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ON ISOMORPHISMS OF LATTICES OF CLOSED SUBSPACES

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1. Introduction. By a *projectivity* of vector spaces X and Y over fields F and G is meant an isomorphism $\psi:\mathcal{M}(X) \rightarrow \mathcal{M}(Y)$ of their lattices of subspaces. A basic theorem of projective geomtry [**2**, p. 44] asserts that, for spaces of dimension at least 3, any such projectivity is of the form $\psi(M) = SM$ for a bijection $S:X \rightarrow Y$ which is semi-linear in the sense that S is an additive mapping for which there exists an isomorphism $\sigma:F \rightarrow G$ such that

$$S(\lambda x) = \sigma(\lambda)Sx \quad \text{for all } \lambda \in F \text{ and all } x \in X.$$

In [**12**] Mackey obtained a continuous version of this result: for real normed linear spaces X and Y, the lattices $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ of closed subspaces are isomorphic if and only if X and Y are isomorphic (i.e., via a bicontinuous linear bijection). Furthermore, examination of his proof shows that, except in dimension 2, the bijection may be chosen to induce the lattice isomorphism. He does not consider the case of complex scalars, and indeed essential to the result is the fact that the only automorphism of the real field is the identity. In contrast to this, there are 2^c automorphisms of the complex field; of which, however, only the identity and complex conjugation are continuous [**8**, p. 49]. Thus the corresponding result for complex scalars should assert that any such isomorphism is induced by a bicontinuous linear or conjugate linear bijection. This we obtain as Theorem 1. Note that finite-dimensional spaces have to be excluded because of the existence of discontinuous automorphisms and because, in this case, all subspaces are closed.

Mackey was motivated by the following result of Eidelheit [**4**]: for real Banach spaces X and Y, any isomorphism $\gamma:\mathcal{B}(X) \rightarrow \mathcal{B}(Y)$ of the rings of continuous linear transformations is of the form $\gamma(T) = STS^{-1}$ for some bicontinuous linear bijection S. Eidelheit remarks that, to obtain this result in the complex case, it is necessary to assume explicitly that γ is homogeneous (i.e., an isomorphism of complex algebras). Several years later Arnold [**1**] clarified this situation by showing that for infinite-dimensional complex normed linear spaces, any ring isomorphism of their rings of continuous linear transformations is induced by a bicontinuous linear or conjugate linear bijection. We give in Theorem 2 a simple proof of this result using the methods of Theorem 1.

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We note in passing that a number of investigators have considered versions of Eideheit's theorem for suitable subalgebras; consult Radjavi [**14**] and the references therein. Also McAsey and Muhly [**13**] have described the structure of a large class of isomorphisms of invariant subspace lattices (in the finite-dimensional case).

In the last two sections, we consider lattice automorphisms of $\mathscr{C}(H)$ where H is infinite-dimensional complex Hilbert space. Theorem 1 is applied to show (Theorem 3) that every such automorphism is uniformly, strongly and semi-strongly continuous and preserves the properties of being uniformly (respectively strongly, semi-strongly, weakly) closed. Finally, Theorem 1 is again applied to show (Theorem 4) that each such automorphism preserves reflexivity and transitivity.

2. The lattice theorem. Consider an isomorphism $\phi:\mathscr{C}(X) \to \mathscr{C}(Y)$ of the lattices of closed subspaces of complex normed linear spaces X and Y. This means that ϕ is a bijection such that

$$M \subseteq N \text{ if and only if } \phi(M) \subseteq \phi(N)$$

for all $M, N \in \mathscr{C}(X)$. Our first task is to show that ϕ extends to a projectivity; i.e., an isomorphism of the lattices of all (not necessarily closed) subspaces.

LEMMA 1. *For an isomorphism* $\phi:\mathscr{C}(X) \to \mathscr{C}(Y)$,

$$\psi(M) = \cup \{\phi(L) \mid L \subseteq M \text{ and } \dim L < \infty\}$$

defines an isomorphism $\psi:\mathscr{M}(X) \to \mathscr{M}(Y)$ *which extends* ϕ.

Proof. Let $M \in \mathscr{M}(X)$. Then $\psi(M)$ is a subspace of Y. For if $u, v \in \psi(M)$ then $u \in \phi(K)$ and $v \in \phi(L)$ for some finite-dimensional subspaces $K, L \subseteq M$, so that for any $\lambda \in \mathbf{C}$,

$$\lambda u + v \in \phi(K) + \phi(L) \subseteq \phi(K + L) \subseteq \psi(M).$$

To see that ψ is surjective, let $N \in \mathscr{M}(Y)$ and consider

$$M = \cup \{\phi^{-1}(K) \mid K \subseteq N \text{ and } \dim K < \infty\}.$$

As above we have $M \in \mathscr{M}(X)$. Because this expression for M evidently displays it as the union of its finite-dimensional subspaces, we have

$$\psi(M) = \cup \{\phi(\phi^{-1}(K)) \mid K \subseteq N \text{ and } \dim K < \infty\} = N.$$

Next let $M, N \in \mathscr{M}(X)$. Obviously $M \subseteq N$ implies $\psi(M) \subseteq \psi(N)$. For the converse let L be a finite-dimensional subspace of M. Then

$$\phi(L) \subseteq \psi(M) \subseteq \psi(N),$$

so $\phi(L) = \phi(K)$ for some finite-dimensional subspace K of N, and consequently $L = K \subseteq N$. Therefore $M \subseteq N$ and ψ is an isomorphism.

Finally, to see that ψ extends ϕ, observe that, for $M \in \mathscr{C}(X)$, both $\psi(M)$ and $\phi(M)$ have the same finite-dimensional subspaces, namely the subspaces $\phi(L)$ for L a finite-dimensional subspace of M. Hence $\psi(M) = \phi(M)$.

LEMMA 2. [**9**, Lemma 2, Cor.]. *If $S:X \rightarrow Y$ is a bijective semi-linear transformation of infinite-dimensional complex normed linear spaces that carries closed hyperplanes to closed hyperplanes, then S is either linear or conjugate linear.*

Proof. We give a sketch for the convenience of the reader. The problem is to show that the automorphism ϕ of $\mathbf{C}$, associated with S is continuous, so assume that there is a bounded sequence $\{\lambda_n\}$ of complex numbers such that $\{\sigma(\lambda_n)\}$ is unbounded. To obtain a contradiction, fix an infinite biorthogonal system $\{x_n, f_n\}$ in X. We can suppose that

$$c = \sum \|f_n\| < \infty,$$

and that

$$|\sigma(\lambda_n)| \geqq n\|Sx_n\| \quad \text{for all } n.$$

Construct $f \in X^*$ as follows: for

$$x = \sum_{n=1}^{k} f_n(x)x_n$$

define

$$f(x) = \sum_{n=1}^{k} f_n(x)\lambda_n;$$

because

$$|f(x)| \leqq \sum_{n=1}^{k} \|f_n\| \, \|x\| \, |\lambda_n| \leqq c(\sup |\lambda_n|) \|x\|$$

the Hahn-Banach Theorem provides a continuous linear extension of f to all of X. Fix $\bar{x} \in X$ with $f(\bar{x}) = 1$ and let

$$y_n = x_n - \lambda_n\bar{x},$$

so that $y_n \in \ker f$ for all n. Then

$$S\bar{x} + S\left(\lambda_n^{-1}y_n\right) = \sigma(\lambda_n)^{-1}Sx_n \rightarrow 0$$

so $S\bar{x} \in \overline{S(\ker f)}$ by hypothesis, and therefore $\bar{x} \in \ker f$, a contradiction.

The real linear case of the next lemma is [**12**, Lemma B].

LEMMA 3. *If $S:X \to Y$ is a linear or conjugate linear bijection of complex normed linear spaces such that S and S^{-1} carry closed hyperplanes to closed hyperplanes, then S is bicontinuous.*

Proof. Let $0 \neq g \in Y^*$ be arbitrary. By hypothesis $S^{-1}(\ker g)$ is a closed hyperplane, so we can choose $f \in X^*$ and $u \in X$ such that

$$\ker f = S^{-1}(\ker g) \quad \text{and} \quad f(u) = 1.$$

Then any $x \in X$ can be written in the form

$$x = f(x)u + v \quad \text{for some } v \in \ker f,$$

so we have

$$g(Sx) = g(S(f(x)u)) + g(Sv) = \alpha(f(x))g(Su)$$

where α is the identity or complex conjugation. If follows that $g \circ S$ is bounded and thus that S is bounded, so continuous, because g is arbitrary. Similarly S^{-1} is continuous.

THEOREM 1. *If $\phi:\mathscr{C}(X) \to \mathscr{C}(Y)$ is an isomorphism of the lattices of closed subspaces of infinite-dimensional complex normed linear spaces X and Y, then there exists a bicontinuous linear or conjugate linear bijection $S:X \to Y$ such that $\phi(M) = SM$ for all $M \in \mathscr{C}(X)$.*

Proof. By Lemma 1 we can extend ϕ to an isomorphism

$$\psi:\mathscr{M}(X) \to \mathscr{M}(Y),$$

and then the "First Fundamental Theorem of Projective Geometry" [**2**, p. 44] implies that there is a semi-linear bijection $S:X \to Y$ that induces ψ. In particular S and S^{-1} carry closed hyperplanes to closed hyperplanes, so S is either linear or conjugate linear by Lemma 2, and bicontinuous by Lemma 3.

Remark. Lemma 1 shows that any isomorphism $\mathscr{F}(X) \to \mathscr{F}(Y)$ of the lattices of finite-dimensional subspaces extends to an isomorphism $\mathscr{M}(X) \to \mathscr{M}(Y)$. Which isomorphisms $\rho:\mathscr{F}(X) \to \mathscr{F}(Y)$ extend to isomorphisms $\mathscr{C}(X) \to \mathscr{C}(Y)$? It is not difficult to see that a necessary and sufficient condition is that for all $F \in \mathscr{F}(X)$ and families $\{F_i|i \in I\} \subseteq \mathscr{F}(X)$,

$$F \subseteq \vee \{F_i|i \in I\} \text{ if and only if } \rho(F) \subseteq \vee \{\rho(F_i)|i \in I\};$$

where $\vee$ denotes closed linear span. Hence, for infinite-dimensional complex normed linear spaces, such isomorphisms are induced by bicontinuous linear or conjugate linear bijections.

3. The ring theorem. We emphasize that we do not assume that the isomorphism γ of the following theorem is homogeneous. The first part of the argument that γ is spatial is essentially due to Eidelheit [**4**].

THEOREM 2. [**1**]. *If* $\gamma:\mathscr{B}(X) \to \mathscr{B}(Y)$ *is an isomorphism of the rings of continuous linear transformations on infinite-dimensional complex normed linear spaces* X *and* Y, *then there exists a bicontinuous linear or conjugate linear bijection* $S:X \to Y$ *such that* $\gamma(T) = STS^{-1}$ *for all* $T \in \mathscr{B}(X)$.

Proof. Fix an idempotent $P_0 \in \mathscr{B}(X)$ of rank 1 and non-zero vectors $x_0 \in P_0X$ and $y_0 \in \gamma(P_0)Y$. Then define $S:X \to Y$ by

$$SUx_0 = \gamma(U)y_0, \quad \text{for all } U \in \mathscr{B}(X).$$

It is a matter of straight-forward verification that S is a well-defined and additive bijection. Moreover S induces γ, because for all T, $U \in \mathscr{B}(X)$ we have

$$STUx_0 = \gamma(TU)y_0 = \gamma(T)\gamma(U)y_0 = \gamma(T)SUx_0$$

so that $ST = \gamma(T)S$ or $\gamma(T) = STS^{-1}$. Now the centre of $\mathscr{B}(X)$ is $\{\lambda I_X | \lambda \in \mathbf{C}\}$, so by restricting γ to the centre we obtain an automorphism σ of $\mathbf{C}$ such that

$$\gamma(\lambda I_X) = \sigma(\lambda)I_Y.$$

Consequently

$$\gamma(\lambda T) = \gamma(\lambda I_X)\gamma(T) = \sigma(\lambda)\gamma(T)$$

and S is semi-linear. Finally, S carries closed hyperplanes to closed hyperplanes (and, by symmetry, so does S^{-1}). For if $0 \neq f \in X^*$, select $z_0 \in X$ with $f(z_0) \neq 0$ and define $T \in \mathscr{B}(X)$ by

$$Tx = f(x)z_0.$$

Then $STS^{-1} = \gamma(T) \in \mathscr{B}(Y)$, so

$$\ker STS^{-1} = \ker TS^{-1} = S(\ker T) = S(\ker f)$$

is a closed hyperplane. Lemmas 2 and 3 now imply that S is bicontinuous and either linear or conjugate linear.

4. Continuity of automorphisms. For the rest of the paper we will confine our attention to infinite-dimensional complex Hilbert space H. In this case we have the usual identification of a closed subspace M and the orthogonal projection P_M of H onto M. In particular the uniform, strong, and weak operator topologies induce topologies on $\mathscr{C}(H)$. There has also been recent interest in the semi-strong topology on $\mathscr{C}(H)$ [**5**, **7**]. A sequence $\{M_n\}$ of closed subspaces converges *semi-strongly* to a closed subspace M, denoted $M_n \to M(ss)$, if

$$\liminf_n M_n = M = \limsup_n M_n.$$

Here $\liminf_n M_n$ denotes the set of vectors $x \in H$ with the property that there is a sequence $\{x_n\}$ with $x_n \in M_n$, converging to x in norm, and $\limsup_n M_n$ denotes the set of vectors which are the (norm) limit of some subsequence of such a sequence $\{x_n\}$. A map $\phi:\mathscr{C}(H) \to \mathscr{C}(H)$ is *semi-strongly continuous* if

$$M_n \to M(ss) \text{ implies } \phi(M_n) \to \emptyset(M)(ss).$$

For any bicontinuous linear or conjugate linear bijection S on H, denote by ϕ_S the automorphism of $\mathscr{C}(H)$ defined by

$$\phi_S(M) = SM.$$

In [**10**] and [**11**] it is shown that if S is linear then ϕ_S is uniformly and strongly continuous and preserves the properties of being uniformly (or strongly, or weakly) closed. Theorem 1 suggests that every automorphism of $\mathscr{C}(H)$ has these properties, and this is indeed the case as we now show.

Consider first those automorphisms induced by conjugations (i.e., conjugate linear isometries J such that $J^2 = I$).

LEMMA 4. *Let J be a conjugation on H. The automorphism ϕ_J of $\mathscr{C}(H)$ is uniformly and strongly continuous. If $\mathscr{F}$ is a uniformly* (*respectively*, *strongly*, *weakly*) *closed family of subspaces of H so is $\phi_J(\mathscr{F})$.*

Proof. For every $M \in \mathscr{C}(H)$, $P_{JM} = JP_MJ$ and the uniform and strong continuity of ϕ_J follow immediately. It is readily verified that $\phi_J(\mathscr{F})$ is uniformly (respectively, strongly) closed if $\mathscr{F}$ is. Suppose $\mathscr{F} \subseteq \mathscr{C}(H)$ is weakly closed (as a subset of $\mathscr{B}(H)$). Let the operator $E \in \mathscr{B}(H)$ be the weak limit of a net $\{P_{JM_\alpha}\}$ with $M_\alpha \in \mathscr{F}$ for every α. By the weak compactness of the unit ball of $\mathscr{B}(H)$, the net $\{P_{M_\alpha}\}$ possesses a subnet converging weakly to an operator $F \in \mathscr{B}(H)$. Since $\mathscr{F}$ is weakly closed, $F = P_M$ for some $M \in \mathscr{F}$. This subnet converges strongly to P_M so P_{JM} is the strong limit of a subnet of $\{P_{JM_\alpha}\}$. But this subnet converges weakly to E, so $E = P_{JM}$. Thus $\phi_J(\mathscr{F})$ is weakly closed.

THEOREM 3. *Every automorphism ϕ of $\mathscr{C}(H)$ is uniformly*, *strongly and semi-strongly continuous. If $\mathscr{F} \subseteq \mathscr{C}(H)$ is a uniformly* (*respectively*, *strongly*, *semi-strongly*, *weakly*) *closed family of subspaces so is $\phi(\mathscr{F})$.*

Proof. By Theorem 1, $\phi = \phi_S$ for some bicontinuous linear or conjugate linear bijection S. For any bicontinuous conjugate linear bijection A and any conjugation J, the map $B = AJ$ is a bijective element of $\mathscr{B}(H)$. Since $J^2 = I$, $A = BJ$ so

$$\phi_A = \phi_B \circ \phi_J.$$

With this in mind, the uniform and strong continuity of ϕ_S follow from the preceding lemma and Theorem 1 of [**11**]. Let $M_n \to M(ss)$. It will follow that

$$\phi_S(M_n) \to \phi_S(M)(ss)$$

if we show

(i) $SM \subseteq \liminf_n SM_n$ and

(ii) $\limsup_n SM_n \subseteq SM$.

Consider (i). Let $x \in M$. Since $x \in \liminf_n M_n$, there is a sequence $\{x_n\}$ with $x_n \in M_n$ such that $x_n \to x$ (in norm). Then $Sx_n \to Sx$ so

$$Sx \in \liminf_n SM_n.$$

Consider (ii). Let

$$y \in \limsup_n SM_n.$$

Then there is a sequence $\{y_n\}$ with $y_n \in SM_n$ such that $y_{n_j} \to y$ for some subsequence $\{y_{n_j}\}$. Then

$$S^{-1}y_{n_j} \to S^{-1}y$$

so, since $S^{-1}y_n \in M_n$,

$$S^{-1}y \in \limsup_n M_n = M.$$

Thus $y \in SM$ and the semi-strong continuity of ϕ_S follows.

Let $\mathscr{F} \subseteq \mathscr{C}(H)$. That $\phi_S(\mathscr{F})$ is uniformly (respectively, strongly, weakly) closed if $\mathscr{F}$ is, follows from Theorem 2 of [**11**], the preceding lemma and our above remark concerning the decomposition of conjugate linear maps. A subset $\mathscr{G} \subseteq \mathscr{C}(H)$ is semi-strongly closed (as a subset of 2^H) if and only if $M_n \to M(ss)$, $M_n \in \mathscr{G}$, $M \in \mathscr{C}(H)$ implies $M \in \mathscr{G}$ (see remark p. 294 [7]). Let $\mathscr{F}$ be semi-strongly closed and suppose that $SM_n \to N(ss)$ with $M_n \in \mathscr{F}$ and $N \in \mathscr{C}(H)$. Then $M_n \to S^{-1}N(ss)$ so $S^{-1}N \in \mathscr{F}$ and

$$N = S(S^{-1}N) \in \phi_S(\mathscr{F}).$$

Hence $\phi_S(\mathscr{F})$ is semi-strongly closed and the proof is complete.

Remarks. (1) It is perhaps worth noting that the formula for P_{SM} given in [**10**] extends to the conjugate linear case. That is, if S is a bicontinuous conjugate linear bijection on H and $M \in \mathscr{C}(H)$, then

$$T_M = 1 + SP_M(S^* - S^{-1})$$

is linear and bijective and has inverse

$$1 - P_{SM} + P_{SM}S^{*-1}S^{-1}.$$

Moreover

$$P_{SM} = T_M^{-1} S P_M S^* \quad \text{and} \quad \|T_M^{-1}\| \leqq 1 + \|S^{*-1} S^{-1}\|.$$

Here S^* is the *adjoint* of S, which is defined as the unique conjugate linear map on H such that

$$(S^* y|x) = (Sx|y) \quad \text{for all } x, y \in H.$$

(2) For separable H, the strong continuity of automorphisms of $\mathscr{C}(H)$ can be proved, assuming their semi-strong continuity, as follows. Let S be a bicontinuous linear or conjugate linear bijection on H. Suppose that the sequence $\{M_n\}$ of subspaces converges strongly to the subspace M. By Theorem 1 of [7] the strong convergence of $\{\phi_S(M_n)\}$ to $\phi_S(M)$ will follow if

$$SM_n \rightarrow SM(ss) \quad \text{and} \quad (SM_n)^{\perp} \rightarrow (SM)^{\perp}(ss).$$

Since $M_n \rightarrow M(ss)$, the former follows from the semi-strong continuity of ϕ_S. Since $M_n^{\perp} \rightarrow M^{\perp}(ss)$, the latter follows from the semi-strong continuity of $\phi_{S^{*-1}}$.

5. Reflexivity and transitivity. For any non-empty subset $\mathscr{F} \subseteq \mathscr{C}(H)$, Alg $\mathscr{F}$ denotes the set of operators in $\mathscr{B}(H)$ leaving every member of $\mathscr{F}$ invariant. For any non-empty subset $\mathscr{A} \subseteq \mathscr{B}(H)$, Lat $\mathscr{A}$ denotes the set of subspaces in $\mathscr{C}(H)$ which are left invariant by each member of $\mathscr{A}$. We say that $\mathscr{F}$ is *reflexive* if

Lat Alg $\mathscr{F} = \mathscr{F}$ and that it is *transitive* if Lat Alg $\mathscr{F} = \mathscr{C}(H)$ [**6**].

THEOREM 4. *For every automorphism ϕ of $\mathscr{C}(H)$ and every non-empty subset $\mathscr{F} \subseteq \mathscr{C}(H)$,*

$$\text{Lat Alg } \phi(F) = \phi(\text{Lat Alg } \mathscr{F}).$$

Consequently $\mathscr{F}$ is reflexive (respectively, transitive) if and only if $\phi(F)$ is.

Proof. Let ϕ be an automorphism, so that $\phi = \phi_S$ for some bicontinuous linear or conjugate linear bijection S on H. Let $\mathscr{F} \subseteq \mathscr{C}(H)$ be non-empty. It readily follows that

$$\text{Alg } \phi(\mathscr{F}) = S(\text{Alg } \mathscr{F})S^{-1}$$

and thus that

$$\text{Lat Alg } \phi(\mathscr{F}) = \phi(\text{Lat Alg } \mathscr{F}).$$

From this, $\mathscr{F} =$ Lat Alg $\mathscr{F}$ implies that

$$\phi(\mathscr{F}) = \text{Lat Alg } \phi(\mathscr{F}).$$

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The reverse implication follows by applying ϕ^{-1}. Also,

$$\text{Lat Alg } \mathscr{F} = \mathscr{C}(H)$$

implies that

$$\text{Lat Alg } \phi(\mathscr{F}) = \mathscr{C}(H)$$

and, again, the reverse implication follows by applying ϕ^{-1}.

For an automorphism ϕ of $\mathscr{C}(H)$ it seems natural to call an element $M \in \mathscr{C}(H)$ *invariant under* ϕ if $\phi(M) \subseteq M$, and just as natural to denote the set of invariant subspaces of ϕ by Lat ϕ. In partial order language Lat ϕ is the largest subset of $\mathscr{C}(H)$ on which ϕ is decreasing (cf. [**3**, p. 3]). Lat ϕ is indeed a sublattice of $\mathscr{C}(H)$. (It is even strongly closed, hence complete [**6**, p. 923].) Does every automorphism ϕ have a non-trivial invariant subspace? An affirmative answer would solve the famous invariant subspace problem. Now the result referred to in [**6**, p. 927] that every self-conjugate lattice is transitive may be stated as: For every conjugation J on H, Alg Lat $J = \mathbf{C}I$ or, equivalently, $S \in \mathscr{B}(H)$ and S non-scalar implies $SM \not\subseteq M$ for some $M \in$ Lat J. Since conjugations are characterized by the relations $J^2 = I$ and $J^* = J$ the following result seems to be the conjugate linear version of the result. In the proof, $\langle x \rangle$ denotes the subspace spanned by $x \in H$.

PROPOSITION. *For every symmetry $U \in \mathscr{B}(H)$ such that the ranges of $1 + U$ and $1 - U$ are each of dimension $\neq 1$, and every non-zero continuous conjugate linear map S on H, there exists $M \in$ Lat U such that $SM \not\subseteq M$.*

Proof. Let N be the range of $1 + U$. Then

$$N = \ker(1-U)$$

and the range of $1-u$ is $N^{\perp} = \ker(1 + U)$. Clearly $x \in N \cup N^{\perp}$ implies $\langle x \rangle \in$ Lat U.

Suppose $SM \subseteq M$ for every $M \in$ Lat U, where S is continuous and conjugate linear. The $S\langle x \rangle \subseteq \langle x \rangle$ for every $x \in N$ so, for every such x, there exists a scalar $\lambda(x) \in \mathbf{C}$ such that $Sx = \lambda(x)x$. If dim $N = 0$, then $Sx = 0$ for every $x \in N$. Suppose dim $N > 1$. Then, if $x, y \in N$ are linearly independent,

$$\lambda(x) = \lambda(y)(= \lambda(x + y))$$

and it follows that there is a scalar λ such that $Sx = \lambda x$ for every $x \in N$. Since S is conjugate linear, $\lambda = 0$ (consider $S(ix)$). Thus $Sx = 0$ for every $x \in N$. Similarly $Sy = 0$ for every $y \in N^{\perp}$. It follows that S is zero. This completes the proof.

The condition that the ranges of $1 \pm U$ each be of dimension $\neq 1$

cannot be dropped in the above proposition. For example, let $e \in H$ be a unit vector and define S on H by

$$Sx = (e|x)e, \quad x \in H.$$

Let $U \in \mathscr{B}(H)$ be the symmetry defined by

$$Ux = 2(x|e)e - x, \quad x \in H.$$

$$\text{Lat } U = \{M \in \mathscr{C}(H): e \in M \cup M^{\perp}\}$$

and it is clear that $SM \subseteq M$ for every $M \in \text{Lat } U$.

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