty\}$. If $s \neq s_\infty$, there is a state-morphism $s_0 \in \mathcal{SM}(M)$ such that $s = \tilde{s}_0$. Then there is β_0 such that for each $\beta > \beta_0$, $s_{\alpha_\beta} \in X$. Therefore, $s_0 \in X$ and $s = \phi(s_0) \in \phi(X) \cup \{s_\infty\}$. We note that $\tilde{t} \notin \phi(X) \cup \{s_\infty\}$.

For each $a \in A$, let $\widetilde{W}(a) := \{s \in \mathcal{SM}(N) \mid s(a) < 1/2\}$. Then $\tilde{t}(a) = t(a) > 1/2$ and $0 = s_\infty(a) < 1/2$, so that $s_\infty \in \widetilde{W}(a)$ for each $a \in A$. Then $\{\widetilde{W}(a) \mid a \in A\}$ is an open covering of the compact set $\phi(X) \cup \{s_\infty\}$. There are $a_1, \ldots, a_n \in A$ such that $\phi(X) \cup \{s_\infty\} \subseteq \widetilde{W}(a_1) \cup \cdots \cup \widetilde{W}(a_n)$, consequently, $X \subseteq W(a_1) \cup \cdots \cup W(a_n)$. Put $a = a_1 \wedge \cdots \wedge a_n$. Then $a \in A$ and for each $s \in X$, we have $s(a) \leq s(a_i) < 1/2$ for $i = 1, \ldots, n$, which proves $X \subseteq W(a)$, that is, $s(a) < 1/2$ for all $s \in X$.

(2) By the first part of the present proof, there exists an $a \in M$ such that $t(a) > 1/2$ while $s(a) < 1/2$ for all $s \in X$. In addition, there is an idempotent b of M with $a \leq b$ and

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

$t(b) = 1$. Then $t(a \wedge \lambda_b(a)) = t(\lambda_b(a))$ and $t(a \odot \lambda_b(a \wedge \lambda_b(a))) = t(a) - t(a \wedge \lambda_b(a)) = t(a) - t(\lambda_b(a)) = 2t(a) - 1 > 0$.

Now let s be an arbitrary element of X. If $s(a) = 0$, then $s(a \odot \lambda_b(a \wedge \lambda_b(a))) = 0$. If $s(a) > 0$, there is an integer m_s such that $s(m_s.a) = m_s.s(a) = 1$ and since $m_s.a \leq m_s.b = b$, we have $s(b) = 1$. Hence, $s(a \wedge \lambda_b(a)) = s(a)$, so that $s(a \odot \lambda_b(a \wedge \lambda_b(a))) = s(a) - s(a \wedge \lambda_b(a)) = 0$. In any case, the element $a \odot \lambda_b(a \wedge \lambda_b(a))$ is an element of $\bigcap\{\mathrm{Ker}(s) \mid s \in X\}$ for which $t(a \odot \lambda_b(a \wedge \lambda_b(a))) > 0$.

(3) From (1) and (2), we have concluded that if we use (2.2), then $a \odot \lambda_b(a \wedge \lambda_b(a)) = a \odot a$ and $s(a \odot a) = 0$ for each $s \in X$. In addition, $t(a \odot a) > 0$. There is an integer r such that $t(r.(a \odot a)) = r.t(a \odot a) = 1$ and $s(r.(a \odot a)) = 0$ for each $s \in X$. Hence, for $x = r.(a \odot a)$, we have $\hat{x}(X) = 0$ and $\hat{x}(t) = 1$. Consequently, for the continuous function f on $\mathcal{SM}(M)$ defined by $f(s) = 1 - \hat{x}(s)$, we have $f(X) = 1$ and $f(t) = 0$, so that $\mathcal{SM}(M)$ is completely regular. □

THEOREM 4.8. *Let M be an EMV-algebra. The mapping $\theta : \mathcal{SM}(M) \to \mathrm{MaxI}(M)$, defined by $s \mapsto \mathrm{Ker}(s)$, is a homeomorphism. In addition, the following statements are equivalent:*

(i) *M has a top element;*
(ii) *$\mathcal{SM}(M)$ is compact in the weak topology of state-morphisms;*
(iii) *$\mathrm{MaxI}(M)$ is compact in the hull-kernel topology.*

PROOF. Define a mapping θ on the set of state-morphisms $\mathcal{SM}(M)$ with values in $\mathrm{MaxI}(M)$ as follows $\theta(s) = \mathrm{Ker}(s)$, $s \in \mathcal{SM}(M)$. By [10, Theorem 4.2], θ is a bijection. Let $C(I)$ be any closed subspace of $\mathrm{MaxI}(M)$. Then

$$\theta^{-1}(C(I)) = \{s \in \mathcal{SM}(M) \mid s(x) = 0 \text{ for all } x \in I\},$$

which is a closed subset of $\mathcal{SM}(M)$. Therefore, θ is continuous.

Given a nonempty subset X of $\mathcal{SM}(M)$, we set

$$\mathrm{Ker}(X) := \{x \in M \mid s(x) = 0 \text{ for all } s \in X\}.$$

Then $\mathrm{Ker}(X)$ is an ideal of M. If, in addition, X is a closed subset of $\mathcal{SM}(M)$, we assert

$$\theta(X) = C(\mathrm{Ker}(X)). \tag{4.2}$$

The inclusion $\theta(X) \subseteq C(\mathrm{Ker}(X))$ is evident. By Proposition 4.7, if $t \notin X$, there is an element $a \in M$ such that $s(a) = 0$ for each $s \in X$ and $t(a) = 1$. Consequently, $t \notin X$ implies $\theta(t) \notin C(\mathrm{Ker}(X))$, and $C(\mathrm{Ker}(X)) \subseteq \theta(X)$. As a result, we conclude θ is a homeomorphism.

(i) ⇒ (ii) If 1 is a top element of M, then $s(1) = 1$ for each state-morphism s, therefore, $\mathcal{SM}(M)$ is a closed subspace of $[0, 1]^M$, consequently, it is compact.

(ii) ⇒ (iii) Let $\{O_\alpha\}$ be an open cover of $\mathrm{MaxI}(M)$. It is enough to take a cover of the form $\{O(x_\alpha)\}$. Then $\mathcal{SM}(M) = \theta^{-1}(\mathrm{MaxI}(M)) = \bigcup_\alpha \theta^{-1}(O(x_\alpha))$. Hence, there are finitely many indices $\alpha_1, \ldots, \alpha_n$ such that $\mathcal{SM}(M) = \bigcup_{i=1}^n \theta^{-1}(O(x_{\alpha_i}))$ and consequently, $\bigcup_{i=1}^n O(x_{\alpha_i})$, which entails $\mathrm{MaxI}(M)$ is compact.

(iii) ⇔ (i) It was proved in Lemma 4.2(v). □

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

Theorem 4.9. *Let M be an EMV-algebra with the general comparability property. Then the mapping $\xi : \mathrm{MaxI}(M) \to \mathrm{MaxI}(\mathcal{I}(M))$ defined by $\xi(A) = A \cap \mathcal{I}(M)$, $A \in \mathrm{MaxI}(M)$, is a homeomorphism.*

In addition, the spaces $\mathcal{SM}(\mathcal{I}(M))$, $\mathcal{SM}(M)$, $\mathrm{MaxI}(\mathcal{I}(M))$ and $\mathrm{MaxI}(\mathcal{I}(M))$ are mutually homeomorphic topological spaces.

Any of the topological spaces is compact if and only if M has a top element.

Proof. Let I be any ideal of $\mathcal{I}(M)$, and let $\hat{I}$ be the ideal of M generated by I. Then (i) $I = \hat{I} \cap \mathcal{I}(M)$, (ii) $I \subseteq J$ if and only if $\hat{I} \subseteq \hat{J}$, (iii) if $\hat{I}$ is a maximal ideal M, then so is I in $\mathcal{I}(M)$ (if I is maximal, then $\hat{I}$ is not necessarily maximal in M), and (iv) if A is a maximal ideal of M such that $A \supseteq \hat{I}$, then $A \cap \mathcal{I}(M) = I$ (see [10, Theorem 3.24]).

The mapping $\xi : A \mapsto A \cap \mathcal{I}(M)$, $A \in \mathrm{MaxI}(M)$, gives an ideal of $\mathcal{I}(M)$ which is prime because A is prime. Then $\xi(A)$ has to be a maximal ideal of $\mathrm{MaxI}(\mathcal{I}(M))$. In fact, if $a, b \notin \xi(A)$, $a \leq b$, then $b = a \vee \lambda_b(a)$, so that $a \wedge \lambda_b(a) = 0$ and $\lambda_b(a) \in A \cap \mathcal{I}(M)$. Due to [10, Theorem 4.4], the mapping ξ is injective, and in view of [10, Theorem 4.3], ξ is invertible, that is, given maximal ideal I of $\mathcal{I}(M)$, there is a unique extension of I onto a maximal ideal A of M such that $\xi(A) = I$.

Now let I be an ideal of $\mathcal{I}(M)$. We assert

$$\xi^{-1}(C(I)) = C(\hat{I}).$$

Indeed, if A is a maximal ideal of $\mathcal{I}(M)$ such that $A \supseteq I$, then $\xi^{-1}(A) \supseteq \hat{I}$. Conversely, if A is a maximal ideal of M such that $A \supseteq \hat{I}$, then $\xi(A) \supseteq \hat{I} \cap \mathcal{I}(M) = I$. As a result, we have that ξ is continuous.

According to Theorem 4.8, the spaces $\mathcal{SM}(M)$ and $\mathrm{MaxI}(M)$ are homeomorphic; the mapping $\theta : s \mapsto \mathrm{Ker}(s)$, $s \in \mathcal{SM}(M)$, is a homeomorphism. Similarly, $\mathcal{SM}(\mathcal{I}(M))$ and $\mathrm{MaxI}(\mathcal{I}(M))$ are homeomorphic under the homeomorphism $\theta_0(s) = \mathrm{Ker}(s)$, $s \in \mathcal{SM}(\mathcal{I}(M))$. If we define $\eta = \theta_0^{-1} \circ \xi \circ \theta$, then η is a bijective mapping from $\mathcal{SM}(M)$ onto $\mathcal{SM}(\mathcal{I}(M))$ such that if s is a state-morphism of M, then $\eta(s) = s_0 := s_{|\mathcal{I}(M)}$, the restriction of s onto $\mathcal{I}(M)$. Conversely, if s is a state-morphism on $\mathcal{I}(M)$, then $\eta^{-1}(s) = \bar{s}$, the unique extension of s onto M. We see that η is a continuous mapping.

Now take an EMV-algebra N with top element such that M can be embedded into N as its maximal ideal, and every element x of N either belongs to M or $\lambda_1(x) \in M$. Given a state-morphism s on M, let $\tilde{s}$ be its extension to N defined by (4.1). According to the proof of Proposition 4.7, the mapping $\phi : \mathcal{SM}(M) \to \mathcal{SM}(N)$ given by $\phi(s) = \tilde{s}$ is injective and continuous, and a net $\{s_\alpha\}_\alpha$ of states of $\mathcal{SM}(M)$ converges weakly to a state-morphism $s \in \mathcal{SM}(M)$ if and only if $\{\phi(s_\alpha)\}_\alpha$ converges weakly to the state-morphism $\phi(s)$ on N.

Take a closed nonvoid subset X of state-morphisms on M, then $\phi(X)$ is a closed subset of $\mathcal{SM}(N)$, consequently, $\phi(X)$ is compact. Let $\{s_\alpha\}_\alpha$ be a net of state-morphisms from X and let its restriction $\{\bar{s}_\alpha\}_\alpha$ to $\mathcal{I}(M)$ converge weakly to a state-morphism s_0 on $\mathcal{I}(M)$. Since the net $\{\tilde{s}_\alpha\}_\alpha$ is from the compact $\phi(X)$, there is a subnet $\{\bar{s}_{\alpha_\beta}\}_\beta$ of the net $\{\tilde{s}_\alpha\}_\alpha$ which converges weakly to a state-morphism $t \in \phi(X)$ on N, that is, $\lim_\beta \tilde{s}_{\alpha_\beta}(x) = t(x)$ for each $x \in N$. Since $s_\infty \notin \phi(X)$, there is a state-morphism $s \in X$

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

with $\tilde{s} = t$. Then $\lim_\beta s_{\alpha_\beta}(x) = s(x)$ for each $x \in M$. In particular, this is true for each $x \in \mathcal{I}(M)$, so that $\eta(s) = s_0$. In other words, we have proved that η is a closed mapping, and whence, η is a homeomorphism.

Since $\xi = \theta_0 \circ \eta \circ \theta^{-1}$, we see that ξ is a homeomorphism, and in view of Theorem 4.8, the spaces $\mathcal{SM}(\mathcal{I}(M))$, $\mathcal{SM}(M)$, $\mathrm{MaxI}(\mathcal{I}(M))$, and $\mathrm{MaxI}(\mathcal{I}(M))$ are mutually homeomorphic topological spaces.

Consequently, according to Theorem 4.8, any of the topological spaces is compact if and only if M has a top element. □

Theorem 4.10. *Let M be an EMV-algebra. Then the topological spaces $\mathcal{SM}(M)$ and $\mathrm{MaxI}(M)$ are locally compact Hausdorff spaces such that if a is an idempotent, then $S(a)$ and $M(a)$ are compact clopen subsets. If M has a top element, then $\mathcal{SM}(M)$ and $\mathrm{MaxI}(M)$ are compact spaces.*

Proof. Due to basic representation theorem 2.1, either M has a top element, and M is termwise equivalent to the MV-algebra $(M; \oplus, \lambda_1, 0, 1)$, or M can be embedded into N as its maximal ideal, and every $x \in N$ either belongs to M or $\lambda_1(x)$ belongs to M. If M has a top element, then $\mathcal{SM}(M)$ and $\mathrm{MaxI}(M)$ are compact and homeomorphic, see Theorem 4.8.

Let us assume M has no top element. Given $x \in M$ and $y \in N$, let $S(x) = \{s \in \mathcal{SM}(M) \mid s(x) > 0\}$ and $S_N(y) = \{s \in \mathcal{SM}(N) \mid s(y) > 0\}$, they are open sets.

Define a mapping $\phi : \mathcal{SM}(M) \to \mathcal{SM}(N)$ by $\phi(s) = \tilde{s}$, $s \in \mathcal{SM}(M)$, where $\tilde{s}$ is defined by (4.1). Then ϕ is an injective mapping such that $\phi(S(x)) = S_N(x)$ for each $x \in M$. Take an idempotent $a \in \mathcal{I}(M)$. Then $S(a) = \{s \in \mathcal{SM}(M) \mid s(a) > 0\} = \{s \in \mathcal{SM}(M) \mid s(a) = 1\}$ is both open and closed. The same is true for $S_N(a) = \{s \in \mathcal{SM}(N) \mid s(a) > 0\}$, in addition $S_N(a)$ is compact because $\mathcal{SM}(N)$ is compact.

For each $x \in M$ and u, v real numbers with $u < v$, the sets $S(x)_{u,v} = \{s \in \mathcal{SM}(M) \mid u < s(x) < v\}$ and $S_N(x)_{u,v} = \{s \in \mathcal{SM}(N) \mid u < s(x) < v\}$, where $x \in N$, are open and they form a subbase of the weak topologies. Then $\phi(S(x)_{u,v}) = S_N(x)_{u,v}$ and $\phi(S(x)) = S_N(x)$ whenever $x \in M$.

Now we show that $S(a)$ is a compact set in $\mathcal{SM}(M)$. Take an open cover of $S(a)$ in the form $\{S(x_\alpha)_{u_\alpha,v_\alpha} \mid \alpha \in A\}$, where $x_\alpha \in M$ and u_α, v_α are real numbers such that $u_\alpha < v_\alpha$ for each $\alpha \in A$. Then

$$S(a) \subseteq \bigcup_\alpha S(x_\alpha)_{u_\alpha,v_\alpha}$$

$$\phi(S(a)) \subseteq \bigcup_\alpha \phi(S(x_\alpha)_{u_\alpha,v_\alpha})$$

$$S_N(a) \subseteq \bigcup_\alpha \phi(S(x_\alpha)_{u_\alpha,v_\alpha}).$$

The compactness of $S_N(a)$ entails a finite subset F of A such that $S_N(a) \subseteq \bigcup\{\phi(S(x_\alpha)_{u_\alpha,v_\alpha}) \mid \alpha \in F\}$, whence, $S(a) \subseteq \bigcup\{S(x_\alpha)_{u_\alpha,v_\alpha} \mid \alpha \in F\}$. Since the system of all open sets $S(x)_{u,v}$ forms a subbase of the weak topology of $\mathcal{SM}(M)$, we have by [14, Theorem 5.6], $S(a)$ is compact and clopen as well. In addition, given a state-morphism

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

$s \in \mathcal{SM}(M)$, there is an element $x \in M$ with $s(x) = 1$, and there is an idempotent $a \in M$ such that $x \leq a$ which entails $s \in S(x) \subseteq S(a)$. Whence, $\mathcal{SM}(M)$ is locally compact.

Claim. $M(a)$ and $M_N(a)$ are both clopen and compact.

Define a mapping $\theta_N : \mathcal{SM}(N) \to \mathrm{MaxI}(N)$ by $\theta_N(s) := \mathrm{Ker}(s)$, $s \in \mathcal{SM}(N)$. Since N has a top element, θ_N is a homeomorphism, see Theorem 4.8. Therefore, $M_N(a)$ is clopen and compact.

Whence $M_N(a)$ is compact in $\mathrm{MaxI}(N)$. We show that also $M(a)$ is compact in $\mathrm{MaxI}(M)$. Take an open covering $\{M(x_\alpha) \mid \alpha \in A\}$ of $M(a)$, where each $x_\alpha \in M$. Given $I \in \mathrm{MaxI}(M)$, there is a unique state-morphism s on M such that $I = \mathrm{Ker}(s) = \theta^{-1}(s)$, therefore, we define the mapping $\psi : \mathrm{MaxI}(M) \to \mathrm{MaxI}(N)$ by $\psi(I) = \theta_N^{-1}(\tilde{s})$.

Then $\{\psi(M(x_\alpha)) \mid \alpha \in A\}$ is an open covering of $\psi(M(a)) = M_N(a)$ which is a compact set. Whence, there is a finite subcovering $\{\psi(M(x_{\alpha_i})) \mid i = 1, \ldots, n\}$ of $\psi(M(a))$, consequently $\{M(x_{\alpha_i}) \mid i = 1, \ldots, n\}$ is a finite subcovering of $M(a)$, consequently, $M(a)$ is compact and clopen as well. □

Corollary 4.11. *Let M be an EMV-algebra with the general comparability property. Then the spaces $\mathcal{SM}(\mathcal{I}(M))$, $\mathcal{SM}(M)$, $\mathrm{MaxI}(\mathcal{I}(M))$, and $\mathrm{MaxI}(\mathcal{I}(M))$ are totally disconnected, locally compact and completely regular spaces.*

Proof. By Theorem 4.9, all spaces are mutually homeomorphic, and by Theorem 4.10, they are completely regular, locally compact and totally disconnected. □

We say that a topological space Ω is *Baire* if, for each sequence of open and dense subsets $\{U_n\}$, their intersection $\bigcap_n U_n$ is dense.

Corollary 4.12. *Let M be an EMV-algebra. The spaces $\mathcal{SM}(M)$ and $\mathrm{MaxI}(M)$ are Baire spaces.*

Proof. Both spaces are homeomorphic, see Theorem 4.8, due to Theorem 4.10, both spaces are locally compact, and by Proposition 4.7, they are completely regular. Therefore, they are also regular. Applying the Baire theorem, [14, Theorem 6.34], the spaces are Baire spaces. □

Motivated by Example 4.5, we have the following result which describes the state-morphisms spaces of M and N from the topological point of view.

Theorem 4.13. *Let M be an EMV-algebra without a top element which is a maximal ideal of the EMV-algebra $N = \{x \in N \mid$ either $x \in M$ or $\lambda_1(x) \in M\}$. Then $\mathcal{SM}(N)$ and $\mathrm{MaxI}(N)$ are the one-point compactifications of the spaces $\mathcal{SM}(M)$ and $\mathrm{MaxI}(M)$, respectively.*

Proof. In what follows, we use the result and notation from Proposition 4.4. By Theorem 4.8, $\mathcal{SM}(N)$ is a compact Hausdorff topological space, whereas $\mathcal{SM}(M)$ is, according to Theorem 4.10, a locally compact Hausdorff topological space. Due to the Alexander theorem, see [14, Theorem 4.21], there is the one-point compactification

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

of $\mathcal{SM}(M)$. We are going to show that the one-point compactification of $\mathcal{SM}(M)$ is $\mathcal{SM}(N)$.

We proceed in five steps.

(1) If O_N is an open set of $\mathcal{SM}(N)$ such that $s_\infty \notin O_N$, then $O_N = \phi(O)$ for some open subset O of $\mathcal{SM}(M)$.

(2) Now take an open set O_N containing s_∞ and $O_N = S_N(x)_{u,v}$, where $x \in M$ and u, v are real numbers with $u < v$. Since $s_\infty(x) = 0$, $u < 0 < v$ and we have $S_N(x)_{u,v} = \{s_\infty\} \cup \{\tilde{s} \mid s \in \mathcal{SM}(M), s(x) < v\} = \{s_\infty\} \cup \phi(\{s \in \mathcal{SM}(N) \mid s(x) < v\})$. If $X := \phi(\mathcal{SM}(M))\setminus(\{s_\infty\} \cup \phi(\{s \in \mathcal{SM}(N) \mid s(x) < v\})$, then $X = \{s \in \mathcal{SM}(M) \mid s(x) \geq v\} \subseteq \{s \in \mathcal{SM}(N) \mid s(a) \geq v\}$, where $a \in \mathcal{I}(M)$ such that $x \leq a$. If $u \geq 1$, then $X = \emptyset$ which is a compact set and if $u < 1$, then $X \subseteq \{s \in \mathcal{SM}(M) \mid s(a) = 1\}$. Since the latter set is compact, see Theorem 4.10, we see that X is closed, and consequently, X is compact, too.

(3) Now let $s_\infty \in O_N = S_N(x)_{u,v}$, where $x \in M$ and u, v are real numbers with $u < v$ and $x = \lambda_1(x_0)$, where $x_0 \in M$. Since $s_\infty(x) = 1$, we have $v > 1$. Then $S_N(x)_{u,v} = \{s_\infty\} \cup \{\tilde{s} \mid s \in \mathcal{SM}(M), u < \tilde{s}(x)\} = \{s_\infty\}\cup, \phi(\{s \in \mathcal{SM}(M) \mid s(x_0) < 1 - u\})$. Therefore, $\phi(\mathcal{SM}(M))\setminus(\{s_\infty\} \cup \phi(\{s \in \mathcal{SM}(M) \mid s(x_0) < 1 - u\}) = \phi(\mathcal{SM}(M)\setminus\{s \in \mathcal{SM}(M) \mid s(x_0) < 1 - u\}) = \phi(\{s \in \mathcal{SM}(M) \mid s(x_0) \geq 1 - u\})$ and $X = \{s \in \mathcal{SM}(M) \mid s(x_0) \geq 1 - u\} = \emptyset$, which is a compact set, if $u < 0$, and $X \subseteq \{s \in \mathcal{SM}(M) \mid s(a) \geq 1 - u\} = \{s \in \mathcal{SM}(M) \mid s(a) = 1\}$ if $u \geq 0$ and a is an idempotent of M with $x_0 \leq a$. Therefore, X is a closed subset which is a subset of a compact set, see Theorem 4.10, and we have that X is a compact set.

(4) Let $s_\infty \in O_N = \bigcap_{i=1}^n S_N(x_i)_{u_i,v_i}$, where $u_i \in N$, $u_i < v_i$ and $s_\infty \in S_N(x_i)_{u_i,v_i}$ for each $i = 1, \ldots, n$. Then $S_N(x_i)_{u_i,v_i} = \{s_\infty\} \cup \phi(S(x'_i)_{u'_i,v'_i})$ where if $x_i \in M$, then $x'_i = x_i$ and $u'_i = u_i$, $v'_i = v_i$ and if $x_i \in N\setminus M$, then $x'_i = \lambda_1(x_i)$ and $u'_i = 1 - v_i$, $v'_i = 1 - u_i$. Hence,

$$\begin{aligned}\phi(\mathcal{SM}(M))\setminus\bigcap_{i=1}^n S_N(x_i)_{u_i,v_i} &= \phi\Big(\mathcal{SM}(M)\setminus\Big(\{s_\infty\} \cup \phi\Big(\bigcap_{i=1}^n S(x'_i)_{u'_i,v'_i}\Big)\Big)\Big)\\ &= \phi\Big(\bigcup_{i=1}^n(\mathcal{SM}(M)\setminus S(x'_i)_{u'_i,v'_i})\Big),\end{aligned}$$

so that $\bigcup_{i=1}^n(\mathcal{SM}(M)\setminus S(x'_i)_{u'_i,v'_i})$ is a compact set in view of (3).

(5) $O_N = \bigcup_\alpha O_\alpha^N$, where each O_α^N is the set of the form (4). Then $O_\alpha^N = \{s_\infty\} \cup \phi(O_\alpha)$ if $s_\infty \in O_\alpha^N$, otherwise $O_\alpha^N = O_\alpha$, where O_α is an open set in $\mathcal{SM}(M)$.

Then $\phi(\mathcal{SM}(M)\setminus \bigcup_\alpha O_\alpha^N) = \phi(\mathcal{SM}(M)\setminus \bigcup_\alpha O_\alpha)$, where O_α is a subset of $\mathcal{SM}(M)$ such that $O_\alpha^N = \phi(O_\alpha)$. Whence, $\mathcal{SM}(M)\setminus \bigcup_\alpha O_\alpha = \bigcap_\alpha(\mathcal{SM}(M)\setminus O_\alpha) \subseteq \mathcal{SM}(M)\setminus O_{\alpha_0}$, where α_0 is an index α such that $s_\infty \in O_{\alpha_0}^N$, which is by (4) a compact set, consequently, $\bigcap_\alpha(\mathcal{SM}(M)\setminus O_\alpha)$ is a compact set.

Therefore, $\mathcal{SM}(N)$ is the one-point compactification of $\mathcal{SM}(M)$.

Since the spaces $\mathcal{SM}(M)$ and $\mathrm{MaxI}(M)$ are homeomorphic, see Theorem 4.8, the same is true for $\mathcal{SM}(N)$ and $\mathrm{MaxI}(N)$. If we define $I_\infty = M$, I_∞ is a maximal ideal of N, and $I_\infty = \mathrm{Ker}(s_\infty)$. In addition, if $s \in \mathcal{SM}(M)$, then $\mathrm{Ker}(\tilde{s}) \cap M = \mathrm{Ker}(s)$. Therefore,

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

we get that the one-point compactification of MaxI(M) is MaxI(N) = {Ker($\tilde{s}$) | $s \in \mathcal{SM}(M)\} \cup \{I_\infty\}$. □

In a dual way as we did for the set of maximal ideals, we define the hull-kernel topology on the set MaxF(M) of maximal filters on EMV-algebras M. Thus given a filter F from the set Fil(M) of all filters on M, we define

$$O_1(F) := \{B \in \mathrm{MaxF}(M) \mid F \subsetneq B\}.$$

Then (i) $F_1 \subseteq F_2$ implies $O_1(F_1) \subseteq O_1(F_2)$, (ii) $\bigvee_\alpha O_1(F_\alpha) = O_1(\bigvee_\alpha F_\alpha)$, (iii) $\bigcup\{O_1(F) \mid F \in \mathrm{Fil}(M)\} = O_1(M) = \mathrm{Fil}(M)$, (iv) $\bigcap_{i=1}^n O_1(F_i) = O_1(\bigcap_{i=1}^n F_i)$. Hence, the system $\{O_1(F) \mid F \in \mathrm{Fil}(M)\}$ defines the so-called hull-kernel topology on the set MaxF(M). Every closed set is of the form $C_1(F) = \{B \in \mathrm{MaxF}(M) \mid F \subseteq B\}$. If given $x \in M$, we set $M_1(x) = \{B \in \mathrm{MaxF}(M) \mid x \notin B\}$, then the system $\{M_1(x) \mid x \in M\}$ is a base for the hull-kernel topology of maximal filters.

The following result is dual to the one from Proposition 4.7.

Proposition 4.14. *Let X be a nonempty set of state-morphisms closed in the weak topology of state-morphisms of an EMV-algebra M. Let t be a state-morphism such that $t \notin X$. There exists an element $a \in M$ such that $t(a) = 0$ and $s(a) = 1$ for all $s \in X$.*

Proof. Since the proof of the statement is dually similar to the one of Proposition 4.7, we outline only the main steps.

Let t be a state-morphism such that $t \notin X$. We assert that there exists an $a \in M$ such that $t(a) < 1/2$ while $s(a) > 1/2$ for all $s \in X$.

Indeed, set $A = \{a \in M : \ t(a) < 1/2\}$, and for all $a \in A$, let

$$W(a) := \{s \in \mathcal{SM}(M)) \mid \ s(a) > 1/2\},$$

which is an open subset of $\mathcal{SM}(M)$. We note that $A \neq \emptyset$ and A is upward directed and closed under $\odot$.

We assert that these open subsets cover X. Consider any $s \in X$. Since Ker(s) and Ker(t) are noncomparable subsets of M, there exists $x \in \mathrm{Ker}(t)\setminus\mathrm{Ker}(s)$. Hence $t(x) = 0$ and $s(x) > 0$. There exists an integer $n \geq 1$ such that $s(n.x) > 1/2$. Then $t(n.x) = 0$. If we put $a = n.x$, then $s \in W(a)$. Therefore, $\{W(a) \mid a \in A\}$ is an open covering of X.

Similarly as in the proof of Proposition 4.7, we pass to $\mathcal{SM}(N)$, where N is an EMV-algebra with a top element such that M is an EMV-subalgebra of N and we take the compact space $\phi(X) \cup \{s_\infty\}$. For each $a \in A$, we define $\widetilde{W}(a) = \{s \in \mathcal{SM}(N) \mid s(a) > 1/2\}$. Then each $\widetilde{W}(a)$ is an open subset of $\mathcal{SM}(N)$ not containing s_∞. Therefore, let $b \in M$ be an arbitrary element and we set $\widetilde{W}(b) = \{s \in \mathcal{SM}(N) \mid s(b) < 1/2\}$. Then $\widetilde{W}(b)$ is an open set containing the state-morphism s_∞, and $\widetilde{W}(b)$ is disjoint with $\widetilde{W}(a)$ for each $a \in A$. Since $\{\widetilde{W}(a) \mid a \in A\} \cup \{\widetilde{W}(b)\}$ is an open covering of $\phi(X) \cup \{s_\infty\}$, so that there are $a_1, \ldots, a_n \in A$ such that $\phi(X) \cup \{s_\infty\} \subseteq \bigcup_{i=1}^n \widetilde{W}(a_i) \cup \widetilde{W}(b)$. Therefore $X \subseteq W(a_1) \cup \cdots \cup W(a_n)$ for some $a_1, \ldots, a_n \in A$. Put $a_0 = a_1 \vee \cdots \vee a_n$. Then $a_0 \in A$ and for each $s \in X$, we have $s(a_0) \geq s(a_i) > 1/2$ for $i = 1, \ldots, n$, which proves $X \subseteq W(a_0)$, that is, $s(a_0) > 1/2$ for all $s \in X$. If we put $a = a_0 \oplus a_0$, then $t(a) = 0$ and $s(a) = 1$ for each $s \in X$. □

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

Theorem 4.15. *Let M be an EMV-algebra. Then the spaces $\mathcal{SM}(M)$,* $\mathrm{MaxI}(M)$ *and* $\mathrm{MaxF}(M)$ *are mutually homeomorphic spaces.*

Proof. According to Theorem 4.8, the spaces $\mathcal{SM}(M)$ and $\mathrm{MaxI}(M)$ are homeomorphic and the mapping $\theta : \mathcal{SM}(M) \to \mathrm{MaxI}(M)$, defined by $\theta(s) = \mathrm{Ker}(s)$, is a homeomorphism. According to Lemma 3.5, the mapping $\zeta : \mathcal{SM}(M) \to \mathrm{MaxF}(M)$ given by $\zeta(s) = \mathrm{Ker}_1(s)$, $s \in \mathcal{SM}(M)$, is bijective.

Let $C_1(F)$ be any closed subspace of $\mathrm{MaxF}(M)$. Then

$$\theta^{-1}(C_1(F)) = \{s \in \mathcal{SM}(M) \mid s(x) = 1 \text{ for all } x \in F\}$$

is a closed subspace of $\mathcal{SM}(M)$, so that ζ is continuous.

Given a nonempty subset X of $\mathcal{SM}(M)$, we define

$$\mathrm{Ker}_1(X) := \{x \in M \mid s(x) = 1 \text{ for all } s \in X\}.$$

Then $\mathrm{Ker}_1(X)$ is a filter of M. If, in addition, X is a closed subset of $\mathcal{SM}(M)$, we assert

$$\zeta(X) = C_1(\mathrm{Ker}_1(X)).$$

The inclusion $\zeta(X) \subseteq C_1(\mathrm{Ker}_1(X))$ is evident. By Proposition 4.14, if $t \notin X$, there is an element $a \in M$ such that $s(a) = 1$ for each $s \in X$ and $t(a) = 0$. Consequently, $t \notin X$ implies $\zeta(t) \notin C(\mathrm{Ker}_1(X))$, and $C(\mathrm{Ker}_1(X)) \subseteq \zeta(X)$. As a result, we conclude ζ is a homeomorphism. □

Lemma 4.16. *Let M be an EMV-algebra, $x \in M$, and $b \in \mathcal{I}(M)$ with $x \leq b$.*

(i) *Then*

$$M(b)\backslash M(x) \subseteq M(\lambda_b(x)).$$

(ii) *If $x \in \mathcal{I}(M)$, then*

$$M(b)\backslash M(x) = M(\lambda_b(x)).$$

(iii) *If $x, y \in M$, $x, y \leq b \in \mathcal{I}(M)$, then*

$$M(y)\backslash M(x \wedge y) = M(y)\backslash M(x) \subseteq M(y \odot \lambda_b(x)) \subseteq M(\lambda_b(x)).$$

(iv) *Let M be semisimple, $x \in M$, and $x \leq b \in \mathcal{I}(M)$. Then $x \in M$ is an idempotent if and only if $M(b)\backslash M(x) = M(\lambda_b(x))$.*

(v) *Let M be semisimple, $x, y \in \mathcal{I}(M)$, and $x, y \leq b \in \mathcal{I}(M)$. Then*

$$M(y)\backslash M(x \wedge y) = M(y)\backslash M(x) = M(y \odot \lambda_b(x)).$$

(vi) *If M is an arbitrary EMV-algebra having a top element 1, then for each idempotent $a \in \mathcal{I}(M)$, we have $M(\lambda_1(a)) = M(1)\backslash M(a) = M(a)^c$, where $M(a)^c$ is the set complement of $M(a)$ in* $\mathrm{MaxI}(M)$.

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

Proof.

(i) Let $x \leq b \in \mathcal{I}(M)$ and take $A \in M(b)\backslash M(x)$. Then $b \notin A$ and $x \in A$. We assert $\lambda_b(x) \notin A$. If not then from $b = x \oplus \lambda_b(x)$ we get a contradiction.

(ii) Assume that x is also an idempotent and take $A \in M(\lambda_b(x))$. Due to $b = x \oplus \lambda_b(x)$, we have $\lambda_b(x) \notin A$ and $b \notin A$. Since A is a prime ideal of M, then $0 = \lambda_a(x) \odot x = \lambda_b(x) \wedge x \in A$ entails $x \in A$ so that $A \in M(b)\backslash M(x)$.

(iii) Let $x, y \leq b \in \mathcal{I}(M)$. We have $M(y)\backslash M(x \wedge y) = M(x)\backslash(M(x) \wedge M(y)) = M(x)\backslash M(y)$. Choose $A \in M(x)\backslash M(y)$. Then $x \notin A$ and $y \in A$. Due to (2.2), we have $y = (x \wedge y) \oplus (y \odot \lambda_b(x))$ so that we get $y \odot \lambda_b(x) \notin A$. It is evident that $M(y \odot \lambda_b(x)) \subseteq M(\lambda_b(x))$.

(iv) Now let M be semisimple and $x \leq b \in \mathcal{I}(M)$. If x is idempotent, we have already established in (ii) $M(b)\backslash M(x) = M(\lambda_b(x))$. Conversely, let $M(b)\backslash M(x) = M(\lambda_b(x))$. Then for each $A \in M(b)$, we have either $x \in A$ or $\lambda_b(x) \notin A$. Whence $x \wedge \lambda_b(x) \in A$, and since $A \cap [0, b]$ is a maximal ideal of the MV-algebra $[0, b]$, [10, Proposition 3.23], we have $x \wedge \lambda_b(x) \in [0, b] \cap A$; the same is true if $A \notin M(b)$, whence it holds for each maximal ideal A of M. Since M is semisimple, $x \wedge \lambda_b(x) = 0$ and x is an idempotent in the MV-algebra $[0, b]$, so it is an idempotent in M, too.

(v) Let $A \in M(y \odot \lambda_b(x))$. Then $y \odot \lambda_b(x) \notin A$ and $y, \lambda_b(x) \notin A$. Due to (2.2), we have $y = (x \wedge y) \oplus (y \odot \lambda_b(x))$ and $(x \wedge y) \wedge (y \odot \lambda_b(x)) = (x \odot y) \odot (y \odot \lambda_b(x)) = 0 \in A$ (x, y and also $\lambda_b(x)$ are idempotents). Then $x \wedge y \in A$ and in addition, $x \in A$. Therefore, $A \in M(y)\backslash M(x)$.

(vi) If 1 is a top element of M, $a \in \mathcal{I}(M)$, then the assertion follows from the above proved equality. □

Proposition 4.17. *Let M be a semisimple EMV-algebra. If $x = \bigvee_t x_t \in M$, then*

$$M(x)\backslash \bigcup_t M(x_t)$$

is a nowhere dense subset of $\mathrm{MaxI}(M)$.

Proof. Let $x = \bigvee_t x_t$ and suppose $M(x)\backslash \bigcup_t M(x_t)$ is not nowhere dense. Since $\{M(y) \mid y \in M\}$ is a base of the topological space $\mathcal{T}_M$, there exists a nonzero element $b \in M$ such that $\emptyset \neq M(b) \subseteq M(x)\backslash \bigcup_t M(x_t)$. Due to $M(b) = M(b) \cap M(x) = M(b \wedge x)$, we take $b_0 := b \wedge x$ which is a nonzero element of M. Then $M(b_0) \cap M(x_t) = \emptyset$ for any t, so that $M(b_0 \wedge x_t) = \emptyset$ and the semisimplicity of M yields $b_0 \wedge x_t = 0$ for any t.

Using Proposition 3.4, we have

$$b_0 = b_0 \wedge a = b_0 \wedge \bigvee_t x_t = \bigvee_t (b_0 \wedge x_t) = 0,$$

which gives $M(b) = \emptyset$, a contradiction, so that our assumption was false, and consequently, $M(x)\backslash \bigcup_t M(x_t)$ is a nowhere dense set. □

Proposition 4.18. *Let M be a semisimple EMV-algebra and let $x_t \leq x \leq a \in \mathcal{I}(M)$ for any t. If $\bigcap_t M(x \odot \lambda_a(x_t))$ is a nowhere dense subset of* $\mathrm{MaxI}(M)$, *then $x = \bigvee_t x_t$.*

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

PROOF. It is clear that in order to prove $x = \bigvee_t x_t$ it is sufficient to verify that $x_t \le y \le x$ for any t implies $y = x$.

So let $\bigcap_t M(x \odot \lambda_a(x_t))$ be a nowhere dense set, and let $y \neq x$ for some $y \ge x_t$, $y \le x$. Then $x \odot \lambda_a(y) \neq 0$ and $M(x \odot \lambda_a(y))$ is a nonempty open subset of $\mathrm{MaxI}(M)$. By assumptions, there exists a nonzero open subset $O \subseteq M(x \odot \lambda_a(y))$ such that $O \cap \bigcap_t M(x \odot \lambda_a(x_t)) = \emptyset$. Consequently, there is a nonzero element $z \in M$ such that $M(z) \subseteq O$. Hence, for any $A \in M(z) \subseteq M(x \odot \lambda_a(y))$, we have $z \notin A$, $x \odot \lambda_a(y) \notin A$ and $A \notin \bigcap_t M(x \odot \lambda_a(x_t))$. This entails that there is an index t such that $x \odot \lambda_a(x_t) \in A$. Since $x_t \le y$, we have $x \odot \lambda_a(y) \le x \odot \lambda_a(x_t) \in A$ which implies $x \odot \lambda_a(y) \in A$, and this is a contradiction with $x \odot \lambda_a(y) \notin A$. Finally, our assumption $y < x$ was false, and whence $y = x$ and $x = \bigvee_t x_t$. □

COROLLARY 4.19. *Let M be a generalized Boolean algebra. Let $\{x_t\}$ be a system of elements of M which is majorized by $x \in M$. Then $x = \bigvee_t x_t$ if and only if $M(x) \setminus \bigcup_t M(x_t)$ is a nowhere dense set of* $\mathrm{MaxI}(M)$.

PROOF. By [10, Lemma 4.8], M is a semisimple EMV-algebra. If $x = \bigvee_t x_t$, the statement follows from Proposition 4.17. Conversely, let $M(x) \setminus \bigcup_t M(x_t)$ be nowhere dense. Then by Lemma 4.16(v), we have $M(\lambda_x(x_t)) = M(x \wedge \lambda_x(x_t)) = M(x \odot \lambda_x(x_t)) = M(x) \setminus M(x_t)$, so that $\bigcup_t M(\lambda_x(x_t)) = M(x) \setminus \bigcap_t M(x_t)$ is a nowhere dense set and applying Proposition 4.18, $x = \bigvee_t x_t$. □

COROLLARY 4.20. *A generalized Boolean algebra M is Dedekind σ-complete if and only if, for each sequence $\{a_n\}$ of elements of M which is majorized by an element $a \in M$, we have $\bigvee_n a_n = a$ if and only if $M(a) \setminus \bigcup_n M(a_n)$ is a nowhere dense set of* $\mathrm{MaxI}(M)$.

PROOF. It follows from Corollary 4.19. □

PROPOSITION 4.21. *Let M be an EMV-algebra. For each $x \in M$, we have*

$$M(x) = \bigcup_{n=1}^{\infty} (M(a) \setminus M(\lambda_a(n.x))), \tag{4.3}$$

where a is an idempotent of M such that $x \le a$.

PROOF. If $x \in \mathrm{Rad}(M)$, then $M(x) = \emptyset$. If $a \in \mathrm{Rad}(M)$, then $M(a) = M(\lambda_a(n.x)) = \emptyset$ and (4.3) holds. If $a \notin \mathrm{Rad}(M)$, then $M(a) \neq \emptyset$. From $a = n.x \oplus \lambda_a(n.x)$ we conclude $A \in M(a)$ if and only if $\lambda_a(n.x) \notin A$, so that $M(a) = M(\lambda_a(n.x))$ for each $n \ge 1$, henceforth (4.3) holds.

Now let $x \notin \mathrm{Rad}(M)$. Then $M(x) \neq \emptyset$ and let $A \in M(x)$. Again from $a = n.x \oplus \lambda_a(n.x)$, we conclude $A \notin M(a)$ and there is an integer $n \ge 1$ such that $\lambda_a(n.x) \in A$. Therefore, $M(x) \subseteq \bigcup_{n=1}^{\infty} (M(a) \setminus M(\lambda_a(n.x)))$.

Now, if $\bigcup_{n=1}^{\infty} (M(a) \setminus M(\lambda_a(n.x)))$ is empty, then $M(x) = \emptyset$ and the equality holds. Thus let $A \in \bigcup_{n=1}^{\infty} (M(a) \setminus M(\lambda_a(n.x)))$. There is an integer $n \ge 1$ such that $A \in M(a) \setminus M(\lambda_a(n.x))$ which means $a \notin A$ and $\lambda_a(n.x) \in A$. From $a = n.x \oplus \lambda_a(n.x)$, we have $n.x \notin A$, so that $x \notin A$ and $A \in M(x)$ which proves (4.3). □

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

5. The Loomis–Sikorski theorem for σ-complete EMV-algebras

In this section, we define a stronger notion of σ-complete EMV-algebras than Dedekind complete EMV-algebras and for them we establish a variant of the Loomis–Sikorski theorem which will say that every σ-complete EMV-algebra is a σ-homomorphic image of some σ-complete EMV-tribe of fuzzy sets, where all operations are defined by points.

We say that an EMV-algebra M is *σ-complete* if any countable family $\{x_n\}$ of elements of M has the least upper bound in M. Clearly, every σ-complete EMV-algebra is Dedekind σ-complete. Therefore, all results of the previous section concerning Dedekind σ-complete EMV-algebras are valid also for σ-complete ones. We note that both notions coincide if M has a top element. In the opposite case, these notions may be different. Indeed, let $\mathcal{T}$ be the set of all finite subsets of the set $\mathbb{N}$ of natural numbers. Then $\mathcal{T}$ is a generalized Boolean algebra that is Dedekind σ-complete but not σ-complete. On the other hand, if $\mathcal{T}$ is a system of all finite or countable subsets of the set of reals, then $\mathcal{T}$ is a σ-complete generalized Boolean algebra without a top element.

Lemma 5.1. *Let M be a σ-complete EMV-algebra. Then no nonempty open set of $\mathcal{SM}(M)$ can be expressed as a countable union of nowhere dense sets.*

Proof. By Proposition 3.3, M satisfies the general comparability property, and by Theorem 4.9, the spaces $\mathcal{SM}(M)$, $\mathrm{MaxI}(M)$, $\mathcal{SM}(\mathcal{I}(M))$, and $\mathrm{MaxI}(\mathcal{I}(M))$ are mutually homeomorphic spaces. In addition, $\mathcal{I}(M)$ is σ-complete. Therefore, we prove the lemma for $\mathrm{MaxI}(\mathcal{I}(M))$. We note, that given $x \in \mathcal{I}(M)$, $M(x) = \{I \in \mathrm{MaxI}(\mathcal{I}(M)) \mid x \notin I\}$, and by Theorem 4.10, $M(x)$ is clopen and compact.

Let $O \neq \emptyset$ be an open set of $\mathrm{MaxI}(\mathcal{I}(M))$ and let $O = \bigcup_n S_n$, where each S_n is a nowhere dense subset of $\mathrm{MaxI}(\mathcal{I}(M))$. Let O_0 be a nonempty open set, there is $x_1 \neq 0$ such that $M(x_1) \subseteq O_0$ and $M(x_1) \cap S_1 = \emptyset$. Since also S_2 is nowhere dense, in the same way, there is $0 < x_2 \in M$ such that $M(x_2) \subseteq M(x_1)$ and $M(x_2) \cap S_2 = \emptyset$. By induction, we obtain a sequence of nonzero elements $\{x_n\}$ such that $M(x_{n+1}) \subseteq M(x_n)$ and $M(x_n) \cap S_n = \emptyset$. We define $y_n = x_1 \wedge \cdots \wedge x_n$ for each $n \geq 1$. Then $M(y_n) = M(x_n)$, $n \geq 1$, and $M(y_n) \subseteq M(y_1)$. Put $y_0 = \bigwedge_n y_n$. Since $M(y_1)$ is compact, $\bigcap_n M(y_n) \neq \emptyset$, otherwise there is an integer n_0 such that $M(y_{n_0}) = \bigcap_{i=1}^{n_0} M(y_i) = \emptyset$, a contradiction.

Therefore, there is a maximal ideal I belonging to each $M(y_n)$ and $I \notin S_n$, so that $I \notin \bigcup_n S_n$ which is absurd, and the lemma is proved. □

Given an element $x \in M$, the set $S(x)$ was defined as $S(x) = \{s \in \mathcal{SM}(M) \mid s(x) > 0\}$.

Theorem 5.2. *Let M be a σ-complete EMV-algebra. For each $x \in M$, we define*

$$a_0(x) := \bigvee_n n.x. \tag{5.1}$$

Then $a_0(x)$ is an idempotent of M such that $a_0(x) \geq x$ and

$$a_0(x) = \bigwedge \{a \in \mathcal{I}(M) \mid a \geq x\}. \tag{5.2}$$

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

In addition, $\overline{S(x)} = S(a_0(x))$, *and if* $\overline{S(x)} = S(b)$ *for some idempotent* $b \in \mathcal{I}(M)$, *then* $a_0(x) = b$.

On the other hand, there is an idempotent $b_0(x)$ *of* M *such that*

$$b_0(x) = \bigwedge_n x^n$$

and

$$b_0(x) = \bigvee\{b \in \mathcal{I}(M) \mid b \le x\}. \tag{5.3}$$

(1) *If* y *is an element of* M *such that* $x \le y$ *and if* b *is an idempotent with* $\overline{S(y)} = S(b)$, *then* $a_0(x) \le b$.

(2) *Let* $x, x_1, \ldots$ *and* $a, a_1, \ldots$ *be a sequence of elements of* M *and* $\mathcal{I}(M)$, *respectively, such that* $\overline{S(x)} = S(a)$ *and* $\overline{S(x_n)} = S(a_n)$ *for each* $n \ge 1$. *If* $x = \bigvee_n x_n$, *then* $a = \bigvee_n a_n$.

Proof. Since M is σ-complete, the element $a_0(x) = \bigvee_n n.x$ exists in M for each $x \in M$. Using [12, Proposition 1.21], we have $a_0(x) \oplus a_0(x) = a_0(x) \oplus \bigvee_n n.x = \bigvee_n (a_0(x) \oplus n.x) = \bigvee_n \bigvee_m (n+m).x = a_0(x)$, so that $a_0(x)$ is an idempotent of M. Now let $b \in \mathcal{I}(M)$ be an idempotent such that $x \le b$. Then $n.x \le b$ for each integer n, so that $a_0(x) \le b$ which yields (5.2).

Since $S(x.n) = S(x)$ for each $n \ge 1$, we have $\bigcup_n S(n.x) \subseteq S(a_0(x))$, which by Proposition 4.17 means that $S(a_0(x)) \setminus \bigcup_n S(n.x) = S(a_0(x)) \setminus S(x)$ is a nowhere dense subset of $\mathcal{SM}(M)$. Then $\overline{S(x)} = \overline{S(n.x)} \subseteq S(a_0(x))$. Because $S(a_0(x))$ is compact and clopen by Theorem 4.10, $S(a_0(x)) \setminus \overline{S(x)} \subseteq S(a_0(x)) \setminus S(x)$, which gives that $S(a_0(x)) \setminus \overline{S(x)}$ is nowhere dense and open. Lemma 5.1 yields $S(a_0(x)) \setminus \overline{S(x)} = \emptyset$ and $S(a_0(x)) = \overline{S(x)}$.

Assume that b is another idempotent of M such that $\overline{S(x)} = S(b)$. First, let $a := a_0(x) \le b$. Then $b = a \vee \lambda_b(a)$, and $\lambda_b(a)$ is an idempotent of M, which entails $s(\lambda_b(a)) = 0$ for each state-morphism s of M. The semisimplicity of M yields $\lambda_b(a) = 0$ and $a = b$. In general, we have $S(a) = S(a) \cup S(b) = S(a \vee b)$, that is, $a = a \vee b = b$.

Let $a = a_0(x)$. Then $a = x \oplus \lambda_a(x)$. By (5.2), there is an idempotent $c_0 = \bigwedge\{c \in \mathcal{I}(M) \mid \lambda_a(x) \le c\}$. Then for the idempotent $\lambda_a(c_0)$ we have $\lambda_a(c_0) = \bigvee\{b \in \mathcal{I}(M) \mid b \le x\}$. Clearly, $n.\lambda_a(x) \le c_0$, so that $\lambda_a(c_0) \le x^n$ for each $n \ge 1$, and whence $\lambda_a(c) \le y_0 := \bigwedge_n x^n$. Using [12, Proposition 1.22], we have $y_0 \odot y_0 = y_0$ so that y_0 is an idempotent of M with $y_0 \le x$. Therefore, $y_0 \le \lambda_a(c_0)$.

(1) Now let $x \le y$. There is a unique idempotent b of M such that $\overline{S(y)} = S(b)$. Then $S(b) = \overline{S(y)} \supseteq \overline{S(x)} = S(a)$ and $S(b \vee a) = S(b) \cup S(a) = S(b)$, that is, $a \vee b = b$ and $a \le b$.

(2) By the above parts, the idempotents a and a_n with $\overline{S(x)} = S(a)$ and $\overline{S(x_n)} = S(a_n)$ are determined unambiguously, where $x = \bigvee_n x_n$. Put $a_0 = \bigvee_n a_n$. Then $a_0 \ge a_n \ge x_n$, $a_0 \ge x$, so that $a_0 \ge a_0(x) := a$. Now let b be any idempotent of M with $b \ge x$. Then $b \ge x_n$ for each $n \ge 1$, so that $b \ge a_n$ for each $n \ge 1$, and $b \ge a_0$ which by (5.2) yields $a_0 = a_0(x) = a$. □

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

The elements $a_0(x)$ and $b_0(x)$ defined in the latter theorem are said to be the *least upper idempotent* of x and the *greatest lower idempotent* of x, respectively, and for them, we have

$$b_0(x) \le x \le a_0(x).$$

Proposition 5.3. *Let M be a σ-complete EMV-algebra and let $\mathcal{B}(M)$ be the system of all compact and open subsets of* $\mathrm{MaxI}(M)$*. Then $\mathcal{B}(M) = \{M(a) \mid a \in \mathcal{I}(M)\}$. Moreover, for $a, b \in \mathcal{I}(M)$, we have $M(a) = M(b)$ if and only if $a = b$, and the closure of the union of countably many elements of $\mathcal{B}(M)$ belongs to $\mathcal{B}(M)$.*

In particular, for every sequence $\{a_n\}$ of elements of $\mathcal{I}(M)$,

$$\overline{\bigcup_n M(a_n)} = M(a), \tag{5.4}$$

where $a = \bigvee_n a_n$ and $a \in \mathcal{I}(M)$. Similarly, $\overline{\bigcup_n S(x_n)} = S(a)$.

Proof. Due to Theorem 4.10, every $M(a)$ is open and compact for each idempotent $a \in \mathcal{I}(M)$. Therefore, each $M(a)$ belongs to $\mathcal{B}(M)$.

If K is a compact and open subset of $\mathrm{MaxI}(M)$, we assert there is an element $x_0 \in M$ such that $K = O(x_0)$. Indeed, we have $K = C(J) = O(I)$ for some ideals J and I of M. Since $I = \bigvee\{I_x \mid x \in I\}$, where I_x is the ideal of M generated by an element x, then $O(I) = \bigcup\{O(I_x) \mid x \in I\}$, and the compactness of K provides us with finitely many elements $x_1, \ldots, x_n$ of I such that if $x_0 = x_1 \vee \cdots \vee x_n \in I$, then $K = O(I) = \bigcup_{i=1}^n O(I_{x_i}) = O(I_{x_0}) = M(x_0)$. Define $a_0(x_0)$ by (5.1). Then by Theorem 5.2, $K = M(x_0) = \overline{M(x_0)} = M(a_0(x_0))$. From the same theorem, we conclude that for two idempotents $a, b \in \mathcal{I}(M)$, $M(a) = M(b)$ implies $a = b$.

Now let $\{K_n\}$ be a sequence of elements from $\mathcal{B}(M)$. For each K_n, there is a unique idempotent $a_n \in \mathcal{I}(M)$ such that $K_n = M(a_n)$. Put $a = \bigvee_n a_n$; then $a \in \mathcal{I}(M)$. By Proposition 4.17, $M(a)\setminus \bigcup_n M(a_n)$ is nowhere dense. Since $M(a)\setminus\overline{\bigcup_n M(a_n)} \subseteq M(a)\setminus \bigcup_n M(a_n)$, the set $M(a)\setminus\overline{\bigcup_n M(a_n)}$ is open and nowhere dense which by Lemma 5.1 yields $M(a)\setminus\overline{\bigcup_n M(a_n)} = \emptyset$, that is, $M(a) = \overline{\bigcup_n M(a_n)} = \overline{\bigcup_n K_n}$.

The second equality $\overline{\bigcup_n S(x_n)} = S(a)$ follows from Theorem 4.10. □

An important notion of this section is an EMV-tribe of fuzzy sets which is a σ-complete EMV-algebra where all operations are defined by points.

Definition 5.4. A system $\mathcal{T} \subseteq [0, 1]^\Omega$ of fuzzy sets of a set $\Omega \neq \emptyset$ is said to be an *EMV-tribe* if

(i) $0_\Omega \in \mathcal{T}$ where $0_\Omega(\omega) = 0$ for each $\omega \in \Omega$;

(ii) $a \in \mathcal{T}$ is a characteristic function, then (a) if $f \in \mathcal{T}$ and $f(\omega) \le a(\omega)$ for each $\omega \in \Omega$, then $a - f \in \mathcal{T}$ (b) if $\{f_n\}$ is a sequence of functions from $\mathcal{T}$ with $f_n(\omega) \le a(\omega)$ for each $\omega \in \Omega$ and each $n \ge 1$, where $a \in \mathcal{T}$ is a characteristic function, then $\bigoplus_n f_n \in \mathcal{T}$, where $\bigoplus_n f_n(\omega) = \min\{\sum_n f_n(\omega), a(\omega)\}$, $\omega \in \Omega$;

(iii) for each $f \in \mathcal{T}$, there is a characteristic function $a \in \mathcal{T}$ such that $f(\omega) \le a(\omega)$ for each $\omega \in \Omega$;

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

(iv) given $\omega \in \Omega$, there is $f \in \mathcal{T}$ such that $f(\omega) = 1$.

Proposition 5.5. *Every EMV-tribe of fuzzy sets is a Dedekind σ-complete EMV-clan where all operations are defined by points. If $\{g_n\}$ is a sequence from $\mathcal{T}$, then $g = \bigwedge_n g_n$ exists in $\mathcal{T}$ and $g(\omega) = \inf_n g_n(\omega)$, $\omega \in \Omega$.*

If for a sequence $\{f_n\}$ from $\mathcal{T}$, $f = \bigvee_n f_n$ exists in $\mathcal{T}$, then $f(\omega) = \sup_n f_n(\omega)$, $\omega \in \Omega$. An EMV-tribe is σ-complete if and only if, for each sequence $\{f_n\}$ of elements of $\mathcal{T}$, there is a characteristic function $a \in \mathcal{T}$ such that $f_n(\omega) \le a(\omega)$, $\omega \in \Omega$.

Proof. By [10, Proposition 4.10], we see that $\mathcal{T}$ is an EMV-clan of fuzzy sets of Ω which is closed under $\vee$ and $\wedge$, defined by points. We have to show that the operation $\bigoplus$ is correctly defined. Let $\{f_n\}$ be any sequence for which there are two characteristic functions $a, b \in \mathcal{T}$ such that $f_n(\omega) \le a(\omega), b(\omega)$, $\omega \in \Omega$ and $n \ge 1$. There is another characteristic function $c \in \mathcal{T}$ with $a(\omega), b(\omega) \le c(\omega)$, $\omega \in \Omega$. We denote $(\bigoplus_n^a f_n)(\omega) := \min\{\sum_n f_n(\omega), a(\omega)\}$ for each $\omega \in \Omega$. In the same way we define $\bigoplus_n^b f_n$ and $\bigoplus_n^c f_n$. Then

$$\Big(\bigoplus_n^a f_n\Big)(\omega) = \begin{cases} \sum_n f_n(\omega) & \text{if } \sum_n f_n(\omega) \le a(\omega) \\ a(\omega) & \text{if } \sum_n f_n(\omega) > a(\omega), \end{cases} \qquad \omega \in \Omega,$$

and

$$\Big(\bigoplus_n^c f_n\Big)(\omega) = \begin{cases} \sum_n f_n(\omega) & \text{if } \sum_n f_n(\omega) \le c(\omega) \\ c(\omega) & \text{if } \sum_n f_n(\omega) > c(\omega), \end{cases} \qquad \omega \in \Omega.$$

If $a(\omega) = 0$, then $f_n(\omega) = 0$ for each n and $(\bigoplus_n^a f_n)(\omega) = 0 = (\bigoplus_n^c f_n)(\omega)$. If $a(\omega) = 1$, then $c(\omega) = 1$ and $(\bigoplus_n^a f_n)(\omega) = (\bigoplus_n^c f_n)(\omega)$. In the same way we have $(\bigoplus_n^b f_n) = (\bigoplus_n^c f_n)$, so that $(\bigoplus_n^a f_n) = (\bigoplus_n^b f_n)$, and $\bigoplus_n f_n$ is well-defined.

Choose an arbitrary sequence $\{f_n\}$ from $\mathcal{T}$ which is dominated by some characteristic function $a \in \mathcal{T}$. Without loss of generality we can assume that $f_n(\omega) \le f_{n+1}(\omega)$, $\omega \in \Omega$, $n \ge 1$. We set $h_1 = f_1$ and $h_n = f_n - f_{n+1}$ for $n \ge 1$. Then each h_n belongs to $\mathcal{T}$ and it is dominated by a. Therefore, $\bigoplus_n h_n \in \mathcal{T}$ and $(\bigoplus_n h_n)(\omega) = \sum_n h_n(\omega) = \sup_n f_n(\omega)$, which proves that $\mathcal{T}$ is Dedekind σ-complete. Consequently, $\mathcal{T}$ is σ-complete if and only if for each sequence $\{f_n\}$ we can find a characteristic function $a \in \mathcal{T}$ which dominates each f_n.

Now let $\{g_n\}$ be any sequence from $\mathcal{T}$. Since $\mathcal{T}$ is a lattice where $(f \wedge g)(\omega) = \min\{f(\omega), g(\omega)\}$, $\omega \in \Omega$, without loss of generality, we can assume that $g_{n+1} \le g_n$ for each $n \ge 1$. Then there is a characteristic function $a \in \mathcal{T}$ such that $g_n(\omega) \le a(\omega)$, $\omega \in \Omega$, $n \ge 1$, and $a - g_n \in \mathcal{T}$, $a - g_n \le a - g_{n+1}$. Whence, $(\bigvee_n (a - g_n))(\omega) = \sup_n (a - g_n)(\omega)$ for each $\omega \in \Omega$. Consequently $(\bigwedge_n g_n)(\omega) = a(\omega) - (\bigvee_n (a - g_n))(\omega) = a(\omega) - \sup_n (a(\omega) - g_n(\omega)) = \inf_n g_n(\omega)$, $\omega \in \Omega$. □

We note that a *tribe* is a system $\mathcal{T} \subseteq [0,1]^\Omega$ of fuzzy sets on $\Omega \ne \emptyset$ such that (i) $1_\Omega \in \mathcal{T}$, (ii) if $f \in \mathcal{T}$, then $1 - f \in \mathcal{T}$, and (iii) for any sequence $\{f_n\}$ of elements of $\mathcal{T}$, the function $\bigoplus_n f_n$ belongs to $\mathcal{T}$, where $(\bigoplus_n f_n)(\omega) = \min\{\sum_n f_n(\omega), 1\}$, $\omega \in \Omega$. Then the notion of an EMV-tribe is a generalization of the notion of a tribe because

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

an EMV-tribe $\mathcal{T}$ is a tribe if and only if $1_\Omega \in \mathcal{T}$. We note that in [7, 18], it was proved that every σ-complete MV-algebra is a σ-homomorphic image of some tribe of fuzzy sets.

We say that an EMV-homomorphism $h : M_1 \to M_2$ is a *σ-homomorphism*, where M_1 and M_2 are EMV-algebras, if for any sequence $\{x_n\}$ of elements from M_1 for which $x = \bigvee_n x_n$ is defined in M_1, then $\bigvee_n h(x_n)$ exists in M_2 and $h(x) = \bigvee_n h(x_n)$.

Let f be a real-valued function on $\Omega \neq \emptyset$. We define $N(f) := \{\omega \in \Omega \mid |f(\omega)| > 0\}$, $N^+(f) = \{\omega \in \Omega \mid f(\omega) > 0\}$ and $N^-(f) = \{\omega \in \Omega \mid f(\omega) < 0\}$. Then $N(f) = N^+(f) \cup N^-(f)$.

Suppose that $\mathcal{T}$ is a system of fuzzy sets on Ω, containing 0_Ω, such that, for each $f \in \mathcal{T}$, there is a characteristic function $a \in \mathcal{T}$ with $f(\omega) \le a(\omega)$, $\omega \in \Omega$. If $f, g \le a$ for some characteristic function from $\mathcal{T}$, we can define $(f \oplus g)(\omega) = \min\{f(\omega) + g(\omega), a(\omega)\}$, $(f \odot g)(\omega) = \max\{f(\omega) + g(\omega) - a(\omega), 0\}$, and $(f * g)(\omega) = \max\{f(\omega) - g(\omega), 0\}$ for each $\omega \in \Omega$, and these operations do not depend on a.

Then for all $f, g \in \mathcal{T}$ we have:

(i) $N(f \oplus g) = N(f) \cup N(g)$;
(ii) $N(f * g) = \{\omega \in \Omega \mid f(\omega) > g(\omega)\}$;
(iii) $(f * g) \oplus (g * f) = (f * g) + (g * f)$;
(iv) $N((f * g) \oplus (g * f)) = N(f - g)$;
(v) $N(f) \subseteq N(g)$ if $f \le g$;
(vi) $N(f \odot g) = \{\omega \in \Omega \mid f(\omega) + g(\omega) > 1\}$.

Now we formulate the Loomis–Sikorski theorem for σ-complete EMV-algebras.

Theorem 5.6 (The Loomis–Sikorski theorem). *Let M be a σ-complete EMV-algebra. Then there are an EMV-tribe $\mathcal{T}$ of fuzzy sets on some $\Omega \neq \emptyset$ and a surjective σ-homomorphism h of EMV-algebras from $\mathcal{T}$ onto M.*

Proof. If $M = \{0\}$, the statement is trivial. So let $M \neq \{0\}$.

By Proposition 3.3, M is a semisimple EMV-algebra, and by the proof of [10, Theorem 4.11], M is isomorphic to $\widehat{M} = \{\hat{x} \mid x \in M\}$, where $\hat{x} : \mathcal{SM}(M) \to [0, 1]$ is defined by $\hat{x}(s) = s(x)$, $s \in \mathcal{SM}(M)$.

Let $\mathcal{T}$ be the system of fuzzy sets f on $\Omega = \mathcal{SM}(M)$ such that (i) for some $x \in M$, $N(f - \hat{x})$ is a meager set (that is, it is a countable union of nowhere dense subsets) in the weak topology of state-morphisms, and we write $f \sim x$, and (ii) there is $a \in \mathcal{I}(M)$ such that $f \le \hat{a}$. It is clear that $\mathcal{T}$ contains $\widehat{M}$.

If x_1 and x_2 are two elements of M such that $N(f - \hat{x}_i)$ is a meager set for $i = 1, 2$, then

$$N(\hat{x}_1 - \hat{x}_2) \subseteq N(\hat{x}_1 - f) \cup N(f - \hat{x}_2)$$

is a meager set. By Lemma 5.1, we conclude that $N(\hat{x}_1 - f) \cup N(f - \hat{x}_2) = \emptyset$ from which we get $\hat{x}_1 = \hat{x}_2$, that is $x_1 = x_2$. Therefore, if $f \sim x_1$ and $f \sim x_2$, then $x_1 = x_2$.

Claim 1. The set $\mathcal{T}$ is an EMV-clan.

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

Let $f, g, h \in \mathcal{T}$ and let $N(g-h)$ be a meager set. We assert $N_0 := N((f\oplus g) * (f\oplus h))$ is a meager set. Set $N_1 = \{s \mid \min\{f(s)+g(s), 1\} > \min\{f(s)+h(s), 1\}\}$ and check

$$\begin{aligned} N_1 &= (N_1 \cap \{s \mid g(s) = h(s)\}) \cup (N_1 \cap \{s \mid g(s) > h(s)\}) \cup (N_1 \cap \{s \mid g(s) < h(s)\}) \\ &= (N_1 \cap \{s \mid g(s) > h(s)\}) \cup (N_1 \cap \{s \mid g(s) < h(s)\}) \subseteq N_1 \cap N(g-h), \end{aligned}$$

which shows that N_0 is a meager set. Similarly, $N((f\oplus h) * (f\oplus g))$ is a meager set.

In a similar way, if $N_3 := N((f\vee g) * (f\vee h)) = \{s \mid f(s)\vee g(s) > f(s)\vee h(s)\}$, then

$$\begin{aligned} N_3 &= (N_3 \cap \{s \mid g(s) = h(s)\}) \cup (N_3 \cap \{s \mid g(s) > h(s)\}) \cup (N_3 \cap \{s \mid g(s) < h(s)\}) \\ &= (N_3 \cap \{s \mid g(s) > h(s)\}) \cup (N_3 \cap \{s \mid g(s) < h(s)\}) \\ &\subseteq N_3 \cap N(g-h) \subseteq N(g-h), \end{aligned}$$

which establishes N_3 is a meager set. In the same way, the set $N((f\vee h)-(f\vee g))$ is meager, consequently, $N((f\vee g)-(f\vee h))$ is a meager set, too.

Therefore, if $f, g \in \mathcal{T}$ and $f \sim x$ and $g \sim y$ for unique $x, y \in M$, there is an idempotent $a \in \mathcal{I}(M)$ such that $x, y \le a$ and $f, g \le \hat{a}$. This implies $N((f\oplus g) * (\hat{x}\oplus\hat{y})) \subseteq N((f\oplus g) * (f\oplus\hat{y})) \cup N((f\oplus\hat{y}) * (\hat{x}\oplus\hat{y}))$ is a meager set. Similarly $N((\hat{x}\oplus\hat{y}) * (f\oplus g))$ is also a meager set. Therefore, $f\oplus g \sim x\oplus y$ which proves $\mathcal{T}$ is an EMV-clan and $\mathcal{T}$ is closed also under $\vee$ and $\wedge$ with pointwise ordering. In the same way, we have also $f\vee g \sim x\vee y$.

We note that if $f \in \mathcal{T}$ is a characteristic function such that $f \sim x \in M$, $f \le \hat{a}$ for some $a \in \mathcal{I}(M)$, then $f = f\oplus f \sim x\oplus x = x$, so that x is an idempotent of M.

Let $f \in \mathcal{T}$, $f \sim x$, $f \le b$ for some characteristic function $b \in \mathcal{T}$. Then there is a unique idempotent $a \in \mathcal{I}(M)$ such that $b \sim a$, in addition, $x \le a$. Then we have $\widehat{\lambda_a(x)} = \hat{a}-\hat{x}$, and $N((b-f)-(\widehat{\lambda_a(x)})) = N((b-f)-(\hat{a}-\hat{x})) = N((b-\hat{a})-(f-\hat{x})) \subseteq N(b-\hat{a}) \cup N(f-\hat{x})$, which is a meager set. Hence,

$$\lambda_b(f) = b-f \sim \lambda_a(x). \tag{5.5}$$

We note that if $f, g \in \mathcal{T}$, and if a is an idempotent of M such that $f, g \le \hat{a}$, then $1-f, f\vee g, f\oplus g$ are dominated by $\hat{a}$. Consequently, $\mathcal{T}$ is an EMV-clan.

Claim 2. The set $\mathcal{T}$ is closed under pointwise limits of nondecreasing sequences from $\mathcal{T}$.

Let $\{f_n\}_n$ be a sequence of nondecreasing functions from $\mathcal{T}$. Choose $x_n \in M$ such that $f_n \sim x_n$ for each $n \ge 1$. Since $f_n = f_1 \vee \cdots \vee f_n \sim x_1 \vee \cdots \vee x_n$ for each $n \ge 1$, we have $x_n \le x_{n+1}$. Denote $f = \lim_n f_n$, $x = \bigvee_{n=1}^{\infty} x_n$, and $b_0 = \lim_n \hat{x}_n$. Then $x \in M$. It is easy to see that there is an idempotent a such that $x, x_1, \ldots \le a$ and $f_1, f_a \le \hat{a}$.

We have

$$N(f-\hat{x}) \subseteq N(f-b_0) \cup N(\hat{x}-b_0)$$

and $N(f-b_0) = \{s \mid f(s) < b_0(s)\} \cup \{s \mid b_0(s) < f(s)\}$.

If $s \in \{s \mid f(s) < b_0(s)\}$, then there is an integer $n \ge 1$ such that $f(s) < \hat{x}_n(s) \le b_0(s)$. Hence, $f_n(s) \le f(s) < \hat{x}_n(s) \le b_0(s)$ so that $s \in \{s \mid f_n(s) < \hat{x}_n(s)\}$.

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

Similarly we can prove that if $s \in \{s \mid b_0(s) < f(s)\}$, then there is an integer $n \geq 1$ such that $s \in \{s \mid \hat{b}_n(s) < f_n(s)\}$.

The last two cases imply

$$N(f - b_0) \subseteq \bigcup_{n=1}^{\infty} N(\hat{x}_n - f_n)$$

is a meager set.

Now it is necessary to show that $N(\hat{x} - b_0)$ is a meager set. We have

$$\begin{aligned} N(\hat{x} - b_0) &= (N(\hat{x} - b_0) \cap \{s \mid s(x) > 0\}) \cup (N(\hat{x} - b_0) \cap \{s \mid s(x) = 0\}) \\ &= N(\hat{x} - b_0) \cap S(x) \\ &= \Big(N(\hat{x} - b_0) \cap \Big(S(x) \setminus \bigcup_n S(x_n)\Big)\Big) \cup \Big(N(\hat{x} - b_0) \cap \Big(S(x) \cap \bigcup_n S(x_n)\Big)\Big). \end{aligned}$$

By Proposition 4.17, we have that $N(\hat{x} - b_0) \cap (S(x) \setminus \bigcup_n S(x_n))$ is a meager set. Therefore, it is necessary to prove that $N_0 := N(\hat{x} - b_0) \cap (S(x) \cap \bigcup_n S(x_n)) = N(\hat{x} - b_0) \cap \bigcup_n S(x_n)$ is a meager set.

To prove it, take an arbitrary open nonempty set O in $\mathcal{SM}(M)$. Then there is an ideal I of M such that $O = \{s \in \mathcal{SM}(M) \mid I \subsetneq \mathrm{Ker}(s)\}$. The ideal I contains a nonzero element $z \in I$. There is an idempotent $a \in \mathcal{I}(M)$ such that $x, z \leq a$. We note that in such a case, $a_0(x) \leq a$, where $a_0(x)$ is the least upper idempotent of x defined in Theorem 5.2. The restriction of any state-morphism $s \in S(a)$ onto the MV-algebra $M_a = [0, a]$ is a state-morphism on M_a; we denote the set of those restrictions by $S_0(a)$. Then $S_0(a) \subseteq \mathcal{SM}(M_a)$. It is clear that M_a is a σ-complete MV-algebra, whence $x, x_1, \ldots \in M_a$ and x is the least upper bound of $\{x_n\}$ taken in the MV-algebra M_a. By the proof of [7, Theorem 4.1], $S_0 := \{s \in \mathcal{SM}(M_a) \mid s(x) > \lim_n s(x_n)\}$ is a meager set in the weak topology of $\mathcal{SM}(M_a)$. Then $\{s_{|M_a} \mid s \in S(x) \cap N(\hat{x} - b_0)\} \subseteq S_0$ is also a meager set of $\mathcal{SM}(M_a)$.

The element z belongs to $[0, a]$, and let $I_a = I \cap [0, a]$. Clearly I_a is an ideal of M_a containing z, and let $O_a(I_a) = \{s \in \mathcal{SM}(M_a) \mid I_a \subsetneq \mathrm{Ker}(s)\}$. Then $O_a(I_a)$ is a nonzero open set of $\mathcal{SM}(M_a)$. Therefore, there is an element $0 < y \in M_a$ such that $S_a(y) = \{s \in \mathcal{SM}(M_a) \mid s(y) > 0\} \subseteq O_a(I_a)$ and it has the empty intersection with S_0. Define $S(y) = \{\mathcal{SM}(M) \mid s(y) > 0\}$. Since $y \leq a$, we have $S(y) \subseteq M(a)$. For each state-morphism s on M, let s_a be the restriction to s onto M_a. Take $s \in S(y)$, then $s_a(y) > 0$, s_a is a state-morphism on M_a, $s_a \in S_a(y)$, and $s_a \in O_a(I_a)$. That is, there is a nonzero element $t \in I_a$ such that $s_a(t) = 0$, that is, $s(t) = 0$ for some $t \in I$ which gets $s \in O$. We have proved that $S(y) \subseteq O$. We assert $S(y) \cap S(x) \cap N(\hat{x} - b_0) = \emptyset$. If not, there is a state-morphism s belonging to the intersection. Then $s(a) = 1$ since $s \in S(y)$, so that s_a is a state-morphism on M_a, $s_a(y) = s(y) > 0$, and $\hat{x}(s) - b_0(s) = s_a(x) - \lim_n s_a(x_n) > 0$ which is an absurd, and the intersection is empty. Therefore, the set $S(x) \cap N(\hat{x} - b_0)$ is a meager set.

Hence, given a nondecreasing sequence $\{f_n\}$, for the function f defined by $f(s) = \sup_n f_n(s)$, $s \in \mathcal{SM}(M)$, we have $f \sim x$, where $x = \bigvee_n x_n$, and clearly $f \in \mathcal{T}$.

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

Claim 3. The set $\mathcal{T}$ is an EMV-tribe.

Now let $\{f_n\}$ be an arbitrary sequence of functions from $\mathcal{T}$ such that $f_n \sim x_n \in M$. By the previous step, there is an idempotent $a \in M$ such that $x_1, x_2, \ldots \leq a$ and $f_1, f_2, \ldots \leq \hat{a}$. Then for each $n \geq 1$, $g_n = f_1 \oplus \cdots \oplus f_n = \min\{f_1 + \cdots + f_n, \hat{a}\} \sim x_1 \oplus \cdots \oplus x_n$ and it does not depend on a. Then $\bigoplus_n f_n$ is a pointwise limit of the nondecreasing sequence $\{g_n\}$, that is, $\bigoplus_n f_n = \lim_n g_n$, which by Claim 2 means, $\bigoplus_n f_n \sim \bigvee_n (x_1 \oplus \cdots \oplus x_n)$. In addition, $\bigoplus_n f_n \leq \hat{a}$, so that, we have shown that $\bigoplus_n f_n \in \mathcal{T}$ and $\mathcal{T}$ is an EMV-tribe of fuzzy sets on $\mathcal{SM}(M)$. Since by the construction of $\mathcal{T}$, for each $f \in \mathcal{T}$, there is an idempotent $a \in \mathcal{I}(M)$ such that $f \leq \hat{a}$, Proposition 5.5 says that $\mathcal{T}$ is an EMV-tribe.

Claim 4. M is a σ-homomorphic image of the EMV-tribe $\mathcal{T}$.

Define a mapping $h : \mathcal{T} \to M$ by $h(f) = x$ if and only if $f \in \mathcal{T}$ and $f \sim x \in M$. By the first part of the present proof, h is a well-defined mapping that is surjective. It preserves $\oplus, \vee, \wedge$, and $h(0_\Omega) = 0$. In addition, if $f = \bigvee_n f_n = \sup_n f_n$, then by Step 2, $f_n \sim x_n$ and $f \sim x = \bigvee_n x_n$, that is $h(f) = \bigvee_n h(f_n)$.

Now let $f \leq b$, where $f \in \mathcal{T}$ and b is a characteristic function from $\mathcal{T}$. There are unique elements $x \in M$ and $a \in \mathcal{I}(M)$ such that $f \sim x$ and $b \sim a$. Clearly, $x \leq a$. Then $b = f \oplus \lambda_b(f)$, and by (5.5), we have $b - f \sim \lambda_a(x)$, that is, $h(b - f) = h(\lambda_b(f)) = \lambda_a(x)$, so that $a = h(b) = h(f) \oplus h(\lambda_b(f)) = h(f) \oplus \lambda_{h(b)}(h(f)) = x \oplus \lambda_a(x)$. By definition of $\lambda_{h(b)}$ in M, we have $\lambda_a(x) = \lambda_{h(b)}(h(f)) \leq h(\lambda_b(f)) = \lambda_a(x)$, that is $h(\lambda_b(f)) = \lambda_{h(b)}(h(f))$, which proves that h is a homomorphism of EMV-algebras. Consequently, h is a surjective σ-homomorphism as we needed.

The theorem is proved. □

We recall that if Ω is a nonvoid set, then a *ring* is a system $\mathcal{R}$ of subsets of Ω such that (i) $\emptyset \in \mathcal{R}$, (ii) if $A, B \in \mathcal{R}$, then $A \cup B, A \backslash B \in \mathcal{R}$. A ring $\mathcal{R}$ is a *σ-ring* if given a sequence $\{A_n\}$ of subsets from $\mathcal{R}$, $\bigcup_n A_n \in \mathcal{R}$. Clearly, every ring is an EMV-algebra and a generalized Boolean algebra of subsets.

We recall that due to the Stone theorem, see for example [16, Theorem 6.6], every generalized Boolean algebra is isomorphic to some ring of subsets.

A corollary of the Loomis–Sikorski theorem 5.6 is the following result.

Corollary 5.7. *Let M be a σ-complete EMV-algebra. Then there are a σ-ring $\mathcal{R}$ of subsets of some set $\Omega \neq \emptyset$ and a surjective σ-homomorphism from $\mathcal{R}$ onto $\mathcal{I}(M)$.*

Proof. Since M is σ-complete, by Proposition 3.3, $\mathcal{I}(M)$ is a σ-complete subalgebra of M, in other words, $\mathcal{I}(M)$ is a σ-complete generalized Boolean algebra.

Use the system $\mathcal{T}$ defined in the proof of Theorem 5.6, that is $f \in \mathcal{T}$ if and only if there is an element $x \in M$ with $f \sim x$ and there is an idempotent $a \in M$ such that $f \leq \hat{a}$; $\mathcal{T}$ is a σ-complete EMV-tribe of fuzzy functions on $\Omega = \mathcal{SM}(M)$. Then the mapping $h : \mathcal{T} \to M$ defined by $h(f) = x$ ($f \in \mathcal{T}$) if $f \sim x \in M$, is a surjective σ-homomorphism.

Denote by $\mathcal{R}_0$ the class of all characteristic functions from $\mathcal{T}$. As proved in Theorem 5.6, for each $f \in \mathcal{R}_0$, there is a unique $x \in \mathcal{I}(M)$ such that $f \sim x$. If (i) $\chi_A, \chi_B \in \mathcal{R}_0$, then $\chi_A \vee \chi_B = \chi_A \oplus \chi_B = \chi_{A \cup B} \in \mathcal{R}_0$, (ii) if $\chi_A, \chi_B \in \mathcal{R}_0$ and $\chi_A \leq \chi_B$,

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

then $\chi_B - \chi_A \in \mathcal{R}_0$, (iii) if $\chi_A, \chi_B \in \mathcal{R}_0$, then $\chi_A \wedge \chi_B = \chi_{A\cap B} \in \mathcal{R}_0$, and (iv) if $\{\chi_{A_n}\}$ is a sequence of characteristic functions from $\mathcal{R}_0$, then $\bigoplus_n \chi_{A_n} = \chi_A \in \mathcal{R}_0$, where $A = \bigcup_n A_n$.

We note here, that in Claim 2 of the proof of the Loomis–Sikorski theorem, it was necessary to prove that $N(\hat{x} - b_0)$ is a meager set. We show that if the nondecreasing sequence $\{x_n\}$ of elements of M with $x = \bigvee_n x_n$ and $b_0 = \lim_n \hat{x}_n$ consists only of idempotent elements, the proof of the fact $N(\hat{x} - b_0)$ is a meager set is very easy. Indeed, if $s \in N_0 := N(\hat{x} - b_0) \cap (S(x) \cap \bigcup_n S(x_n)) = N(\hat{x} - b_0) \cap \bigcup_n S(x_n)$, there is an integer n_0 such that $s \in S(x_{n_0})$. Then we have $1 \geq s(x) \geq s(x_{n_0}) = 1$ that yields $\hat{x}(s) = 1 = b_0(s)$ and the set N_0 is empty.

Now if $h_0 : \mathcal{R}_0 \to \mathcal{I}(M)$ is the restriction of the σ-homomorphism $h : \mathcal{T} \to M$ onto $\mathcal{R}_0$ we see that h_0 is a σ-homomorphism from $\mathcal{R}_0$ onto $\mathcal{I}(M)$. Now let $\mathcal{R} = \{A \subseteq \Omega \mid \chi_A \in \mathcal{R}_0\}$. Then $\mathcal{R}_0$ is a σ-complete ring of subsets of $\Omega = \mathcal{SM}(M)$. Define a mapping $\iota : \mathcal{R} \to \mathcal{R}_0$ by $\iota(A) = \chi_A$, $A \in \mathcal{R}$. It is clear that ι is a σ-complete isomorphism. If we set $\phi = h_0 \circ \iota : \mathcal{R} \to \mathcal{I}(M)$, then ϕ is a surjective σ-homomorphism from $\mathcal{R}$ onto the set of idempotents $\mathcal{I}(M)$, and the corollary is proved. □

We note that the last result can be found in [14, page 216] using the language of σ-complete Boolean rings. Therefore, Theorem 5.6 is a generalization of the Loomis–Sikorski theorem for Boolean σ-algebras, see [15, 19], σ-complete Boolean rings, [14], and σ-complete MV-algebras, see [1, 7, 18].

We say that an ideal I of an EMV-algebra M is *σ-complete* if, for each sequence $\{x_n\}$ of elements of I, the existence of $\bigvee_n x_n$ in M implies $\bigvee_n x_n \in I$.

Theorem 5.8. *Every σ-complete EMV-algebra M without a top element can be embedded into a σ-complete EMV-algebra N with a top element as its maximal ideal which is also σ-complete. Moreover, this N can be represented as*

$$N = \{x \in N \mid \text{ either } x \in M \text{ or } x = \lambda_1(y) \text{ for some } y \in M\}.$$

Proof. If a σ-complete EMV-algebra M possesses a top element, then it is termwise equivalent to an MV-algebra, so $(M; \oplus, \lambda_1, 0, 1)$ is a σ-complete MV-algebra. Thus, let M have no top element. According to Theorem 2.1, there is an EMV-algebra N with a top element such that M can be embedded into N as its maximal ideal. Without loss of generality let us assume that M is an EMV-subalgebra of N. Let 1 be the top element of N. By the proof of Theorem 2.1, every element $x \in N$ is either from M, or $\lambda_1(x) \in M$. Due to Mundici's result, see [17], there is a unital Abelian ℓ-group (G, u) such that $N = \Gamma(G, u)$ so that $1 = u$. Thus let $\{x_n\}$ be an arbitrary sequence of elements of N.

There are three cases. (1) Every $x_n \in M$. Then there is an element $x = \bigvee_n x_n \in M$, where the supremum x is taken in the σ-complete EMV-algebra M. Thus let $x_n \leq y$ for each n, where $y \in N$. It is enough to assume that $y = \lambda_1(y_0)$ for some $y_0 \in M$. Using the Mundici representation of MV-algebras by unital ℓ-groups, we obtain $x_n \leq \lambda_1(y_0) = u - y_0$, so that $y_0 + x_n \leq u$, where $+$ and $-$ denote the group addition and the group subtraction, respectively, taken in the group (G, u). Hence, $y_0 + x_n = y_0 \oplus x_n \in M$, so

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

that there is $\bigvee_n(y_0 \oplus x_n)$ in M, which means $y_0 \oplus \bigvee_n x_n = \bigvee_n(y_0 \oplus x_n) \le u$ as well as $y_0 + \bigvee_n x_n = \bigvee_n(y_0 + x_n) = \bigvee_n(y_0 \oplus x_n) \le u$. Then $\bigvee_n x_n \le u - y_0 = y$ which proves $\bigvee_n x_n$ is also a supremum of $\{x_n\}$ taken in the whole MV-algebra $(N; \oplus, \lambda_1, 0, 1)$.

We note that for each sequence $\{z_n\}$ of elements of M, there is an idempotent $a \in M$ such that $z_n \le a$, so that $z = \bigwedge_n z_n$ exists in M and similarly as for $\bigvee$, we can show that z is also the infimum taken in the whole N.

Case (2), every $x_n = \lambda_1(x_n^0) = u - x_n^0$, where $x_n^0 \in M$ for each $n \ge 1$. Clearly, $\bigwedge_n x_n$ exists in M as well as in $(N; \oplus, \lambda_1, 0, 1)$ and they are the same. Hence, in the unital ℓ-group as well as in the MV-algebra $(N; \oplus, \lambda_1, 0, 1)$, we have $u - \bigwedge_n x_n^0 = \bigvee_n(u - x_n^0) = \bigvee_n x_n \in N$ which says $\bigvee_n x_n$ exists in N.

Case (3), the sequence $\{x_n\}$ can be divided into two sequences $\{y_i\}$ and $\{z_m\}$, where $y_i \in M$, $z_m = \lambda_1(z_m^0)$ with $z_m^0 \in M$ for each n and m. By cases (1) and (2), $y = \bigvee_i y_i$ and $z = \bigvee_m z_m$ are defined in N, so that $y \vee z$ exists in N and clearly, $y \vee z = \bigvee_n x_n$.

Combining (1)–(3), we see that $(N; \oplus, \lambda_1, 0, 1)$ is a σ-complete MV-algebra.

From Theorem 2.1, we conclude M is a maximal ideal of N, and Case (1) says that M is a σ-ideal of N. □

We mention that if M is a σ-complete MV-algebra, then $\mathcal{SM}(M)$ is a basically disconnected space, see [7, Proposition 4.3]. A similar result holds also for σ-complete EMV-algebras as it follows from the following statement.

THEOREM 5.9. *Let M be a σ-complete EMV-algebra. If $\{C_n\}$ is a sequence of compact subsets of $\mathcal{SM}(M)$ such that $A = \bigcup_n C_n$ is open, then the closure of A in the weak topology of state-morphisms on M is open.*

PROOF. If M has a top element, the statement follows from [7, Proposition 4.3]. Thus let M have no top element and let $A = \bigcup_n C_n$ be open, where each C_n is compact. Let N be an EMV-algebra with a top element representing the EMV-algebra given by Theorem 2.1. According to Theorem 4.13, the state-morphism space $\mathcal{SM}(N)$ is the one-point compactification of $\mathcal{SM}(M)$, and the mapping $\phi : \mathcal{SM}(M) \to \mathcal{SM}(N)$ defined by $\phi(s) = \tilde{s}$, $s \in \mathcal{SM}(M)$, given by (2.1), is a continuous embedding of $\mathcal{SM}(M)$ into $\mathcal{SM}(N)$. Then $\mathcal{SM}(N) = \phi(\mathcal{SM}(M)) \cup \{s_\infty\}$. We have $\phi(A) = \bigcup_n \phi(C_n)$. Since $s_\infty \notin \phi(A)$, we see that $\phi(A)$ is open and every $\phi(C_n)$ is closed in the weak topology of state-morphisms on N. Since $(N; \oplus, \lambda_1, 0, 1)$ is by Theorem 5.8 a σ-complete MV-algebra, the state-morphism space $\mathcal{SM}(N)$ is basically disconnected. That is, $\overline{\phi(A)}^N$ is an open set, where $\overline{K}^N$ and $\overline{K}^M$ denote the closure of K in the weak topology on $\mathcal{SM}(N)$ and $\mathcal{SM}(M)$, respectively. If $s_\infty \notin \overline{\phi(A)}^N$, then $\phi^{-1}(\overline{\phi(A)}^N \cap \phi(X)) = \overline{A}^M$, where $X = \mathcal{SM}(M)$, which means that $\overline{A}^M$ is open. If $s_\infty \in \overline{\phi(A)}^N$, then $\overline{\phi(A)}^N = \phi(\overline{A}^M) \cup \{s_\infty\}$, so that $X \backslash \phi^{-1}(\overline{\phi(A)}^N) = X \backslash \overline{A}^M$ is compact, and $\overline{A}^M$ is open. □

Now we present another proof of the Loomis–Sikorski theorem for σ-complete EMV-algebras which is based on Theorem 5.8 and on the Loomis–Sikorski representation of σ-complete MV-algebras, see for example [7, 18]. We note that the proof from Theorem 5.6 gives an interesting and more instructive look into important

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

topological methods which follow from the hull-kernel topology of maximal ideals and the weak topology of state-morphisms than a simple application of the Loomis–Sikorski theorem for σ-complete MV-algebras.

THEOREM 5.10 (Loomis–Sikorski theorem 1). *Let M be a σ-complete EMV-algebra. Then there are an EMV-tribe $\mathcal{T}$ of fuzzy sets on some $\Omega \neq \emptyset$ and a surjective σ-homomorphism h of EMV-algebras from $\mathcal{T}$ onto M.*

PROOF. Let M be a proper σ-complete EMV-algebra. According to Theorem 5.8, M can be embedded into a σ-complete EMV-algebra N with a top element as its maximal ideal which is also σ-complete. Without loss of generality, we can assume that M is an EMV-subalgebra of N, and every element x of N is either from M or $\lambda_1(x)$ is from M. Using Mundici's representation of MV-algebras by unital ℓ-groups, there is a unital Abelian ℓ-group (G, u) such that $N = \Gamma(G, u)$. Hence, if $x \leq a \in \mathcal{I}(M)$, then $\lambda_a(x) = a - x$, where $-$ is the subtraction taken from the ℓ-group G.

By [7, Theorem 5.1], there are a tribe $\mathcal{T}_0$ of fuzzy sets of some set $\Omega \neq \emptyset$ and a σ-homomorphism of MV-algebras h_0 from $\mathcal{T}_0$ onto N. We note that if $\{f_n\}$ is a sequence of functions from $\mathcal{T}_0$ such that there is a characteristic function $a \in \mathcal{T}_0$ with $f_n(\omega) \leq a(\omega)$ for each $\omega \in \Omega$ and each integer n, then

$$\min\Big\{\sum_n f_n(\omega), a(\omega)\Big\} = \min\Big\{\sum_n f_n(\omega), 1\Big\}, \quad \omega \in \Omega.$$

This statement follows the same proof of an analogous equality from the proof of Proposition 5.5. Therefore, $h_0(f \oplus g) = h_0(f) \oplus h_0(g)$. Let $f \in \mathcal{T}_0$ and assume that a is a characteristic function from $\mathcal{T}_0$ such that $f \leq a$. Then $\lambda_a(f) = a - f \in \mathcal{T}_0$ and $a = f + (a - f) = f \oplus (a - f)$ which means $h_0(a) = h_0(f) \oplus h_0(a - f) = h_0(f) + (h_0(a) - h_0(f)) = h_0(f) + \lambda_{h_0(a)}(h_0(f)) = h_0(f) \oplus \lambda_{h_0(a)}(h_0(f))$, where $+$ and $-$ are group addition and subtraction, respectively, taken in the group G. In other words, we have established that h_0 is also a homomorphism of EMV-algebras.

Denote by $\mathcal{T}$ the set of functions $f \in \mathcal{T}_0$ such that (1) there is $x \in M$ with $h_0(f) = x$, and (2) there is a characteristic function $a \in \mathcal{T}_0$ such that $f \leq a$ and $h_0(f) \in \mathcal{I}(M)$. We assert that $\mathcal{T}$ is an EMV-tribe of fuzzy sets. Indeed, if $f, a \in \mathcal{T}$, where $f \leq a$ and a is a characteristic function, then $h_0(f) = x$, $b := h_0(a)$ is an idempotent of M, and $x \leq a$. Then $a - f \in \mathcal{T}_0$ and $a - f \leq a$, and using the fact that h_0 is a homomorphism of EMV-algebras, $a = f + (a - f) = a \oplus (a - f)$ implies $h_0(a - f) = \lambda_{h_0(a)}(f) \in \mathcal{T}$, that is, $h_0(a - f) = \lambda_b(x) \in M$ which means $a - f \in \mathcal{T}$. Clearly $f, g \in \mathcal{T}$ implies $f \oplus g \in \mathcal{T}$, $f \vee g = \max\{f, g\}, f \wedge g = \min\{f, g\} \in \mathcal{T}$, whence, $\mathcal{T}$ is an EMV-tribe.

Now let $\{f_n\}$ be a sequence of functions from $\mathcal{T}$. Since $\mathcal{T}$ is closed under $\vee = \max$, we can assume that $\{f_n\}$ is nondecreasing. For each n, there is a characteristic function $a_n \in \mathcal{T}_0$ such that $f_n \leq a_n$. We can choose $\{a_n\}$ to also be nondecreasing. Assume $h_0(f_n) = x_n \in M$ and $h_0(a_n) = b_n \in \mathcal{I}(M)$. Then $x = \bigvee_n x_n \in M$ and $b = \bigvee_n b_n \in \mathcal{I}(M)$. Define

$$f(\omega) = \lim_n f_n(\omega), \quad a(\omega) = \lim_n a_n(\omega), \quad \omega \in \Omega.$$

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press

Then a is a characteristic function with $f \leq a$, and $h_0(a) = h_0(\bigvee_n a_n) = b$, $h_0(f) = x$ and $f \leq a$, so that $a, f \in \mathcal{T}$.

Now let $\{f_n\}$ be a sequence of arbitrary functions from $\mathcal{T}$ and let each f_n be dominated by a characteristic function $a \in \mathcal{T}$. Then $g_n := f_1 \oplus \cdots \oplus f_n = \min\{f_1 + \cdots + f_n, a\} \in \mathcal{T}$ for each $n \geq 1$, $f = \lim_n g_n \in \mathcal{T}$, and $f = \min\{\sum_n f_n, a\}$. Consequently, $\mathcal{T}$ is an EMV-tribe of fuzzy functions.

Finally, if h is the restriction of h_0 onto $\mathcal{T}$, then h is a σ-homomorphism of EMV-algebras from $\mathcal{T}$ onto M which completes the proof of the theorem. □

6. Conclusion

The main aim of the paper was to formulate and prove a variant of the Loomis–Sikorski theorem for σ-complete EMV-algebras. To do it, we have used some topological methods. The main complication is that an EMV-algebra does not possess a top element, in general. We have introduced the weak topology of state-morphisms and the hull-kernel topology of maximal ideals. We have shown that these spaces are homeomorphic, Theorem 4.8, and they are compact if and only if the EMV-algebra possesses a top element. In general, these spaces are locally compact, completely regular and Hausdorff, Theorem 4.10, and due to Corollary 4.12, they are Baire spaces. Nevertheless if an EMV-algebra M does not possess a top element, due to the basic representation theorem, it can be embedded into an EMV-algebra N with a top element as its maximal ideal and every element of N either belongs to M or is a complement of some element of M. Therefore, the one-point compactification of the state-morphism space is homeomorphic to the state-morphism space of N, a similar result holds for the set of maximal ideals, Theorem 4.13. The main result of the paper is the Loomis–Sikorski theorem for σ-complete EMV-algebras, Theorem 5.6, which says that every σ-complete EMV-algebra is a σ-epimorphic image of some σ-complete EMV-tribe, which is a σ-complete EMV-algebra of fuzzy sets where all EMV-operations are defined by points. We have presented two proofs of the Loomis–Sikorski theorem, see also Theorem 5.10.

The presented paper enriches the class of Łukasiewicz-like algebraic structures where the top element is not assumed.

Acknowledgement

The authors are very indebted to an anonymous referee for his/her careful reading and suggestions which helped to improve the presentation of the paper.

References

[1] G. Barbieri and H. Weber, 'Measures on clans and on MV-algebras', in: *Handbook of Measure Theory,* Vol. II (ed. E. Pap) (Elsevier Science, Amsterdam, 2002), 911–945.

[2] L. P. Belluce, 'Semisimple algebras of infinite valued logic and bold fuzzy set theory', *Can. J. Math.* **38**(6) (1986), 1356–1379.

[3] C. C. Chang, 'Algebraic analysis of many valued logics', *Trans. Amer. Math. Soc.* **88**(2) (1958), 467–490.

[4] R. Cignoli, I. M. L. D'Ottaviano and D. Mundici, *Algebraic Foundations of Many-Valued Reasoning* (Springer Science and Business Media, Dordrecht, 2000).
[5] P. Conrad and M. R. Darnel, 'Generalized Boolean algebras in lattice-ordered groups', *Order* **14**(4) (1997), 295–319.
[6] A. Di Nola and C. Russo, 'The semiring-theoretic approach to MV-algebras', *Fuzzy Sets and Systems* **281** (2015), 134–154.
[7] A. Dvurečenskij, 'Loomis–Sikorski theorem for σ-complete MV-algebras and ℓ-groups', *J. Aust. Math. Soc.* **68**(2) (2000), 261–277.
[8] A. Dvurečenskij, 'Pseudo MV-algebras are intervals in ℓ-groups', *J. Aust. Math. Soc.* **72**(3) (2002), 427–446.
[9] A. Dvurečenskij and S. Pulmannová, *New Trends in Quantum Structures* (Kluwer Academic Publishers, Dordrecht, Ister Science, Bratislava, 2000).
[10] A. Dvurečenskij and O. Zahiri, 'On EMV-algebras', Preprint, 2017, arXiv:1706.00571.
[11] N. Galatos and C. Tsinakis, 'Generalized MV-algebras', *J. Algebra* **283** (2005), 254–291.
[12] G. Georgescu and A. Iorgulescu, 'Pseudo MV-algebras', *Mult.-Valued Logic* **6**(1–2) (2001), 95–135.
[13] K. R. Goodearl, *Partially Ordered Abelian Groups with Interpolation*, Matematical Surveys and Monographs, 20 (American Mathematical Society, Providence, RI, 1986).
[14] J. L. Kelley, *General Topology, Van Nostrand, Toronto 1957*, Graduate Texts in Mathematics (reprinted by Springer, New York, 1975).
[15] L. H. Loomis, 'On the representation of σ-complete Boolean algebras', *Bull. Amer. Math. Soc. (N.S.)* **53**(8) (1947), 757–760.
[16] W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces*, Vol. 1 (North-Holland Publishers, Amsterdam, London, 1971).
[17] D. Mundici, 'Interpretation of AF C^*-algebras in Łukasiewicz sentential calculus', *J. Funct. Anal.* **65**(1) (1986), 15–63.
[18] D. Mundici, 'Tensor products and the Loomis–Sikorski theorem for MV-algebras', *Adv. Appl. Math.* **22**(2) (1999), 227–248.
[19] R. Sikorski, 'On the representation of Boolean algebras as fields of sets', *Fundam. Math.* **35**(1) (1948), 247–258.
[20] M. H. Stone, 'Applications of the theory of Boolean rings to general topology', *Trans. Amer. Math. Soc.* **41**(3) (1937), 375–481.
[21] M. H. Stone, 'Topological representations of distributive lattices and Brouwerian logics', *Časopis pro pěstování matematiky a fysiky* **67**(1) (1938), 1–25.

ANATOLIJ DVUREČENSKIJ, Mathematical Institute,
Slovak Academy of Sciences, Štefánikova 49, SK-814 73 Bratislava,
Slovakia
and
Depart. Algebra Geom., Palacký Univer. 17. listopadu 12,
CZ-771 46 Olomouc, Czech Republic
e-mail: dvurecen@mat.savba.sk

OMID ZAHIRI, University of Applied Science and Technology,
Tehran, Iran
e-mail: zahiri@protonmail.com

https://doi.org/10.1017/S1446788718000101 Published online by Cambridge University Press