**(vii)** *Theory of functions with improvements and additions to the theory of functions (posthumous).* This is a course on rigorous analysis as far as continuity and differentiability, including Taylor's theorem with remainder term; functions of several variables are also considered. An example is given of a continuous function which neither increases nor decreases on any subinterval of its domain and in consequence is nowhere differentiable; this is different from the famous Weierstrass example. The 'Additions' are in note form and take up some ideas of uniform continuity and the convergence, continuity and differentiability of series of functions.

**(viii)** *Dr Bernard Bolzano's paradoxes of the infinite, edited from the writings of the author by Dr F. Přihonský (1851).* Infinite subsets of infinite sets are discussed and the idea of comparing infinite sets by pairing off their elements is introduced. However, much of this item is devoted to a philosophical discussion of the nature of space and similar ideas.

Bolzano intersperses his mathematics with many interesting comments and he is not slow to point out perceived deficiencies in the proofs or ideas of other mathematicians: Cauchy, Laplace, Lagrange and Euler, among others, come in for some criticism. However, he himself is not infallible: for example, in (vii) (§ 155) he appears to assert that the rule for differentiating a sum of two functions extends trivially to an infinite sum of functions; in the 'Additions' (§ 2) he has realized that this is not so straightforward but I do not find his revised version convincing.

Apart from dealing with the mathematics, the translator is faced with two main problems which are not always reconcilable: producing a text which is acceptable in the new language and which preserves the style and underlying philosophy of the author. Dr Russ explains his approach in his introductory 'Note on the translations' and I am sure that he has done an excellent job in making Bolzano's work available in English. However, I feel that a little paraphrasing could have eased the reading: for example, in my experience, the 'one' construction has never been common in mathematical English, whereas phrases such as *man kann zeigen* abound in German mathematics; I wonder if any harm would be done if this were to be translated as *it can be shown* or even as *we can show* rather than *one can show*; for my taste, the 'one' construction is used excessively in the translations. Another personal quibble is that the words *mere* and *merely*, again not common choices for me at least, seem to be much overworked.

The book will of course be of interest to mathematical historians but anyone involved in teaching rigorous analysis could benefit from studying it, especially item (vii), and Bolzano's construction of the real numbers in (vi) might provide an interesting alternative to the approaches of Dedekind and Cantor.

I. TWEDDLE

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Schechter, M. *An introduction to nonlinear analysis* (Cambridge University Press, 2005), 376 pp., 0 521 84397 9 (hardback), £40.

This book is no ordinary introductory text. It is based on lectures where the author's aim is to quickly introduce some techniques of nonlinear analysis that can be used in a variety of situations. In order not to restrict the course to advanced students only, background material from functional analysis is called on as and when needed but is not presented in the main body of the text. However, for reference purposes this material is listed in some appendices.

The style is not 'theorem, proof, example', but begins by posing a problem to be solved and gradually develops the tools to solve it. The emphasis is on developing methods and showing how they are useful in many nonlinear problems. Problems from differential equations have been used to do this because of their familiarity, and accordingly they do not demand too much preparation.

The first part of the book concentrates on critical point theory. This is motivated by the study of periodic solutions of a second-order ordinary differential equation in the first chapter. The question posed is as follows.

> Can the problem be thought of as some equation $G'(u) = 0$ for some differentiable functional in a suitable Hilbert space?

This leads to discussion of how the derivative of a functional should be defined, the notion of Frechét derivative, and then to determining a suitable space to work in. The classical space $C^1$ is a candidate because of the wish to exploit a natural inner product: but it is not complete, so the space $H$ that is used is its completion relative to the norm defined via the inner product. This leads to working with functions only defined almost everywhere and the concept of a weak derivative. Once the set-up is achieved, several results show that a minimum of $G(u)$ can be found which is a critical point and gives a solution to the problem. The existence of approximate extrema is shown by giving a proof of Ekeland's variational principle. This is used to motivate the important Palais–Smale (PS) condition.

The second chapter deals extensively with the PS condition. Additionally, the search for saddle points rather than minima is begun here with some early versions of the *mountain pass lemma*. Some theorems about global existence of solutions of differential equations are needed, and are established via the contraction mapping principle and some other tools.

Chapter 3 studies the differential equation of Chapter 1 but now with the Dirichlet boundary conditions. It is shown how, by modifying the space, very similar methods apply, though some stronger techniques are needed.

Chapter 4 deals with saddle points for functionals which are convex and lower semi-continuous, while Chapter 5 studies problems that arise naturally as minimization problems (the calculus of variations). This is illustrated by the brachistochrone problem. Hamilton's principle and the Euler equations are derived.

Degree theory is covered in Chapter 6: first the Brouwer degree in finite-dimensional spaces and then the Leray–Schauder degree in infinite dimensions. Following the plan of the text, the properties of degree are used to prove some results required in earlier chapters before going into the nitty-gritty of how the degree is defined. The construction of Brouwer degree follows the well-known plan of first carrying out the construction for $C^1$ functions and then using Sard's theorem (proof included) and approximation techniques to extend to all continuous functions. The properties of degree the author wants to use are derived, but some properties are omitted: for example, odd maps are not discussed here. The Leray–Schauder degree is constructed via approximation by finite-dimensional maps.

Extremal problems in the presence of constraints are met in Chapter 7 with a generalized Lagrange multiplier rule being proved. Minimax techniques are covered in Chapter 8.

Chapter 9 discusses the problem

$$\begin{aligned} -u'' &= bu^+ - au^-, \quad \text{on } (0,\pi), \\ u(0) &= 0 = u(\pi). \end{aligned}$$

Finding solutions of this problem leads to the concept of the *Dancer–Fučik spectrum* and this is then used to discuss the equation $-u'' = f(x,u)$ where

$$f(x,u)/u \to \begin{cases} b & \text{as } u \to \infty, \\ a & \text{as } u \to -\infty. \end{cases}$$

Chapter 10 is a long chapter dealing with functions of several variables and correspondingly with partial differential equations. Much more knowledge is needed to study these problems so only the periodic case is studied. Sobolev spaces for periodic functions are defined via formal series. Sobolev inequalities are proved for this case.

Four appendices complete the text. A large number of results from functional analysis and facts about Lebesgue integration are given in the first two, including a proof of the useful result of measurability of $x \mapsto f(x, v(x))$ when $v$ is a continuous function and $f$ satisfies the Carathéodory conditions. A short third appendix introduces metric spaces and some of their properties, one of which, paracompactness, is used in the fourth appendix to discuss the construction of pseudo-gradients, which made their entry in Chapter 2.

Each chapter ends with a selection of exercises, these sometimes filling gaps deliberately left in some of the proofs.

The topics chosen make a very nice graduate course and an interesting book, adopting a rather different approach from most other texts. As this is a self-contained text it would have been a useful addition to have, at the end of each chapter, some historical comments about how the subject has evolved, and some guide to further reading.

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Cuntz, J., Skandalis, G. and Tsygan, B. *Cyclic homology in non-commutative geometry* (Springer, 2004), 137 pp., 3 540 40469 4 (hardback), £65.50.

Cyclic homology was discovered in the early 1980s independently by Alain Connes and Boris Tsygan. It has since become one of the most important tools in Connes's noncommutative geometry, where it plays the role of (co)homology for noncommutative spaces. The main ingredients of cyclic homology are cyclic-type theories of an algebra $A$: the classical Hochschild homology $H_*(A, A)$, cyclic homology $HC_*(A)$, and the periodic cyclic homology $HP_*(A)$ together with their dual cohomology theories. The Hochschild and cyclic homology are $\mathbb{N}$-graded, whereas the periodic cyclic homology, similarly to topological $K$-theory of $C^*$-algebras, is $\mathbb{Z}/2\mathbb{Z}$-graded.

The basic idea of noncommutative geometry is to treat all algebras as algebras of functions on spaces; the space in question is often not known explicitly, but the properties of the algebra correspond to certain features of the space. For example, in the case of a manifold $M$, properties of $M$, and various analytic and geometric objects associated with it, can be described in terms of a suitable algebra of functions of $M$: measurable, continuous, smooth, holomorphic and so on. One can say that Hochschild and cyclic homology generalize in some sense the differential and integral calculus on $M$. An important theorem of Connes shows that periodic cyclic homology is an extension of de Rham cohomology. In the case of the algebra of smooth functions on a differentiable manifold $M$, where both theories can be defined, we have the following isomorphisms:

$$HP_0(C^\infty(M)) \simeq H^{\mathrm{even}}(M) = \bigoplus_{i=0}^{k} H^{2i}(M),$$
$$HP_1(C^\infty(M)) \simeq H^{\mathrm{odd}}(M) = \bigoplus_{i=0}^{m} H^{2i+1}(M),$$

where $2k$ (respectively, $2m+1$) is the largest even (respectively, odd) integer not greater than the dimension of $M$. The advantage of cyclic theory is that it can be defined for algebras of functions that are not differentiable, whereas the de Rham theory exists in the smooth category.

This basic pattern was studied in a great variety of contexts, and has been extended to crossed product algebras, which are the non-commutative replacement for quotient spaces, groupoids, etc. The power of cyclic cohomology as a computational tool has been greatly augmented with the proof of the excision theorem for periodic cyclic homology and cohomology. This theorem, due to Cuntz and Quillen, states that any algebra extension $0 \to J \to A \to A/J \to 0$ leads to an exact sequence of length six in both periodic cyclic homology and cohomology, which is analogous to an excision sequence in $K$-theory.

The slim volume by Cuntz, Tsygan and Skandalis covers the foundations of the theory and its main results in the past two decades. Following the format of other books in the series it provides only sketches of proofs; however, this allows for a quick tour of the main points of the cyclic homology.

Cuntz's contribution, 'Cyclic theory, bivariant $K$-theory and the bivariant Chern character', provides a survey of the results achieved through a far-reaching development of the initial idea