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GROUP RINGS WHOSE TORSION UNITS FORM A SUBGROUP*

by SÔNIA P. COELHO and C. POLCINO MILIES

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In this note, we determine fields K and groups G that are either nilpotent or FC and such that the set of torsion elements of the group ring KG forms a subgroup.

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1. Introduction

Let $U(KG)$ denote the group of units of the group ring of a given group G over a field K. Also, we shall denote by $T=T(G)$ and $TU(KG)$ the set of elements of finite order in G and $U(KG)$ respectively.

In this note, we shall consider groups G that are either nilpotent or FC and determine conditions on G and K for $TU(KG)$ to be closed under multiplication, i.e. to be a subgroup of $U(KG)$. This question was first studied in [**4**] but the answer was incomplete because it depended on the fact that every idempotent of KT is central in KG, a condition not fully understood at that time. Using the results in [**1**, **2**], we are able to give a complete answer to this question. In particular, we do not need a technical hypothesis assumed in [**4**, Theorem 4.4] and we correct a gap in [**4**, Theorem 5.2]. In what follows, if a ring R is such that its torsion units form a subgroup, we shall say, briefly, that R has the t.p.p. (*torsion product property*).

2. Group rings in characteristic $p>0$

We remark first that, if G is either a nilpotent or FC group, then T is locally finite and that if $G \neq T$, then G contains a central element of infinite order (see [**5**, 5.2.22 and 14.5.6]). We denote the Jacobson radical of a given ring R by $J(R)$.

Lemma 2.1. *Let G be a group such that $T=T(G)$ is locally finite, and assume that either G contains a central element of infinite order or K is not algebraic over its prime field $\mathcal{P}(K)$. If $TU(KG)$ is a subgroup then, for every finite subgroup $T_1 \subset T$, the quotient ring $KT_1/J(KT_1)$ is a direct sum of fields.*

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Proof. Let x denote be a central element of infinite order in G. Denote by $KT_1[x]$ the smallest subring of KG containing KT_1 and $\{x\}$ and let ϕ: $(KT_1)[x]\to(KT_1/J(KT_1))[x]$ the epimorphism induced by the natural map $KT_1\to KT_1/J(KT_1)$. Since $J(KT_1)$ is nilpotent and x is central, it follows that $Ker(\phi)=J(KT_1)[x]$ is a nilpotent ideal. Hence, ϕ induces, by restriction, epimorphisms of the respective unit groups and also of the respective sets of torsion elements. Then, it is easily seen that $(KT_1/J(KT_1))[x]$ also has the t.p.p.

Since $KT_1/J(KT_1)$ is semisimple artinian, we have that

$$\frac{KT_1}{J(KT_1)}\cong\bigoplus_{i=1}^{t} M_{n_i}(D_i),$$

where D_i is a division ring containing K, $1\leqq i\leqq t$.

For each index i we have:

$$\begin{aligned}(M_{n_i}(D_i))[x]&\cong(M_{n_i}(D_i)\otimes_K K)[x]\\&\cong(D_i\otimes_K M_{n_i}(K)\otimes K)[x]\cong D_i\otimes_K M_{n_i}(K[x]).\end{aligned}$$

Then, also $M_{n_i}(K[x])$ has the t.p.p. It follows from [4, Prop. 2.2] that $n_i=1$. Then:

$$\frac{KT_1}{\mathrm{J}(KT_1)}\cong\bigoplus_{i=1}^{t} D_i.$$

Given any two elements $x,y\in T_1$ we have that $\bar{x},\bar{y}\in\bigoplus_{i=1}^{t}TU(D_i)$ and [4, Prop. 2.1] shows they are central. Hence, $KT_1/J(KT_1)$ is commutative and the result follows.

A similar argument proves the statement in the case where K contains an element x which is transcendental over $\mathscr{P}(K)$. □

Lemma 2.2. *Let K and G be as in the previous lemma. If $TU(KG)$ is a subgroup, then P, the set of p-elements in G, is a normal subgroup of G and $T'\subset P$.*

Proof. Assume that α is a p-element. Then, for some integer $n\geqq 1$ we have that $(\alpha-1)^{p^n}=\alpha^{p^n}-1=0$ i.e. $\alpha-1$ is a nilpotent element. We set $T_1=\langle supp(\alpha)\rangle$. Then T_1 is finite and the image $\overline{\alpha-1}$ in $KT_1/J(KT_1)$ is also nilpotent. Then, Lemma 2.1 shows that $\alpha\in 1+J(KT_1)$.

Given two p-elements $\alpha,\beta\in G$, put $T_1=\langle supp(\alpha), supp(\beta)\rangle$; then we have that $\alpha\beta\in 1+J(KT_1)$, which is a p-group.

Given $x,y\in T$, Lemma 2.1 shows that $K\langle x,y\rangle/J(K\langle x,y\rangle)\cong\oplus_i D_i$, a direct sum of fields; hence, $(x,y)-1=xyx^{-1}y^{-1}-1\in J(K\langle x,y\rangle)$. Thus, there exists an integer $n\geqq 1$ such that $(x,y)^{p^n}=1$. Consequently $T'\subset P$. □

In what follows, we shall denote by $\Delta(G\colon P)$ the kernel of the natural homomorphism $KG\to K(G/P)$.

Theorem 2.3. *Let G be a nilpotent or FC group and let K be a field with $char(K)=p>0$. Then $TU(KG)$ is a subgroup if and only if one of the following conditions holds:*

(*i*) *G is abelian.*

(*ii*) *$G=T$ and K is algebraic over its prime field $\mathscr{P}(K)$.*

(*iii*) *The set P of p-elements in G is a subgroup, $T'\subset P$ and if T/P is non central in G/P then Ω, the algebraic closure of $\mathscr{P}(K)$ in K, is finite and, for all $x\in G$ and all p'-elements $a\in T$, we have that xax^{-1} is of the form $xax^{-1}=a^{p^r}y$, where $r\geqq 0$ and $y\in P$. Furthermore, for every such an exponent r we have that $[\Omega:\mathscr{P}(K)]\,|\,r$.*

Proof. Assume that $TU(KG)$ is a subgroup, that G is not abelian and that either $G\neq T$ or K is not algebraic over $\mathscr{P}(K)$. From Lemma 2.2 we see that P is a subgroup and that $T'\subset P$.

Since $\Delta(G:P)$ is a locally nilpotent ideal, it follows that $K(G/P)$ also has the t.p.p. Since T/P contains no p-elements, [7, Lemma VI.3.12] shows that if there exists a non central idempotent $e\in K(T/P)$, then $U(K(G/P))$ contains a subgroup which is isomorphic to $GL(m,K)$ with $m>1$. If K is not algebraic over $\mathscr{P}(K)$ this yields a contradiction. On the other hand, if $G\neq T$, then [4, Theorem 4.1] shows directly that every idempotent of $K(T/P)$ is central in $K(G/P)$.

In both cases, [1] shows that (*iii*) holds.

To prove sufficiency, we observe that both (*i*) and (*ii*) imply readily that KG has the t.p.p. Thus, assume that (*iii*) holds. Then, [1] shows that every idempotent of $K(T/P)$ is central in $K(G/P)$ and, as in [4, Theorem 4.4] we see that KG has the t.p.p. also in this case. □

3. Group rings in characteristic 0

Our first result holds in a slightly more general setting.

Lemma 3.1. *Let G be a group such that $T(G)$ is locally finite and let K be a field of characteristic 0. If $TU(KG)$ is a subgroup, then T is abelian.*

Proof. To prove our statement, we can assume that T is finite. Then, we can write $KT\cong\bigoplus_{i=1}^{t} M_{n_i}(D_i)$. Since $M_2(\mathbf{Q})$ does not have the t.p.p. (see, for example [6, p. 20]), it follows immediately that $n_i=1$, $1\leqq i\leqq t$.

Thus, $KT\cong\bigoplus_{i=1}^{t} D_i$ contains no nilpotent elements, so [7, Theorem VI.1.11] shows that T_1 is either abelian or a Hamiltonian group. Finally, if T_1 is Hamiltonian, it contains a subgroup of the form

$$\mathscr{Q}=\langle a,b\,|\,a^4=1,a^2=b^2,bab^3=a^3\rangle.$$

Let p be any prime and denote by $\mathbf{Z}_{(p)}$ the localization of $\mathbf{Z}$ at the prime ideal (p). It was shown in [3, Theorem 2] that $\alpha=x+ya$ with $x,y\in\mathbf{Z}$, $p\nmid x, p\,|\,y$, is a unit in $\mathbf{Z}_{(p)}\mathscr{Q}$, and therefore in $\mathbf{Q}G$ and that $b(b^{-1})^{\alpha}=(b,\alpha)$ is not an element of finite order. Hence, T_1 must be abelian. □

We can now correct [4, Theorem 5.2], which should be stated as follows.

Theorem 3.2. *Let G be a nilpotent or FC group and let K be a field of characteristic* 0. *Then, $TU(KG)$ is a subgroup if and only if the following conditions hold:*

(*i*) *T is abelian.*
(*ii*) *For each $t\in T$ and each $x\in G$ there exists a positive integer i such that $xtx^{-1}=t^i$ and, for each non central element $t\in T$, K contains no root of unity of order $o(t)$.*

Proof. Assume that $TU(KG)$ is a subgroup. We know, from the lemma above, that T is abelian.

Also, every idempotent in KT is central in KG, since, if this is not the case, as before, [7, Lemma VI.3.12] shows that $U(KG)$ contains a copy of $GL(m,K)$, with $m>1$. This yields a contradiction, because $M_2(\mathbf{Q})$ does not have the t.p.p.

Now, [2] shows that, for each $t\in T$ and each $x\in G$ we have that $xtx^{-1}=t^i$, as stated, and that for every non central element $t\in T$, K contains no root of unity of order $o(t)$.

To prove sufficiency, notice that we may suppose that G is finitely generated and, therefore, that T is finite. Thus $KT=\bigoplus_{i=1}^{t}K_i$, a direct sum of fields. Let S be a transversal of T in G. Then, we know from [7, Lemma VI.3.22] that every unit $u\in KG$ can be written in the form $u=\sum_i f_i g_i$ where $0\neq f_i\in K_i, g_i\in S$, $1\leqq i\leqq t$.

Since [2] show that the conditions in the statement of our theorem imply that every idempotent of KT is central in KG, we have that $g_i f_i=f'_i g_i$, for some $f'_i\in K$, $1\leqq i\leqq t$. Hence:

$$u^m=\sum_{i=1}^{t}\bar{f}_i g_i^m,$$

where $\bar{f}_i\in K_i$, $1\leqq i\leqq t$. Thus, $u\in TU(KG)$ if and only if there exists an integer m such that $g_i^m=1$, $1\leqq i\leqq t$, i.e. if and only if $u\in U(KT)$. Since T is abelian, it follows easily that KG has the t.p.p. □

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Instituto de Matemática e Estatistíca
Universidade de São Paulo
Caixa Postal 20570—Ag. Iguatemi
01452—990—São Paulo—Brasil