A CHARACTERIZATION OF THE HYPERHOMOLOGY GROUPS OF THE TENSOR PRODUCT

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1. Introduction. If K and L are chain complexes of abelian groups (to which we restrict ourselves throughout this paper), then $\mathscr{L}(K \otimes L)$ denotes the graded hyperhomology group of K and L, as defined in Cartan and Eilenberg **(1)** by means of free double complex resolutions of K and L. Hyperhomology groups have proved convenient in proving various versions of the Künneth theorem (see, for example, **(4; 1; 2)**). The purpose of this note is to present a description of the hyperhomology group $\mathscr{L}(K \otimes L)$ by means of generators and relations. This characterization, though somewhat complicated, does have the advantage of providing an explicit description of $\mathscr{L}(K \otimes L)$ without reference to free double complex resolutions.

The basic method is to make use of the concept of Bockstein spectra (cf. **(3)**) and an isomorphism theorem from **(4)** in order to construct objects which are naturally isomorphic to the hyperhomology groups and which can be defined in terms of generators and relations. For convenience we repeat here some definitions from **(3; 4)**.

Let Z^+ denote the non-negative integers. A Bockstein spectrum is a collection $\{B_m \mid m \in Z^+\}$, where each B_m is a (graded) abelian group, together with homomorphisms $\lambda_m^{mk}\colon B_{mk} \to B_m$ and $\mu_{mk}^m\colon B_m \to B_{mk}$ (for each $(m, k) \in Z^+ \times Z^+$). The maps λ and μ are required to satisfy certain properties which will not be listed here since they will be apparent in the two particular examples of Bockstein spectra we shall be dealing with.

If $\{B_m\}$ and $\{B'_m\}$ are Bockstein spectra, then (for our purposes) the tensor product of these spectra $\{B_m\} \otimes \{B'_m\}$ is the (graded) abelian group $[\sum_{m \geqslant 0} B_m \otimes B'_m]/S$, where S is the subgroup generated by all elements of the form:

$$(1) \qquad \lambda_m^{mk} U_{mk} \otimes V_m - (-1)^{\deg \mu_{mk}^m \cdot \deg U_{mk}} U_{mk} \otimes \mu_{mk}^m V_m \qquad (mk \geqslant 0),$$

$$(2) \qquad \mu_{mk}^m U_m \otimes V_{mk} - U_m \otimes \lambda_m^{mk} V_{mk} \qquad (mk \geqslant 0),$$

where $U_i \in B_i$ and $V_j \in B'_j$. If $U \otimes V \in B_m \otimes B'_m$ represents an element x of $\{B_m\} \otimes \{B'_m\}$, then the degree of x is $\deg U + \deg V - 1$ if $m > 0$ and $\deg U + \deg V$ if $m = 0$.

Each of the groups Z_m ($m \geqslant 0$; with $Z_0 = Z$) can be considered as a complex (in dimension 0). If K is any complex, we denote by $\mathscr{L}(K, m)$ the (graded) hyperhomology group $\mathscr{L}(K \otimes Z_m)$. Note that $\mathscr{L}(K, 0) = \mathscr{L}(K \otimes Z) = H(K)$. Since $\mathscr{L}(K, m) = H(\bar{K} \otimes Z_m)$ ($\bar{K}$ a free double complex resolution of K),

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the canonical maps $Z \to Z_m$, $Z_{mk} \to Z_m$, and $Z_m \to Z_{mk}$ induce (coefficient) homomorphisms λ_m^{mk}: $\mathscr{L}(K, mk) \to \mathscr{L}(K, m)$ $(mk \geqslant 0)$ and μ_{mk}^m: $\mathscr{L}(K, m) \to \mathscr{L}(K, mk)$ $(mk > 0)$. Since $\bar{K}$ is free, the exact sequence

$$0 \to Z \xrightarrow{m} Z \to Z_m \to 0$$

induces a connecting homomorphism of degree -1: $\mu_0^m: \mathscr{L}(K, m) \to \mathscr{L}(K, 0)$. The hyperhomology spectrum of K is the Bockstein spectrum consisting of $\mathscr{L}(K, m)$ $(m \geqslant 0)$ and the maps λ_m^{mk}, μ_{mk}^m (all $m, k \geqslant 0$). It is denoted by $\{\mathscr{L}(K, m)\}$. The chief result of **(4)**, which we shall use below, is

THEOREM 1.1. *If K and L are complexes, then there is a natural isomorphism of graded groups:*

$$\{\mathscr{L}(K, m)\} \otimes \{\mathscr{L}(L, m)\} \cong \mathscr{L}(K \otimes L).$$

Furthermore, if K and L are differential graded rings, then there is induced a ring structure on $\mathscr{L}(K \otimes L)$ and $\{\mathscr{L}(K, m)\} \otimes \{\mathscr{L}(L, m)\}$ so that the isomorphism is a ring isomorphism.

2. The cone spectrum.

Definition 2.1. If f: $K \to L$ is a chain map, then the *cone of f* is the complex given by:

$$(Cf)_n = K_{n-1} + L_n,$$
$$\partial(u, v) = (-\partial u, \partial v + f(u)).$$

If K is any complex, then multiplication by a positive integer m gives a chain map m: $K \to K$; the cone of this map will be denoted by $C(K, m)$. Define chain maps

$$\lambda_m^{mk}: C(K, mk) \to C(K, m),$$
$$\mu_{mk}^m: C(K, m) \to C(K, mk)$$

as follows:

$$\lambda_m^{mk}(u, v) = (ku, v), \qquad (u, v) \in K_{p-1} + K_p = C(K, mk)_p;$$
$$\mu_{mk}^m(u, v) = (u, kv), \qquad (u, v) \in K_{p-1} + K_p = C(K, m)_p.$$

For each $m > 0$, define a chain map λ_m^0: $K \to C(K, m)$ by

$$\lambda_m^0(u) = (0, u), \qquad u \in K_p; (0, u) \in K_{p-1} + K_p = C(K, m)_p.$$

Also, define a map μ_0^m: $C(K, m) \to K$ by

$$\mu_0^m(u, v) = -u, \qquad (u, v) \in K_{p-1} + K_p = C(K, m)_p.$$

μ_0^m is not a chain map; it has degree -1 and anticommutes with the boundary (i.e. $\partial\mu_0^m = -\mu_0^m\partial$). Each of the maps λ_m^{mk}, μ_{mk}^m $(mk \geqslant 0)$ induces a map on the homology groups $HC(K, mk)$, $HC(K, m)$ of $C(K, mk)$, $C(K, m)$; the induced maps will also be denoted by λ_m^{mk} and μ_{mk}^m. If K is denoted by $C(K, 0)$, then the collection of groups $\{HC(K, m)\}$ and maps λ_m^{mk}, μ_{mk}^m $(m, k \geqslant 0)$ will be called the cone spectrum of K; it is a Bockstein spectrum.

Definition 2.2. If K is a differential graded ring and m a positive integer, then a differential graded ring structure is given in $C(K, m)$ by:

$$(u_1, v_1)(u_2, v_2) = (u_1 v_2 + (-1)^{p_1} v_1 u_2, v_1 v_2),$$

where $(u_i, v_i) \in K_{p_i - 1} + K_{p_i} = C(K, m)_{p_i}$ $(i = 1, 2)$.

A tedious but straightforward calculation shows that this product has all the required properties ($(0, 1_K)$ is the identity).

THEOREM 2.3. *If K is a complex of abelian groups, then there is a natural isomorphism of spectra:*

$$\{\mathscr{L}(K, m)\} \cong \{HC(K, m)\}.$$

If K is a differential graded ring, this is a ring isomorphism.

By an isomorphism of spectra is meant a collection of isomorphisms $\{f_m: \mathscr{L}(K, m) \to HC(K, m)\}$ $(m \geqslant 0)$ such that $\mu_{mk}^m f_m = f_{mk} \mu_{mk}^m$ and $\lambda_m^{mk} f_{mk} = f_m \lambda_m^{mk}$ $(mk \geqslant 0)$.

Proof. By definition, $\mathscr{L}(K, 0) = H(K) = HC(K, 0)$. For each $m > 0$, $\mathscr{L}(K, m) = H(K \otimes X_m)$, where X_m is a free resolution of (the complex) Z_m. In particular, we may choose for X_m the resolution:

$$0 \to Z(c_m) \xrightarrow{\partial} Z(c'_m) \xrightarrow{\zeta} Z_m(1_m) \to 0,$$

where $\partial c_m = mc'_m$ and $\zeta(c'_m) = 1_m$. Note that this is a free resolution of both the group and the complex Z_m. Since $K_p \otimes Z \cong K_p$ for all p, $K \otimes X_m$ is the complex given by

$$(K \otimes X_m)_p = K_{p-1} \otimes Z(c_m) + K_p \otimes Z(c'_m) \cong K_{p-1} + K_p;$$
$$\partial(u, v) = (\partial u, (-1)^{p-1} mu + \partial v); (u, v) \in K_{p-1} + K_p.$$

Define a chain map $\bar{f}_m$: $(K \otimes X_m) \to C(K, m)$ by

$$\bar{f}_m(u, v) = ((-1)^{p-1} u, v), \qquad (u, v) \in K_{p-1} + K_p.$$

It is evident that $\bar{f}_m$ is an isomorphism in each dimension and therefore induces (for each $m > 0$) an isomorphism:

$$f_m: \mathscr{L}(K, m) \cong HC(K, m).$$

Let f_0 be the identity map on $\mathscr{L}(K, 0) = H(K) = HC(K, 0)$.

To show that $f = \{f_m\}$ is in fact a map of spectra we need only show that f commutes with λ and μ. Recall that the maps λ_m^{mk}, μ_{mk}^m $(mk > 0)$ on $\mathscr{L}(K, m)$ were induced by the canonical maps $Z_{mk} \to Z_m$ and $Z_m \to Z_{mk}$. Thus we can take λ_m^{mk} and μ_{mk}^m on $\mathscr{L}(K, m)$ to be the maps induced by the following chain maps:

$$\bar{\lambda}_m^{mk}: X_{mk} \to X_m, \quad \bar{\lambda}(c_{mk}) = kc_m, \quad \bar{\lambda}(c'_{mk}) = c_{mk} \qquad (mk > 0);$$
$$\bar{\lambda}_m^0: Z \to X_m, \quad \bar{\lambda}(1) = c'_m; \quad \bar{\mu}_{mk}^m: X_m \to X_{mk}, \quad \bar{\mu}(c_m) = c_{mk},$$
$$\bar{\mu}(c'_m) = kc'_{mk} \qquad (mk > 0).$$

But the maps induced by these maps on $K_{p-1} + K_p = (K \otimes X_{mk})_p$ and $K_{p-1} + K_p = (K \otimes X_m)_p$ are precisely the maps λ_m^{mk} and μ_{mk}^m defined on $K_{p-1} + K_p = C(K, mk)_p$ and $C(K, m)_p$. It follows immediately that f commutes with λ and μ in these cases; the case of f and λ_m^0 is also easy.

It remains only to show that $f_0 \mu_0^m = \mu_0^m f_m$ (for $m > 0$). On $\mathscr{L}(K, m) = H(K \otimes X_m)$ the map μ_0^m is the composition

$$H(K \otimes X_m) \overset{\cong}{\leftarrow} H(\bar{K} \otimes X_m) \overset{\cong}{\rightarrow} H(\bar{K} \otimes Z_m) \overset{\delta}{\rightarrow} H(\bar{K}) \overset{\cong}{\rightarrow} H(K),$$

where $\bar{K}$ is a free resolution of K and the first and last isomorphisms are induced by the augmentation $\epsilon: \bar{K} \to K$; the second isomorphism is induced by the augmentation $\zeta: X_m \to Z_m$; δ_0^m is the usual connecting homomorphism. If $\eta(x)$ denotes the homology class of an element x, then δ_0^m is given on $\eta(\bar{u} \otimes 1_m) \in H(\bar{K} \otimes Z_m)$ by $\eta((1/m)\, \partial \bar{u}) \in H(\bar{K})$. Let (u, v) be a cycle of $K_{p-1} + K_p = (K \otimes X_m)_p$; recall that (u, v) is the element $u \otimes c_m + v \otimes c'_m$. Hence $\mu_0^m[\eta(u, v)]$ is the homology class of the element

$$\begin{aligned} \epsilon\delta_0^m(1 \otimes \zeta)(\epsilon^{-1} \otimes 1)(u \otimes c_m + v \otimes c'_m) \\ &= \epsilon\delta_0^m(\epsilon^{-1} \otimes \zeta)(u \otimes c_m + v \otimes c'_m) \\ &= \epsilon\delta_0^m(\epsilon^{-1}v \otimes 1_m) = \epsilon((1/m)\partial)(\epsilon^{-1}v) = (1/m)\partial v. \end{aligned}$$

But since (u,v) is a cycle, $(-1)^{p-1}mu + \partial v = 0$, and thus $\mu_0^m[\eta(u,v)] = (-1)^p \eta(u)$. It follows immediately that $f_0 \mu_0^m = \mu_0^m f_m$. Hence f is an isomorphism of spectra.

Now suppose that K is a differential graded ring. We need only show that each map f_m $(m > 0)$ preserves products. Note that the resolution X_m of Z_m is a differential graded ring, if a product is defined by

$$c_m \cdot c_m = 0, \qquad c_m \cdot c'_m = c_m = c'_m . c_m, \qquad c'_m \cdot c'_m = c'_m.$$

A direct calculation now shows that the product in $K \otimes X_m$, is given by

$$(u_1, v_1)\cdot(u_2, v_2) = ((-1)^{p_2} u_1 v_2 + v_1 u_2, v_1 v_2),$$

where $(u_i, v_i) \in K_{p_i - 1} + K_{p_i} = (K \otimes X_m)_{p_i}$ $(i = 1, 2)$. Another easy calculation shows that f_m does in fact preserve the product structure. This completes the proof of the theorem.

Note that the first part of the proof actually proves

Corollary 2.4. *For each complex K and integer $m \geqslant 0$, there is an isomorphism (natural in K):*

$$C(K, m) \cong K \otimes X_m.$$

3. Characterization of $\mathscr{L}(K \otimes L)$. The following theorem is now an immediate corollary of Theorems 1.1 and 2.3.

Theorem 3.1. *If K and L are complexes, then there is a natural isomorphism of graded groups:*

$$\{HC(K, m)\} \otimes \{HC(L, m)\} \cong \mathscr{L}(K \otimes L),$$

which is a ring isomorphism if K and L are differential graded rings.

Since the spectra tensor product is defined essentially in terms of generators and relations, we can now use the isomorphism of Theorem 3.1 to obtain a description of $\mathscr{L}(K \otimes L)$.

If (u', u) is a cycle in $C(K, m)$, then $\partial(u', u) = (-\partial u', \partial u + mu') = 0$; hence $\partial u = -mu'$ and $\partial u' = 0$. This leads us to take for the generators of $\mathscr{L}_p(K \otimes L)$ all symbols of the forms:

(1) $[u', u, v', v]_m$, where $m > 0$, $\partial u = -mu'$, $\partial v = -mv'$, $\partial u' = 0$, $\partial v' = 0$, $\deg u + \deg v = p + 1$;

(2) $[0, u, 0, v]_0$, where $\partial u = 0$, $\partial v = 0$, $\deg u + \deg v = p$.

The symbol $[u', u, v', v]_m$ represents the element

$$\eta(u', u) \otimes \eta(v', v) \in HC(K, m) \otimes HC(L, m)$$

whose coset is an element of $\{HC(K, m)\} \otimes \{HC(L, m)\}$. Similarly, $[0, u, 0, v]_0$ represents

$$\eta(u) \otimes \eta(v) \in H(K) \otimes H(L) = HC(K, 0) \otimes HC(L, 0).$$

The requirements on degrees are necessary because of the way degrees are defined in the spectra tensor product (so that the resulting isomorphisms will be isomorphisms of graded groups).

The following relations are required to hold whenever both sides of the equation are defined (in relations (3)–(7) $m \geqslant 0$; if $m = 0$ it is assumed that $u' = 0$ and $v' = 0$; in relations (8)–(11) $m, k \geqslant 0$):

(3) $[-\partial u', \partial u + mu', v', v]_m = 0$;

(4) $[u', u, -\partial v', \partial v + mv']_m = 0$;

(5) $[u' + \bar{u}', u + \bar{u}, v', v]_m = [u', u, v', v]_m + [\bar{u}', \bar{u}, v', v]_m$;

(6) $[u', u, v' + \bar{v}', v + \bar{v}]_m = [u', u, v', v]_m + [u', u, \bar{v}', \bar{v}]_m$;

(7) $[ru', ru, v', v]_m = [u', u, rv', rv]_m$ for $r \in Z$;

(8) $[ku', u, v', v]_m = [u', u, v', kv]_{mk}$;

(9) $[u', u, kv', v]_m = [u', ku, v', v]_{mk}$;

(10) $[0, u, v', v]_m = [0, u, 0, -v']_0$;

(11) $[u', u, 0, v]_m = [0, -u', 0, v]_0$.

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Relations (3) and (4) merely state that the homology class of a boundary in $C(K, m)$ or $C(L, m)$ is zero. Relations (5)–(7) are those necessary to define the usual tensor product $HC(K, m) \otimes HC(L, m)$; for the spectra tensor product is a quotient group of $\sum_{m \geqslant 0} HC(K, m) \otimes HC(L, m)$. Finally, relations (8)–(11) are those required by the definition of the spectra tensor product.

$\mathscr{L}_p(K \otimes L)$, defined in this way, can be made into a covariant bifunctor: if $f\colon K \to K'$ and $g\colon L \to L'$ are chain maps, define

$$\mathscr{L}(f, g)[u', u, v', v]_m = [f(u'), f(u), g(v'), g(v)]_m.$$

We can summarize these facts in

THEOREM 3.2. *If K and L are complexes, then $\mathscr{L}_p(K \otimes L)$ is naturally isomorphic to the group with generators all symbols of the forms* (1) *and* (2), *subject to the relations* (3)–(11).

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