Canad. J. Math. Vol. **54** (4), 2002 pp. 736–756

Chief Factor Sizes in Finitely Generated Varieties

K. A. Kearnes, E. W. Kiss, Á. Szendrei, R. D. Willard

Abstract. Let **A** be a k-element algebra whose chief factor size is c. We show that if **B** is in the variety generated by **A**, then any abelian chief factor of **B** that is not strongly abelian has size at most c^{k-1}. This solves Problem 5 of *The Structure of Finite Algebras*, by D. Hobby and R. McKenzie. We refine this bound to c in the situation where the variety generated by **A** omits type **1**. As a generalization, we bound the size of multitraces of types **1**, **2**, and **3** by extending coordinatization theory. Finally, we exhibit some examples of bad behavior, even in varieties satisfying a congruence identity.

1 Introduction

A *chief factor* of an algebra **A** is a minimal congruence β/α of a factor algebra $\mathbf{A}/\alpha$. The size of this chief factor is the supremum $\#(\beta/\alpha)$ of the cardinalities of its congruence blocks. The *chief factor size* of **A** is the supremum, $\mathrm{c}(\mathbf{A})$, of the sizes of all chief factors of **A**.

In this paper we investigate bounds on the sizes of chief factors of algebras in varieties generated by a finite set of finite algebras. If $\mathcal{V}$ is a variety and $\mathbf{S} \in \mathcal{V}$ is simple, then **S** has only one chief factor and its size is $|S|$. Hence the supremum of the sizes of simple algebras in $\mathcal{V}$ is a lower bound on the supremum of the sizes of chief factors in $\mathcal{V}$. On the other hand, if $\mathbf{B} \in \mathcal{V}$ has a chief factor β/α with $\#(\beta/\alpha) = \kappa$, and γ is a congruence that is maximal for the property of being above α and not above β, then it can be shown that $\mathbf{B}/\gamma$ is subdirectly irreducible with monolith $\mu = (\beta \vee \gamma)/\gamma$ and that $\#(\mu/0) \geq \kappa$. This shows that the largest chief factors in a variety occur as monoliths of subdirectly irreducible algebras, and that any cardinality bound on the sizes of subdirectly irreducible algebras in $\mathcal{V}$ is an upper bound on the sizes of the chief factors in $\mathcal{V}$.

The study of chief factor sizes in universal algebra has been generally related to the investigation of the sizes of simple and subdirectly irreducible algebras, and specifically related to the investigation of the Quackenbush Conjecture. This conjecture from [10] asks if a finitely generated residually finite variety has only finitely many subdirectly irreducible algebras. W. Taylor resolved the Quackenbush Conjecture positively for congruence regular, congruence permutable varieties in [12]. His proof required an analysis of chief factor sizes in finitely generated congruence permutable

Received by the editors December 15, 2000; revised September 7, 2001.
This material is based upon work supported by the National Science Foundation under Grant No. DMS 9802922, by the Hungarian National Foundation for Scientific Research (OTKA), Grant Nos T029525 and T023186, and by the NSERC of Canada (operating grant of R. D. Willard).
AMS subject classification: 08B26.
Keywords: tame congruence theory, chief factor, multitrace.

varieties [12, Lemma 3]. R. Freese and R. McKenzie resolved the Quackenbush Conjecture positively for congruence modular varieties in [2]. Their paper extended earlier results on chief factor sizes [2, Theorems 15 and 16, Corollary 17]. D. Hobby and R. McKenzie resolved the Quackenbush Conjecture positively for varieties that satisfy a nontrivial congruence identity in [4, Chapter 10]. They further extended results on chief factor sizes [4, Chapter 14], and posed the problem which led to this paper [4, Problem 5]. The Quackenbush Conjecture was finally answered negatively in [9].

The theorem of Freese and McKenzie is the model for what one hopes to prove about chief factors in finitely generated varieties:

Theorem 1.1 **([3, Theorem 10.16])** *If* $|A| = k$ *and* $\mathsf{V}(\mathbf{A})$ *is congruence modular, then the chief factor size of any* $\mathbf{B} \in \mathsf{V}(\mathbf{A})$ *is at most* k*.*

It is clear from the proof of this theorem that one can replace the assumption that $|A| = k$ with the assumption that $\mathbf{A}$ is finite and $\mathrm{c}(\mathbf{A}) \leq k$. That is: the biggest chief factor in the variety already occurs in the generating algebra. The theorem remains valid if one replaces the generating algebra $\mathbf{A}$ with a finite set $\{\mathbf{A}_1, \dots, \mathbf{A}_n\}$ of finite algebras with $\mathrm{c}(\mathbf{A}_i) \leq k$ for all i.

Congruence modularity is essential to the proof of Theorem 1.1, and without this assumption it is possible to construct finite algebras that generate varieties containing a proper class of simple algebras (*cf.* [11]). However, one of the results of Hobby and McKenzie is that all big finite simple algebras are of type **5**.

Theorem 1.2 **([4, Theorem 14.5])** *Every finitely generated variety has, up to isomorphism, only a finite set of finite simple algebras of types* **1***,* **2***,* **3***,* **4***.*

This theorem does not bound the sizes of the chief factors in these four types. In fact, it is possible to construct a finite algebra that generates a variety containing algebras with strongly abelian chief factors with no bound on size (see [7, Corollary 10]). Since a chief factor of a finite algebra is strongly abelian if and only if it is of type **1**, the results mentioned show that we cannot expect to bound chief factors whose type is **1** or **5**. On the other hand, tame congruence theory produces the following positive result for types **3** and **4**.

Theorem 1.3 **([4, Lemma 14.4])** *If* $|A| = k$*, then any chief factor of type* **3** *or* **4** *in a finite algebra in* $\mathsf{V}(\mathbf{A})$ *has size at most* k*.*

It is *not* possible to replace the assumption that $|A| = k$ with the assumption that $\mathbf{A}$ is finite and $\mathrm{c}(\mathbf{A}) = k$ in this theorem. (See Example 6.1.) However, the theorem remains valid if one replaces the generating algebra $\mathbf{A}$ with a finite set $\{\mathbf{A}_1, \dots, \mathbf{A}_n\}$ of finite algebras with $|A_i| \leq k$ for all i.

Problem 5 of [4] asks whether there is a finite bound on the sizes of abelian but not strongly abelian chief factors in a finitely generated variety. This is equivalent to asking if there is a finite bound on the sizes of the chief factors of type **2** in the finite members of the variety. The following theorem, one of the main results of the paper, answers this positively. (See Theorem 4.4 for a generalization.)

Theorem 1.4 *Assume that $|A| = k$ and $\mathrm{c}(\mathbf{A}) = c$. If $\mathbf{B} \in \mathsf{V}(\mathbf{A})$, then any abelian but not strongly abelian chief factor of $\mathbf{B}$ has size at most c^{k-1}.*

Problem 1.5 Can the exponential upper bound c^{k-1} of Theorem 1.4 be improved to a polynomial of c and k? (See Example 6.3 for a quadratic lower bound.)

If the variety omits type **1**, then this bound can be improved to c (see Corollary 5.5). This result is proved in a more general form: we bound the sizes of multitraces (see Theorem 5.4). Multitraces were introduced in [5]; they play an important role in the investigation of the Quackenbush Conjecture (see [6]). Lemma 5.6 and Theorem 5.7 establish properties of multitraces of independent interest, which are crucial to our proof of Theorem 5.4.

There is a limit, however, to these improvements to Theorem 1.4. This is demonstrated in Section 6, where counterexamples to some plausible conjectures are exhibited.

Our references for tame congruence theory are the handbook [4] and the introductory paper [8]. We use Section 3 of [5] as a source for some of the information on multitraces in Section 5. Our notation is standard (see [1] and [4]). A boldface lower case letter $\mathbf{x}$ usually denotes a sequence $(x_1, \ldots, x_n)$ of unspecified length. If $p(\mathbf{x}) = t(\mathbf{x}, \mathbf{a})$ is a polynomial on some algebra $\mathbf{A}$, where t is a term and $\mathbf{a}$ is a vector in $\mathbf{A}$, then for a congruence α of $\mathbf{A}$, the notation p/α denotes the polynomial $t(\mathbf{x}, a_1/\alpha, \ldots, a_n/\alpha)$ of $\mathbf{A}/\alpha$.

2 Elementary Observations

To obtain a finite bound on the sizes of abelian-but-not-strongly-abelian chief factors in a finitely generated variety, it suffices to prove a finite bound for the finite algebras in the variety. This observation, which we record, can be proved with arguments from the first paragraph of Lemma 14.1 of [4]

Lemma 2.1 *Let $\mathcal{V}$ be a locally finite variety, and suppose that an algebra $\mathbf{B} \in \mathcal{V}$ has a chief factor β/α such that $\#(\beta/\alpha) \geq n$ for some integer n. Then there exists a finite algebra $\mathbf{C} \in \mathcal{V}$, and a chief factor ν/μ of $\mathbf{C}$ such that $\#(\nu/\mu) \geq n$. The algebra $\mathbf{C}$ can be chosen so that ν/μ is strongly abelian if and only if β/α is, and abelian if and only if β/α is.*

A *congruence quotient* $\langle \alpha, \beta \rangle$ of an algebra $\mathbf{B}$ is a pair of congruences $\alpha < \beta$. Quotients $\langle \alpha, \beta \rangle$ and $\langle \delta, \theta \rangle$ are *perspective* if $\alpha \wedge \theta = \delta$ and $\alpha \vee \theta = \beta$. The fact that they are perspective may be denoted by $\langle \alpha, \beta \rangle \searrow \langle \delta, \theta \rangle$, or $\langle \delta, \theta \rangle \nearrow \langle \alpha, \beta \rangle$. The following lemma is proved in the first paragraph of the text on p. 429 of [2].

Lemma 2.2 *Let $\mathbf{B}$ be a finite algebra.*

(1) *If $\gamma \leq \alpha \prec \beta$, and $\widehat{\alpha} = \alpha/\gamma$ and $\widehat{\beta} = \beta/\gamma$ are the congruences on $\mathbf{B}/\gamma$ corresponding to α and β, then $\#(\beta/\alpha) = \#(\widehat{\beta}/\widehat{\alpha})$.*
(2) *If $\langle \alpha, \beta \rangle \searrow \langle \delta, \theta \rangle$, then $\#(\beta/\alpha) \geq \#(\theta/\delta)$.*
(3) *If $\langle \alpha, \beta \rangle \searrow \langle \delta, \theta \rangle$ and α permutes with θ, then $\#(\beta/\alpha) = \#(\theta/\delta)$.*

Let $\mathcal{K} = \{\mathbf{A}_1, \ldots, \mathbf{A}_m\}$ be a finite set of finite algebras. We intend to prove that there is a finite bound on the sizes of abelian-but-not-strongly-abelian chief factors of algebras in the variety $\mathsf{V}(\mathcal{K})$. Lemma 2.1 shows that it suffices to prove this for the finite members of this variety, and for finite algebras the abelian-but-not-strongly-abelian chief factors are exactly those of type **2** (according to [4, Theorem 4.23]). By Birkhoff's Theorem, a typical finite algebra in $\mathsf{V}(\mathcal{K})$ has the form $\mathbf{B}/\gamma$ for some $\mathbf{B} \in \mathsf{SP}(\mathcal{K})$. Lemma 2.2 (1) proves that the coimage of a chief factor of $\mathbf{B}/\gamma$ is a chief factor in $\mathbf{B}$ of the same size, and the type will be the same according to the way the type is defined at the top of p. 54 of [4]. Thus it suffices for us to bound the chief factors of type **2** in finite algebras in $\mathsf{SP}(\mathcal{K})$. Our approach to this will be to choose an arbitrary finite product $\mathbf{X}$ of algebras in $\mathcal{K}$, and an arbitrary subalgebra $\mathbf{B} \leq \mathbf{X}$ which has a large chief factor β/α of type **2**. Then we choose a congruence $\theta \leq \beta$ that is minimal for $\theta \not\leq \alpha$. Any such θ is join-irreducible. If the lower cover of θ is δ, then $\langle \alpha, \beta \rangle \searrow \langle \delta, \theta \rangle$ and $\mathrm{typ}(\delta, \theta) = \mathrm{typ}(\alpha, \beta) = \mathbf{2}$. The points of $\mathbf{X}$ are separated by homomorphisms into members of $\mathcal{K}$, so at least one of these homomorphisms has kernel not containing θ. We denote one such kernel by $\eta \in \mathrm{Con}(\mathbf{B})$. This situation is pictured in Figure 1.

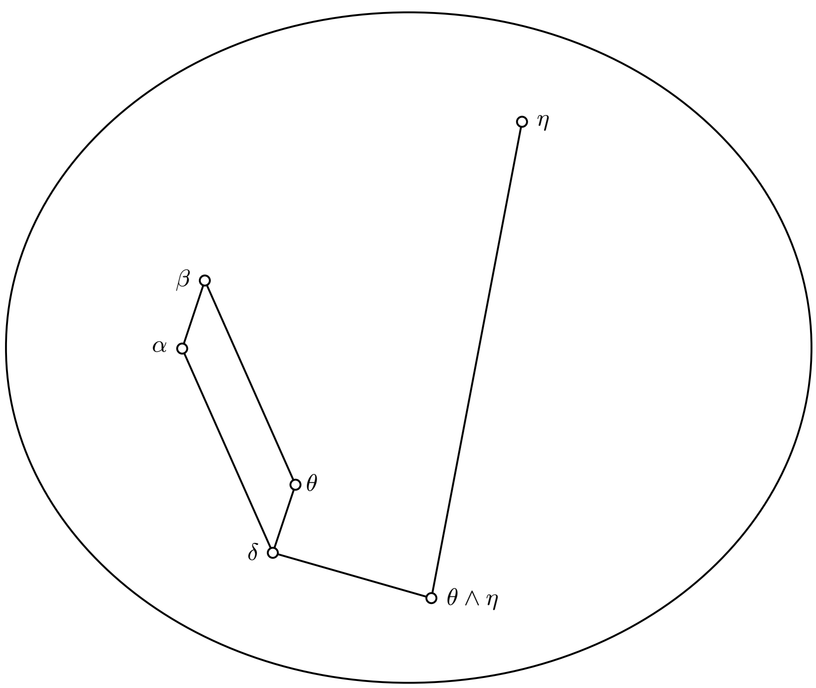

Figure 1: The congruence lattice of a typical **B**.

So, we have a finite algebra $\mathbf{B} \leq \mathbf{X} \in \mathsf{P}_{\mathrm{fin}}(\mathcal{K})$ that has quotients $\langle \alpha, \beta \rangle \searrow \langle \delta, \theta \rangle$ of type **2**, with θ join-irreducible, and a congruence $\eta \not\geq \theta$ that is the kernel of a homomorphism of $\mathbf{B}$ into some $\mathbf{A} \in \mathcal{K}$. Our plan is to compare $\#(\beta/\alpha)$ and

https://doi.org/10.4153/CJM-2002-028-x Published online by Cambridge University Press

$\#(\theta/\delta)$ via the perspectivity $\langle\alpha,\beta\rangle \searrow \langle\delta,\theta\rangle$, and then try to bound $\#(\theta/\delta)$ via the homomorphism of $\mathbf{B}$ into $\mathbf{A}$.

The notation established in the last two paragraphs will be used throughout this section.

Lemma 2.3 *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are finite algebras and* θ *is a join-irreducible congruence of* $\mathbf{B}$ *with lower cover* δ. *If there is a homomorphism* $\pi\colon \mathbf{B} \to \mathbf{A}$ *with kernel* $\eta \not\geq \theta$, *then* $\#(\theta/\delta) \leq \mathrm{c}(\mathbf{A})$.

(It is not assumed here that π is surjective. It would not be sufficient to consider that case only, because it is not true that the chief factor size of an algebra bounds the chief factor sizes of its subalgebras, see Example 6.5.)

Proof If $n = \#(\theta/\delta)$, then it is possible to pick n elements $b_1,\dots,b_n$ that are θ-congruent and pairwise δ-incongruent. Since θ is join-irreducible, each pair (b_k,b_ℓ) with $k \neq \ell$ generates θ. Hence there are Maltsev chains witnessing that $(b_i,b_j) \in \mathrm{Cg}^{\mathbf{b}}(b_k,b_\ell)$ whenever $k \neq \ell$. Let $c_i = \pi(b_i)$ for $i = 1,\dots,n$. Since π maps Maltsev chains of $\mathbf{B}$ to Maltsev chains of $\mathbf{A}$, we get that $(c_i,c_j) \in \mathrm{Cg}^{\mathbf{a}}(c_k,c_\ell)$ whenever $k \neq \ell$. The congruence $\Theta := \mathrm{Cg}^{\mathbf{a}}(c_1,c_2)$ contains all of the pairs (c_i,c_j), and is not the zero congruence since $\theta = \mathrm{Cg}^{\mathbf{b}}(b_1,b_2)$, $\theta \not\leq \eta$ and $\eta = \ker(\pi)$. If Δ is any lower cover of Θ, then Δ contains none of the pairs (c_k,c_ℓ) with $k \neq \ell$, since each of these pairs generate Θ. Thus Θ/Δ has a block containing at least n distinct elements: $c_1/\Delta,\dots,c_n/\Delta$. Hence $\mathrm{c}(\mathbf{A}) \geq \#(\Theta/\Delta) \geq n = \#(\theta/\delta)$. ∎

Recall that an E-trace of an algebra $\mathbf{B}$ with respect to a congruence β is the intersection of the range of an idempotent unary polynomial of $\mathbf{B}$ with a β-block.

Corollary 2.4 *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are finite algebras,* $\langle\alpha,\beta\rangle \searrow \langle\delta,\theta\rangle$ *are perspective prime quotients of* $\mathbf{B}$, *and* θ *is a join-irreducible congruence of* $\mathbf{B}$ *with lower cover* δ. *Assume that* N *is an* E*-trace with respect to* β *such that the induced algebra* $\mathbf{B}|_N$ *has a Maltsev polynomial. If there is a homomorphism* $\pi\colon \mathbf{B} \to \mathbf{A}$ *with kernel* $\eta \not\geq \theta$, *then* $|N/\alpha| \leq \mathrm{c}(\mathbf{A})$.

Proof As N is an E-trace with respect to β, we get $\theta|_N \vee \alpha|_N = \beta|_N = 1_N$. Since $\mathbf{B}|_N$ is Maltsev, Lemma 2.2 (3) implies that $\#(\beta|_N/\alpha|_N) = \#(\theta|_N/\delta|_N)$. By the definition of an E-trace we have $\#(\beta|_N/\alpha|_N) = |N/\alpha|$. Clearly $\#(\theta|_N/\delta|_N) \leq \#(\theta/\delta)$, so by the previous lemma we have $|N/\alpha| = \#(\beta|_N/\alpha|_N) = \#(\theta|_N/\delta|_N) \leq \#(\theta/\delta) \leq \mathrm{c}(\mathbf{A})$. ∎

Corollary 2.5 *Let* $\mathcal{K} = \{\mathbf{A}_1,\dots,\mathbf{A}_m\}$ *be a finite set of finite algebras. In every finite algebra in the variety* $\mathsf{V}(\mathcal{K})$, *every trace of a minimal congruence of type* $\mathbf{2}$ *has size at most* $\max_{1\leq i\leq m} \mathrm{c}(\mathbf{A}_i)$.

Proof Let γ be a minimal congruence of type $\mathbf{2}$ in a finite algebra $\mathbf{C} \in \mathsf{V}(\mathcal{K})$. Then $\mathbf{C} \cong \mathbf{B}/\alpha$, where $\mathbf{B}$ is a subalgebra of some $\mathbf{X} \in \mathsf{P}_{\mathrm{fin}}(\mathcal{K})$ and $\gamma = \beta/\alpha$ for a suitable congruence $\beta \succ \alpha$. By [4, Lemma 2.18], every $\langle 0_C,\gamma\rangle$-trace is of the form N/α, where N is an $\langle \alpha, \beta \rangle$-trace. In particular, N is an E-trace with respect to β. The induced algebra $\mathbf{B}|_N$ is Maltsev by [4, Lemma 4.20].

Choose a prime quotient $\langle \delta, \theta \rangle$ such that $\langle \alpha, \beta \rangle \searrow \langle \delta, \theta \rangle$ and θ is join-irreducible. Since $\mathbf{B} \leq \mathbf{X} \in \mathsf{P}_{\text{fin}}(\mathcal{K})$, there is a homomorphism from $\mathbf{B}$ into some $\mathbf{A}_i \in \mathcal{K}$ with kernel $\eta \ngeq \theta$. Now Corollary 2.4 implies that $\mathrm{c}(\mathbf{A}_i)$ bounds the size of the $\langle 0_C, \gamma \rangle$-trace N/α. ■

This corollary will be extended by Theorem 5.4.

3 Dimension

In this section we analyze linear independence among traces of type **2**, and define the dimension of a simple algebra of type **2**. First we recall, and slightly reformulate Theorem 13.5 of [4].

Let $\mathbf{K}$ be a field, and $n \geq 1$ an integer. The vector space $\mathbf{K}^n$ is made into a module over the full $n \times n$ matrix ring $\mathbf{K}^{n \times n}$ by letting the matrices act via matrix multiplication on the left. The resulting (simple) module will be denoted by $\mathbf{K}_n$.

Theorem 3.1 *Let* $\mathbf{A}$ *be a finite simple algebra of type* **2***, and* N *a trace of* $\mathbf{A}$*. Then the following conditions hold for some integer* $n \geq 1$ *and finite field* $\mathbf{K}$*.*

(1) *There exists a* $\mathbf{K}^{n \times n}$*-module* $\mathbf{M}$ *that is isomorphic to* $\mathbf{K}_n$ *such that:*
   - *A is a subset of M;*
   - *N contains the zero element 0 of* $\mathbf{M}$*;*
   - *every polynomial of* $\mathbf{A}$ *is the restriction of a polynomial of* $\mathbf{M}$*; and*
   - *the set A spans* $\mathbf{M}$ *(considered as a vector space over* $\mathbf{K}$*).*

   We shall identify $\mathbf{M}$ *with* $\mathbf{K}_n$*.*
(2) *N is a one-dimensional subspace of the vector space* $\mathbf{K}^n$*, and the polynomials of* $\mathbf{A}$ *whose range is contained in N are exactly the mappings $r_1x_1 + \cdots + r_mx_m + c$, where $c \in N$ and the r_i are $k \times k$ matrices satisfying $r_iK^n \subseteq N$. In particular, the number of unary polynomials of* $\mathbf{A}$ *mapping into N is $|N|^{n+1}$.*
(3) *Every trace in* $\mathbf{A}$ *is a coset of a one-dimensional subspace of* $\mathbf{K}^n$*.*

Proof The trace N is polynomially equivalent to a one-dimensional vector space over a finite field $\mathbf{K}$. Let 0 denote its zero element, and $+$ its addition. The proof of Theorem 13.5 of [4] proceeds in the following way. The functions from A to N form a vector space under pointwise operations. It is first shown that the subspace of this vector space generated by the idempotent polynomials of $\mathbf{A}$ that fix 0 is equal to the subspace of all polynomials of $\mathbf{A}$ that fix 0. Now a basis $e_1, \dots, e_n$ of $\mathbf{A}$ consisting of idempotent unary polynomials is chosen from this subspace. Then the mapping $\pi(x) = \big(e_1(x), \dots, e_n(x)\big)$ maps A bijectively onto a subset E of N^n. If the operations of $\mathbf{A}$ are transformed over to E using π, then we get restrictions of the polynomials of the matrix power $(\mathbf{A}|_N)^{[n]}$ to E. From the fact that the e_i are linearly independent it follows that the subspace of N^n generated by E is all of N^n.

https://doi.org/10.4153/CJM-2002-028-x Published online by Cambridge University Press

The algebra $(\mathbf{A}|_N)^{[n]}$ is polynomially equivalent to the module $\mathbf{K}_n$. One can obviously choose this equivalence so that $\pi(0) = (0, \ldots, 0)$ becomes the zero element of this module. Then identifying every $a \in A$ with $\pi(a)$ we see that statement (1) of the theorem has been established.

The set N corresponds to $\{(a, \ldots, a) \in K^n \mid a \in K\}$, so it is a one-dimensional subspace of $\mathbf{K}^n$. The fact that the space of unary polynomials of $\mathbf{A}$ that map into N and fix 0 has a basis $\{e_1, \ldots, e_n\}$ of n elements shows that the number of such polynomials is $|K|^n$. On the other hand, every unary polynomial of $\mathbf{A}$ that fixes 0 is the restriction of a module polynomial that fixes 0. That is, it has the form rx where r is an $n \times n$ matrix satisfying $rK^n \subseteq N$. Such matrices are determined by their action on n basis vectors, so the number of these matrices is at most $|N|^n = |K|^n$. We have already established that we have this many polynomials mapping into N and fixing 0, so rx is a polynomial of $\mathbf{A}$ for every matrix r satisfying $rK^n \subseteq N$. The abelian group operations on N are given by polynomials, so all module polynomials mapping into N are polynomials of $\mathbf{A}$. This gives us the statement in (2).

Every trace M of $\mathbf{A}$ is a polynomial image of N, and thus has the form $rN + c$ for some matrix r and $c \in A$. As N is a one-dimensional subspace, so is rN, and so $rN+c$ is a coset, proving (3). ■

The construction of the module $\mathbf{K}_n$ from $\mathbf{A}$ in this proof depended on the choice of a trace N and of a basis $\{e_1, \ldots, e_n\}$. In the rest of this section we will describe some features of the module $\mathbf{K}_n$ which reflect intrinsic properties of $\mathbf{A}$. The notation of the previous proof is used in the next definition.

Definition 3.2 If M is a trace of $\mathbf{A}$, then a *direction* of M is a nonzero vector in $\mathbf{K}^n$ from the subspace of which M is a coset (*cf.* Theorem 3.1 (3)). A set of traces is *linearly independent* if the set of their directions is linearly independent in $\mathbf{K}^n$. The *dimension* of $\mathbf{A}$ is the dimension n of $\mathbf{K}^n$.

The dimension of $\mathbf{A}$ is an intrinsic property of the algebra since $n = \dim(\mathbf{K}^n)$ occurs in Theorem 3.1 (2) in the count of unary polynomials mapping $\mathbf{A}$ into N. This count is an invariant of $\mathbf{A}$ since all traces are polynomially isomorphic. While the notion of "direction" is not intrinsic to $\mathbf{A}$, part (1) of the next lemma proves that linear independence is. Thus, it always makes sense to refer to directions being the same or different. Part (3) of the next lemma suggests an alternate definition of dimension.

Lemma 3.3 *Let* $\mathbf{A}$, N, $\mathbf{K}^n$ *and* $\mathbf{K}_n$ *be as in Theorem 3.1.*

(1) *A set* $\{N_1, \ldots, N_m\}$ *of traces is linearly independent if and only if there exist* m *unary polynomials* f_i *of* $\mathbf{A}$ *mapping* A *to* N *such that* f_i *maps* N_i *onto* N *and is constant on* N_j *when* $j \neq i$.
(2) *If* $N_1, \ldots, N_m$ *are linearly independent, then there exists a unary polynomial* f *of* $\mathbf{A}$ *that maps* A *to* N, *and maps every* N_i *onto* N.
(3) *Every maximal set of linearly independent traces in* $\mathbf{A}$ *has* n *members.*

Proof We still consider $\mathbf{A}$ embedded into $\mathbf{K}_n$. To prove (1), pick elements $c_i \neq d_i$ from N_i. Then these traces are linearly independent if and only if so are the vectors

$b_i = c_i - d_i$. If these vectors are independent, then (although they are not necessarily in A), we can find linear transformations f_j of $\mathbf{K}^n$ whose range is contained in N, and map exactly one of the vectors b_i to a nonzero element. By Theorem 3.1 (2), these are unary polynomials of $\mathbf{A}$, and they clearly satisfy the conditions. Conversely, the existence of such polynomials f_j implies that the vectors b_i are linearly independent. Thus (1) is proved. A similar argument proves (2), one has to choose a linear map that maps each b_i to a nonzero element of N.

To prove (3), pick a set $N_1, \dots, N_m$ of linearly independent traces with $m < n$. Let W be the (m-dimensional) subspace of $\mathbf{K}^n$ generated by the directions of these traces. Since A spans $\mathbf{K}^n$, which properly contains W, there is an element $b \in A - W$. Pick a matrix r mapping $\mathbf{K}^n$ to N such that $rW = \{0\}$ but $rb \neq 0$. The mapping r is a unary polynomial of $\mathbf{A}$, which maps 0 to 0, and b to a nonzero element. It collapses every trace N_i to 0. Connect 0 to b by a chain of traces. Then at least one member of this chain, named M, will not be collapsed to 0 by r. This means that the direction of M is not contained in W, and hence the family $N_1, \dots, N_m, M$ is still independent. ■

We close this section by pointing out that if γ is a minimal congruence of type **2** on a finite algebra $\mathbf{C}$, then the induced algebra on every γ-block is simple of type **2**, and so the concept of independence makes sense for any family of $\langle 0_C, \gamma\rangle$ traces that lie in the same γ-block.

4 Cycles

We are now in the position to prove Theorem 1.4. We shall investigate the following situation. Let $\mathbf{B}$ be a finite algebra, $\langle\alpha, \beta\rangle$ a prime quotient of type **2** of $\mathbf{B}$, and $N_1, \dots, N_n$ a family of $\langle\alpha, \beta\rangle$-traces in the same β-block. Assume that the family N_i/α is linearly independent.

Lemma 4.1 *There exist idempotent unary polynomials $h_1, \dots, h_n$ of $\mathbf{B}$ such that N_i is contained in the range of h_i, the identities $h_1 h_i = h_1$ and $h_i h_1 = h_i$ hold for all i, and $h_1(N_i) = N_1$ and $h_i(N_1) = N_i$ for all i. There also exists an idempotent unary polynomial e of $\mathbf{B}$, which contains N_1 in its range, and which collapses every other N_i into an α-block in N_1.*

Proof Let $N = N_1/\alpha$, and apply Lemma 3.3 (2) to the induced algebra on the β/α-block containing all N_i/α. The lemma shows that there exists a unary polynomial f of $\mathbf{B}/\alpha$ whose range on this β/α-block is N, and maps every N_i/α bijectively onto N.

Let U_i be an $\langle\alpha, \beta\rangle$-minimal set in which N_i is a trace, and let k be an idempotent polynomial of $\mathbf{B}$ whose range is U_1. Let g be a unary polynomial of $\mathbf{B}$ such that $g/\alpha = f$. Then kg maps each N_i into a trace within U_1 (since it does not collapse N_i into an α-block), and the fact that N_i and N_1 are in the same β-block implies that kg maps each N_i bijectively onto N_1. Thus kg is a permutation of U_1. Let h be an idempotent power of kg. Then h is the identity map on U_1, and still maps each N_i onto N_1.

By Theorem 2.8 of [4], there exist unary polynomials k_i of $\mathbf{B}$ for each $i \geq 2$ such that $k_i(B) = U_i$, and k_i is an inverse of $h\colon U_i \to U_1$, that is, $k_i h$ is the identity map

https://doi.org/10.4153/CJM-2002-028-x Published online by Cambridge University Press

on U_i, and hk_i is the identity map on U_1. Letting $h_1 = h$, and h_i be an idempotent power of $k_i h$ for $i \geq 2$, we get the desired polynomials.

To construct e, we apply Lemma 3.3 (1). That is, we pick the polynomial f so that it is the identity map on N, but collapses each N_i/α, $i \neq 1$, to a point within N. The same lifting argument yields now a polynomial kg whose idempotent power e works. ■

The following observation shows that linearly independent traces cannot form a nontrivial cycle in any factor.

Lemma 4.2 *Let η be a congruence of $\mathbf{B}$. Suppose that there exist $c_i, d_i \in N_i$ such that $d_i\ \eta\ c_{i+1}$ for every $1 \leq i \leq n$ (with the understanding that $c_{n+1} = c_1$). Then $c_i\ \eta\ d_i$ for every i.*

Proof If the lemma is not true, then there is a nonzero congruence θ that is minimal among the congruences of $\mathbf{B}$ for the property that $(c_i, d_i)\ in \theta \vee \eta$ for every i. If δ is a lower cover of θ, then the minimality of θ implies that $(c_i, d_i) \notin \delta \vee \eta$ for some i which we may assume to be $i = 1$. We plan to use the fact that the induced algebra on N_1 is E-minimal (see [4, Theorem 4.31]), hence every unary polynomial of $\mathbf{B}$ that maps N_1 to N_1 is either a permutation of N_1, or collapses $\theta|_{N_1}$ to $\delta|_{N_1}$.

By Lemma 4.1, there is a unary polynomial e of $\mathbf{B}$ such that e is the identity map on N_1, but collapses every other N_i into a proper subset of N_1. Consider also the unary polynomials h_i provided by this lemma. Then eh_i is a unary polynomial of $\mathbf{B}$ that collapses N_1 into a proper subset. Thus, by the previous remark, eh_i collapses $\theta|_{N_1}$ to $\delta|_{N_1}$. Therefore $e|_{N_i} = eh_i h_1|_{N_i}$ collapses $\theta|_{N_i}$ to $\delta|_{N_1}$. From $(c_i, d_i) \in (\theta \vee \eta)|_{N_i} = \theta|_{N_i} \vee \eta|_{N_i}$ we get that $\big(e(c_i), e(d_i)\big) \in \delta|_{N_1} \vee \eta|_{N_1}$. On the other hand, $d_i\ \eta\ c_{i+1}$ implies $e(d_i)\ \eta|_{N_1}\ e(c_{i+1})$. Thus

$$e(d_1)\ \eta|_{N_1}\ e(c_2)\ (\delta|_{N_1} \vee \eta|_{N_1})\ e(d_2)\ \eta|_{N_1}\ e(c_3) \cdots e(c_n)\ (\delta|_{N_1} \vee \eta|_{N_1})\ e(d_n)\ \eta|_{N_1}\ e(c_1)$$

shows that $\big(e(d_1), e(c_1)\big) = (d_1, c_1) \in \delta \vee \eta$, which is a contradiction. ■

Lemma 4.3 *Let $\mathbf{B}$ be a finite algebra, $\langle \alpha, \beta \rangle \searrow \langle \delta, \theta \rangle$ be prime quotients of type $\mathbf{2}$ with θ join-irreducible, and η a congruence of $\mathbf{B}$ satisfying $\eta \ngeq \theta$. Then the dimension of the induced algebra on any β/α-block is at most $|\mathbf{B}/\eta| - 1$.*

Proof Let $k = |\mathbf{B}/\eta|$. Choose an arbitrary linearly independent set $\{N_1, \ldots, N_m\}$ of $\langle \alpha, \beta \rangle$-traces from the same β-block. By Lemma 3.3 (3), it is sufficient to show that $m \leq k - 1$.

The facts that $\eta \ngeq \theta$, θ is generated by the square of any $\langle \delta, \theta \rangle$-trace, and $\langle \alpha, \beta \rangle$-traces contain $\langle \delta, \theta \rangle$-traces, jointly imply that $\eta|_{N_i}$ has at least two blocks for every i. We use this to define a graph. The vertices are the different η-blocks $V_1, \ldots, V_k$. For every $1 \leq i \leq m$ we let E_i be an edge that connects two different (arbitrarily chosen) η-blocks that intersect N_i. This produces a graph of k vertices and m edges that has no loops (by definition) and no cycles or multiple edges (by Lemma 4.2). Hence this graph is a forest, and so the number m of its edges is at most $k - 1$. ■

Theorem 4.4 *Let $\mathcal{K} = \{\mathbf{A}_1, \ldots, \mathbf{A}_m\}$ be a finite set of finite algebras. The sizes of abelian but not strongly abelian chief factors of algebras in the variety $\mathsf{V}(\mathcal{K})$ are at most $c = \max_{1 \le i \le m} \mathrm{c}(\mathbf{A}_i)^{|A_i|-1}$.*

Proof As noted earlier, it suffices to show that $\#(\beta/\alpha) \le c$ whenever $\langle \alpha, \beta \rangle$ is a prime quotient of type **2** of some $\mathbf{B} \le \mathbf{X} \in \mathsf{P}_{\text{fin}}(\mathcal{K})$. To show that this is true, choose a join-irreducible congruence θ with lower cover δ such that $\langle \alpha, \beta \rangle \searrow \langle \delta, \theta \rangle$, and choose a congruence $\eta \not\geq \theta$ that is the kernel of a homomorphism of $\mathbf{B}$ into some $\mathbf{A}_i \in \mathcal{K}$.

Let C be any β/α-block. Theorem 3.1 proves that the simple algebra $(\mathbf{B}/\alpha)|_C$ is embeddable into a module, and so C may be identified with a subset of N^n for some $\langle 0, \beta/\alpha \rangle$-trace N and some number n. Corollary 2.4 proves that $|N| \le \mathrm{c}(\mathbf{A}_i)$ while Lemma 4.3 proves that we may take $n \le |B/\eta| - 1 \le |A_i| - 1$. Hence $|C| \le c$. ■

5 Multitraces

The purpose of this section is to prove the refinement of Theorem 1.4 mentioned in the Introduction: the sizes of abelian chief factors in a finitely generated variety that omits type **1** does not exceed the maximum chief factor size in the generating algebras. The arguments require us to recall definitions from [5].

Definition 5.1 Let $\mathbf{A}$ be a finite algebra, $\langle \alpha, \beta \rangle$ a prime quotient of $\mathbf{A}$, and N an $\langle \alpha, \beta \rangle$-trace. For any polynomial f of $\mathbf{A}$, a set of the form $M = f(N, \ldots, N)$ is called an $\langle \alpha, \beta \rangle$-*multitrace* of $\mathbf{A}$.

Definition 5.2 Let $\mathbf{A}$ be a finite algebra containing subsets N and S, and let $g_1, \ldots, g_n \in \mathrm{Pol}_1(\mathbf{A})$, and $f \in \mathrm{Pol}_n(\mathbf{A})$. If properties (i)–(iv) below hold, then we shall say that S is *coordinatizable (of rank n) by N*. If properties (i)–(iii) hold, then we shall say that S is *weakly coordinatizable (of rank n) by N*.

(i) $f(N, \ldots, N) \subseteq S$;
(ii) $g_i(S) \subseteq N$ for $1 \le i \le n$;
(iii) $f\big(g_1(x), \ldots, g_n(x)\big) = x$ for all $x \in S$.
(iv) $g_i f(x_1, \ldots, x_n) = x_i$ for every $x_1, \ldots, x_n \in N$ and $1 \le i \le n$.

The four conditions of this definition mean the following: (i) says that the polynomial f restricts to a function from N^n to S; (ii) says that the sequence of polynomials $\widehat{g} = (g_1, \ldots, g_n)$ may be considered to be a function from S to N^n; (iii) says that $f \circ \widehat{g}$ is the identity function on S; while (iv) says that $\widehat{g} \circ f$ is the identity function on N^n. Thus S is coordinatizable by N precisely when the polynomials f, g_i describe a bijection between N^n and S.

Lemma 5.3 *When S and N are finite, then S is coordinatizable of rank n by N if and only if S is weakly coordinatizable of rank n by N and $|N|^n \le |S|$.*

Proof Clearly rank n coordinatization of S by N implies rank n weak coordinatization and $|S| = |N|^n$. Conversely, if S is weakly coordinatizable by N of rank n, then there are polynomials $f\colon N^n \to S$ and $\widehat{g}\colon S \to N^n$ such that $f \circ \widehat{g} = \mathrm{id}_S$. This equation forces f to be a surjective function from the finite set N^n to the set S, which is not a smaller set. Hence f and $\widehat{g}$ are inverse bijections between S and N^n, establishing that f and $\widehat{g}$ coordinatize S by N. ■

The basic properties of (weak) coordinatization are explored in [5, Lemma 3.6]. It is shown in Lemmas 3.8 and 3.9 of [5] that $\langle 0, \gamma\rangle$-multitraces of types **1** and **2** are coordinatizable with respect to a $\langle 0, \gamma\rangle$-trace, while in Lemma 3.11 of [5] it is shown that $\langle 0, \gamma\rangle$-multitraces of type **3** are weakly coordinatizable with respect to a $\langle 0, \gamma\rangle$-trace.

The main result of this section is the following.

Theorem 5.4 *Let $\mathcal{K} = \{\mathbf{A}_1, \dots, \mathbf{A}_m\}$ be a set of finite algebras, $c = \max_{1\le i\le m} \mathrm{c}(\mathbf{A}_i)$, and $k = \max_{1\le i\le m} |A_i|$. Let $\mathbf{C} \in \mathsf{V}(\mathcal{K})$ be a finite algebra, and M a multitrace for a minimal congruence γ on $\mathbf{C}$.*

(1) *If the type of $\langle 0_C, \gamma\rangle$ is **2** or **3**, then $|M| \le c$.*
(2) *Suppose that the type of $\langle 0_C, \gamma\rangle$ is **1**. Then the rank of M is at most $\log_2(c)$, and $|M| \le (k!)^{\log_2(c)}$.*

This theorem yields the refinement of Theorem 1.4 that we promised in the Introduction.

Corollary 5.5 *Let $\mathcal{K} = \{\mathbf{A}_1, \dots, \mathbf{A}_m\}$ be a finite set of finite algebras, and $c = \max_{1\le i\le m} \mathrm{c}(\mathbf{A}_i)$. If $\mathsf{V}(\mathcal{K})$ omits type **1**, then every abelian chief factor in this variety has size at most c.*

Proof By Lemma 2.1, it suffices to prove that any abelian chief factor in a finite member of $\mathsf{V}(\mathcal{K})$ has size at most c. Let $\mathbf{C}$ be a finite algebra in $\mathsf{V}(\mathcal{K})$ and let γ be a minimal congruence of $\mathbf{C}$ of type **2**. Theorem 9.6 of [4] shows that the γ-blocks are Maltsev, hence they are multitraces by [6, Theorem 4.5]. Thus Theorem 5.4 shows that the size of each γ-block is at most c. ■

We prove Theorem 5.4 by showing that coordinatization and weak coordinatization can be lifted from a factor.

Lemma 5.6 *Let $\mathbf{B}$ be a finite algebra, $\alpha < \beta$ congruences of $\mathbf{B}$, and N an E-trace of $\mathbf{B}$ with respect to β. Suppose that the algebra $\mathbf{B}/\alpha$ has a subset M that is weakly coordinatizable of rank n with respect to N/α. Then there exists a set $S \subseteq A$ that is weakly coordinatizable of rank n with respect to N such that $S/\alpha = M$.*

Proof The fact that M is weakly coordinatizable of rank n with respect to N/α means that there exist unary polynomials g_i of $\mathbf{B}$, and an n-ary polynomial f of $\mathbf{B}$ such that

(1) $(f/\alpha)(N/\alpha, \ldots, N/\alpha) \subseteq M$;
(2) $(g_i/\alpha)(M) \subseteq N/\alpha$ for $1 \leq i \leq n$;
(3) $f\big(g_1(x), \ldots, g_n(x)\big)\ \alpha\ x$ for every $x \in \mathbf{B}$ with $x/\alpha \in M$.

Hence $f/\alpha\colon (N/\alpha)^n \to M$ and $\widehat{g}/\alpha = (g_1/\alpha, \ldots, g_n/\alpha)\colon M \to (N/\alpha)^n$ are functions such that $(f/\alpha) \circ (\widehat{g}/\alpha) = \mathrm{id}_M$. This shows that f/α is a surjective function from $(N/\alpha)^n$ onto M, hence that $f(N, \ldots, N)/\alpha = M$. So, if S_0 is the set of all elements s for which $s/\alpha \in M$, then $f(N, \ldots, N) \subseteq S_0$. It is clear from the definition of S_0 that it is a union of α-blocks, and it is clear from the the fact that f/α maps onto M that each $\alpha|_{S_0}$-block is represented by some element of $f(N, \ldots, N)$. Hence, the fact that the set N is contained within a single β-block implies that the same is true for $f(N, \ldots, N)$, and hence also for S_0. Of course, $S_0/\alpha = f(N, \ldots, N)/\alpha = M$.

We shall modify the polynomials g_i and f. First, as N is an E-trace with respect to β, there exists an idempotent unary polynomial whose range intersects the β-block containing N in N. Prefix each g_i with this idempotent unary polynomial, but retain the notation g_i for this polynomial. Then the conditions in (1)–(3) still hold, but we now have that $g_i(S_0) \subseteq N$ for every i.

Now let $g(x) = f\big(g_1(x), \ldots, g_n(x)\big)$, and choose an integer $k \geq 2$ such that $e = g^k$ is idempotent. In the factor modulo α, the function g is the identity map on M. Hence if we replace $f(\mathbf{x})$ by $g^{2k-1}f(\mathbf{x})$, then the conditions in (1)–(3) will still hold, but for the new f the polynomial $e(x) = f\big(g_1(x), \ldots, g_n(x)\big)$ is idempotent. We also have the identity $ef(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$.

Let $S = e(S_0)$. The facts that S_0 is a union of α-blocks, $S_0/\alpha = M$, and $(e/\alpha)(x) = x$ on M, imply that $S \subseteq S_0$. Moreover, we have $S/\alpha = e(S_0)/\alpha = (e/\alpha)(S_0/\alpha) = (e/\alpha)(M) = M$. From the fact that e is idempotent, we get that $f\big(g_1(s), \ldots, g_n(s)\big) = e(s) = s$ for every $s \in S$, so condition (iii) of Definition 5.2 holds. Since $S \subseteq S_0$ and $g_i(S_0) \subseteq N$ (as we showed above), we get that $g_i(S) \subseteq N$. This verifies condition (ii) of Definition 5.2. We have already shown that $f(N, \ldots, N) \subseteq S_0$, $ef = f$, and $e(S_0) = S$, and these imply that $f(N, \ldots, N) \subseteq S$. This verifies that condition (i) of Definition 5.2 holds. ■

Theorem 5.7 *Let* $\mathbf{B}$ *be a finite algebra,* $\alpha < \beta$ *congruences of* $\mathbf{B}$*, and* N *an* E*-trace of* $\mathbf{B}$ *with respect to* β *such that the induced algebra* $\mathbf{B}|_N$ *is* $\langle \alpha|_N, \beta|_N \rangle = \langle \alpha|_N, 1_N \rangle$*-minimal. Suppose that the algebra* $\mathbf{B}/\alpha$ *has a subset* M *that is coordinatizable of rank* n *with respect to* N/α*. Then there exists a set* $S \subseteq A$ *that is coordinatizable of rank* n *with respect to* N *such that* $S/\alpha = M$*.*

Proof The fact that M is coordinatizable of rank n with respect to N/α means that there exist unary polynomials g_i of $\mathbf{B}$, and an n-ary polynomial f of $\mathbf{B}$ such that conditions (1)–(3) of the proof of the previous lemma hold along with

(4) $g_i f(x_1, \ldots, x_n)\ \alpha\ x_i$ for every $x_1, \ldots, x_n \in N$ and $1 \leq i \leq n$.

The modifications that we made to g_i and f in the previous proof did not change them modulo α, and therefore, doing the same modifications, we can assume that the polynomials g_i and f satisfy (i)–(iii) of Definition 5.2 together with (4).

https://doi.org/10.4153/CJM-2002-028-x Published online by Cambridge University Press

The previous proof provides us with a set S such that $S/\alpha = M$ and S is weakly coordinatizable of rank n by N via the polynomials f and g_i. Under the additional hypotheses of this theorem S is actually coordinatized by N with these polynomials. To show this it suffices (by Lemma 5.3) to prove that $|N|^n \leq |S|$. Our method for showing that $|N|^n \leq |S|$ will be to construct a right inverse to the function

$$\widehat{g} = (g_1, \dots, g_n) \colon S \to N^n.$$

That is, we will construct an n-ary polynomial $h(\mathbf{x})$ satisfying $h(N^n) \subseteq S$ for which $g_i h(\mathbf{x}) = x_i$ on N for $1 \leq i \leq n$.

Before starting the construction we make a remark that will be used below. Suppose that t is an n-ary polynomial such that $t(N, \dots, N) \subseteq N$, and we have $t(x, \mathbf{y}) \ \alpha \ x$ for every $x \in N$ and $\mathbf{y} \in N^{n-1}$. Then using the fact that N is $\langle \alpha|_N, 1_N \rangle$-minimal, we see that t is a permutation in its first variable (and this permutation is the identity map modulo α). Therefore if we iterate t in its first variable in the usual manner (see [4, Lemma 4.4]), we get the identity map on N for every choice of $\mathbf{y}$. Doing one less iteration than this, we get a new polynomial $t^-(x, \mathbf{y})$ such that $t\big(t^-(x, \mathbf{y}), \mathbf{y}\big) = x = t^-\big(t(x, \mathbf{y}), \mathbf{y}\big)$ is an identity on N. We shall call this polynomial t^- the inverse of t in its first variable.

We define auxiliary polynomials in order to arrive at the definition of h. First set

$$e_i(\mathbf{x}) \stackrel{\text{def}}{=} g_i f(\mathbf{x}).$$

From $f(N^n) \subseteq S$ and $g_i(S) \subseteq N$ we get that $e_i(N^n) \subseteq N$. As $f\big(g_1(s), \dots, g_n(s)\big) = s$ holds for every $s \in S$, the fact that $f(N^n) \subseteq S$ implies that the identity

$$e_i\big(e_1(\mathbf{x}), \dots, e_n(\mathbf{x})\big) = e_i(\mathbf{x})$$

holds on N for every $1 \leq i \leq n$. From (4) above we get that $e_i(\mathbf{x}) \ \alpha \ x_i$ holds for every $\mathbf{x} \in N^n$ and $1 \leq i \leq n$.

Next we define n-ary polynomials $t_{j,i}$ $(1 \leq i, j \leq n)$, and f_j $(1 \leq j \leq n)$ by induction on j. Set

$$t_{1,i}(\mathbf{x}) \stackrel{\text{def}}{=} x_i$$

for $1 \leq i \leq n$. It will be clear from the inductive definition we give that $t_{j,i}(\mathbf{x}) \ \alpha \ x_i$ holds on N for all i, j. Thus it will follow that $e_j\big(t_{j,1}(\mathbf{x}), \dots, t_{j,n}(\mathbf{x})\big) \ \alpha \ x_j$ for each j, and so if the polynomials $t_{j,i}$ are already defined for some j and all i we can let f_j be the inverse of

$$e_j\big(t_{j,1}(\mathbf{x}), \dots, t_{j,n}(\mathbf{x})\big)$$

in its j-th variable. Since $e_j(N^n) \subseteq N$, and by induction we will see that $t_{j,i}(N^n) \subseteq N$, it follows that $f_j(N^n) \subseteq N$. Now, still assuming that the polynomials $t_{j,i}$ are already defined for this j and all i, set

$$t_{j+1,i}(\mathbf{x}) \stackrel{\text{def}}{=} t_{j,i}\big(x_1, \dots, x_{j-1}, f_j(\mathbf{x}), x_{j+1}, \dots, x_n\big).$$

Next we recursively define polynomials h_i, $1 \le i \le n$. Let

$$h_n(\mathbf{x}) \stackrel{\text{def}}{=} f_n(\mathbf{x}),$$

and if $h_n, h_{n-1}, \ldots, h_{j+1}$ are already defined, then set

$$h_j(\mathbf{x}) \stackrel{\text{def}}{=} f_j\big(x_1, \ldots, x_j, h_{j+1}(\mathbf{x}), h_{j+2}(\mathbf{x}), \ldots, h_n(\mathbf{x})\big).$$

It is clear by induction that $t_{j,i}$, f_i, and h_i are all equal to the i-th projection modulo α on N, and all map N^n into N. Finally, let

$$h(\mathbf{x}) \stackrel{\text{def}}{=} f\big(h_1(\mathbf{x}), \ldots, h_n(\mathbf{x})\big).$$

Since f is on the outside and all h_i map N^n into N, it is clear that $h(N^n) \subseteq S$. To finish the proof we only have to verify that $g_j h(\mathbf{x}) = x_j$ for $1 \le j \le n$. From the definitions of h and e_i, this is exactly the same as showing that

$$e_j\big(h_1(\mathbf{x}), \ldots, h_n(\mathbf{x})\big) = x_j$$

holds on N for every $1 \le j \le n$.

Claim 5.8 *For all i, j we have $t_{j,i}\big(x_1, \ldots, x_{j-1}, h_j(\mathbf{x}), \ldots, h_n(\mathbf{x})\big) = h_i(\mathbf{x})$.*

From $t_{1,i}(\mathbf{x}) = x_i$ one can prove by induction on j that $t_{j,i}(\mathbf{x}) = x_i$ if $j \le i$. Therefore the claim is true when $j \le i$. If the claim holds for some j, then using the definition of $t_{j+1,i}$ we get that

$$\begin{aligned} &t_{j+1,i}\big(x_1, \ldots, x_j, h_{j+1}(\mathbf{x}), \ldots, h_n(\mathbf{x})\big) \\ &\qquad = t_{j,i}\Big(x_1, \ldots, x_{j-1}, f_j\big(x_1, \ldots, x_j, h_{j+1}(\mathbf{x}), \ldots, h_n(\mathbf{x})\big), h_{j+1}(\mathbf{x}), \ldots, h_n(\mathbf{x})\Big). \end{aligned}$$

But the definition of h_j shows that

$$f_j\big(x_1, \ldots, x_j, h_{j+1}(\mathbf{x}), \ldots, h_n(\mathbf{x})\big) = h_j(\mathbf{x}),$$

and so by the induction hypothesis, the result is indeed $h_i(\mathbf{x})$.

We now apply this claim to complete the proof. Since $f_j(\mathbf{x})$ is the inverse of the polynomial $e_j\big(t_{j,1}(\mathbf{x}), \ldots, t_{j,n}(\mathbf{x})\big)$ in its j-th variable (on N), substituting $f_j(\mathbf{x})$ for x_j into this polynomial produces a polynomial that is projection onto the j-th variable on N. Do this substitution. Now replace each x_i with $h_i(\mathbf{x})$ for every $i > j$. This has no further effect, because our polynomial was already independent of these other variables on N. Thus we have

$$e_j\big(u_{j,1}(\mathbf{x}), \ldots, u_{j,n}(\mathbf{x})\big) = x_j$$

on N, where $u_{j,i}(\mathbf{x})$ equals

$$t_{j,i}\Big(x_1,\dots,x_{j-1}, f_j\big(x_1,\dots,x_{j-1},x_j,h_{j+1}(\mathbf{x}),\dots,h_n(\mathbf{x})\big), h_{j+1}(\mathbf{x}),\dots,h_n(\mathbf{x})\Big).$$

By definition, the inner $f_j\big(x_1,\dots,x_{j-1},x_j,h_{j+1}(\mathbf{x}),\dots,h_n(\mathbf{x})\big)$ equals $h_j(\mathbf{x})$, so $u_{j,i}(\mathbf{x})$ actually equals

$$t_{j,i}\big(x_1,\dots,x_{j-1},h_j(\mathbf{x}),h_{j+1}(\mathbf{x}),\dots,h_n(\mathbf{x})\big).$$

By Claim 5.8 this is just $h_i(\mathbf{x})$. Thus the first displayed line of this paragraph reduces to $e_j\big(h_1(\mathbf{x}),\dots,h_n(\mathbf{x})\big) = x_j$, which we had to show to complete the proof. ■

Now we prove Theorem 5.4. Let $\mathcal{K} = \{\mathbf{A}_1,\dots,\mathbf{A}_m\}$ be a finite set of finite algebras, $c = \max_{1\le i\le m} \mathrm{c}(\mathbf{A}_i)$, and $k = \max_{1\le i\le m} |A_i|$. Suppose that $\mathbf{C} \in \mathsf{V}(\mathcal{K})$ is a finite algebra, and M is a multitrace for a minimal congruence γ on $\mathbf{C}$. Then $\mathbf{C} \cong \mathbf{B}/\alpha$, where $\mathbf{B} \le \mathbf{X} \in \mathsf{P}_{\text{fin}}(\mathcal{K})$, and $\gamma = \beta/\alpha$ for a suitable congruence $\beta \succ \alpha$.

If the type of $\langle\alpha,\beta\rangle$ is **2** or **3**, then by Theorem 3.10 or Lemma 3.11 of [5], M is weakly coordinatizable by an N/α where N is an $\langle\alpha,\beta\rangle$-trace. Hence Lemma 5.6 can be applied to M. We get a set S that is weakly coordinatizable by N and which satisfies $S/\alpha = M$. Since the $\langle\alpha,\beta\rangle$-traces in types **2** or **3** are Maltsev, Lemma 3.6 of [5] implies that $\mathbf{B}|_S$ is Maltsev. Now Corollary 2.4 proves that $|M| = |S/\alpha| \le \mathrm{c}(\mathbf{A}_i)$ for some $\mathbf{A}_i \in \mathcal{K}$.

Now assume that the type of $\langle\alpha,\beta\rangle$ is **1**. By [5, Theorem 3.10], M is coordinatizable by N/α where N is an $\langle\alpha,\beta\rangle$-trace. The argument proving Theorem 14.7 of [4] shows that the trace N/α has size $\le k!$ (the details are left to the reader). To use this to bound the size of M we need to determine how large the rank n of M can be with respect to N/α. Theorem 5.7 yields a set S satisfying $S/\alpha = M$ that is coordinatizable of the same rank n with respect to N. By [5, Corollary 3.7], the induced algebra on S is polynomially equivalent to the full matrix power $(\mathbf{B}|_N)^{[n]}$.

Consider the projection kernels on $\mathbf{X}$ resulting from the fact that $\mathbf{B} \le \mathbf{X} \in \mathsf{P}_{\text{fin}}(\mathcal{K})$. Their intersection is zero, so there is one of them, called η, that does not contain $S\times S$. Let $\theta \ge \eta$ be a congruence on $\mathbf{X}$ that is maximal for not containing $S \times S$. Then $\mathbf{X}/\theta$ is an algebra which contains a set S/θ of size > 1 that supports (matrix) polynomials expressing the fact that this set is the n-th matrix power of a smaller set. It follows that $|S/\theta| = |N/\theta|^n \ge 2^n$. But it also follows from the choice of θ that any minimal congruence of $\mathbf{X}/\theta$ contains S/θ in a block. Thus, some minimal congruence of $\mathbf{X}/\theta$ is a chief factor of size at least 2^n. Since $\mathbf{X}/\theta$ is a quotient of $\mathbf{X}/\eta$, which is isomorphic to some $\mathbf{A}_i \in \mathcal{K}$, it follows from Lemma 2.2 that $2^n \le \mathrm{c}(\mathbf{A}_i)$, or $n \le \log_2\big(\mathrm{c}(\mathbf{A}_i)\big)$. Hence $|M| = |N/\alpha|^n \le (k!)^{\log_2(c)}$. ■

6 Examples

6.1 Example

Let $\mathcal{V}$ be a variety generated by finitely many finite algebras, and let c be the maximum of the chief factor sizes of these generators. By Corollary 5.5, if $\mathcal{V}$ omits type **1**, then the chief factors of type **2** in $\mathcal{V}$ have sizes bounded by c. If $\mathcal{V}$ is congruence modular, then all chief factors are bounded by c. We present an example of a variety satisfying a congruence identity (in fact, omitting types **1**, **2**, and **5**, which is stronger), where the bound c does not bound the chief factor sizes. (They are still bounded by the size of the biggest generating algebra by the refined version of Theorem 1.3.)

Our variety is $\mathcal{V} = \mathsf{V}(\{\mathbf{A}, \mathbf{B}\})$ where $\mathbf{A}$ has underlying set $A = \{0, 1\}$ and the basic operations of $\mathbf{A}$ are the following Boolean operations:

$$d(x, y, z) = x \wedge (y \oplus z)', \quad e(x, y, z) = x \wedge y \wedge z,$$
$$f(x) = 0, \quad g(x) = 1, \quad h(x) = k(x) = x.$$

Here $\vee$ is join, $\wedge$ is meet, prime is complementation, and $\oplus$ is symmetric difference. The underlying set of $\mathbf{B}$ is $B = \{0, 1, 2, 3\}$. The basic operations of $\mathbf{B}$ are defined with reference to the lattice operations of the order $0 < 1 < 2 < 3$. Let $d(x, y, z) = x$, let $e(x, y, z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$ be the median operation, and define the operations f, g, h, k by the following table.

x	$f(x)$	$g(x)$	$h(x)$	$k(x)$
0	0	1	3	1
1	1	1	2	0
2	0	2	0	0
3	1	2	1	0

Both $\mathrm{c}(\mathbf{A})$ and $\mathrm{c}(\mathbf{B})$ equal 2. However, in the algebra $\mathbf{A} \times \mathbf{B}$ there is a chief factor of type **3** that has size 3. This can be seen in Figure 2 below that illustrates the congruence lattice of $\mathbf{A} \times \mathbf{B}$. If we throw out the operation k, then the only difference is that the type of this chief factor of size 3 changes from **3** to **4** (but the arguments below will still work).

We now justify the claim that $\mathcal{V}$ omits types **1**, **2** and **5**. The argument depends on a modified version of Theorem 9.11 of [4], which asserts that a locally finite variety omits types **1**, **2** and **5** if and only if it satisfies an idempotent *linear* Maltsev condition that fails in the variety of semilattices and in every nontrivial variety of vector spaces over a finite field. The modification of this statement that we need is:

Claim 6.1.1 *A locally finite variety omits types* **1**, **2** *and* **5** *if and only if it satisfies an idempotent Maltsev condition that fails in the variety of semilattices and in every nontrivial variety of modules over a finite ring.*

The claim differs from Theorem 9.11 of [4] in that linearity of the Maltsev condition is not assumed, at the expense of considering varieties of modules rather than vector spaces.

To prove Claim 6.1.1, note that if $\mathcal{V}$ omits types **1**, **2** and **5**, then by Theorem 9.11 of [4] is satisfies an idempotent linear Maltsev condition that fails in the variety of semilattices and in every nontrivial variety of vector spaces over a finite field. This Maltsev condition must also fail in every nontrivial variety of modules over a finite

https://doi.org/10.4153/CJM-2002-028-x Published online by Cambridge University Press

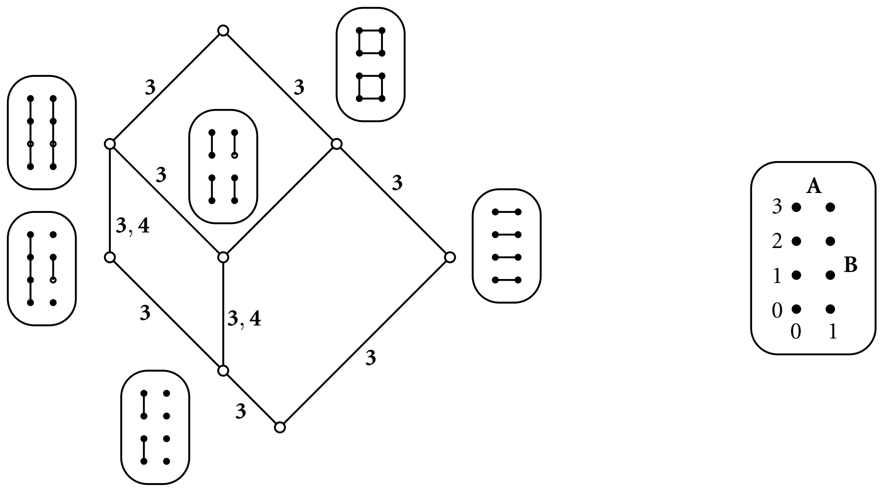

Figure 2: The congruence lattice of $\mathbf{A} \times \mathbf{B}$.

ring. (The indirect argument for this is that if the Maltsev condition forces the omission of type **2**, then it must fail in every locally finite variety of modules since modules have type **2**. The direct argument is supplied by Lemma 9.2 of [4].)

Now suppose that $\mathcal{V}$ is a locally finite variety that satisfies an idempotent Maltsev condition that fails in the variety of semilattices and in every nontrivial variety of modules over a finite ring. By [4, Lemma 9.5], $\mathcal{V}$ satisfies a linear idempotent Maltsev condition that fails in the variety of semilattices, so $\mathcal{V}$ omits types **1** and **5** according to [4, Theorem 9.8]. But if $\mathcal{V}$ omits type **1**, then Theorem 7.12 of [4] implies that the restrictions of idempotent operations of any $\mathbf{A} \in \mathcal{V}$ to a block of a minimal abelian congruence are idempotent module operations. Thus, since $\mathcal{V}$ omits type **1** and satisfies an idempotent Maltsev condition that fails in every variety of modules, we get that no algebra in $\mathcal{V}$ can have a nontrivial abelian congruence. Hence $\mathcal{V}$ omits type **2**. This proves Claim 6.1.1.

Now we apply Claim 6.1.1 to show that $\mathcal{V}$ omits types **1**, **2** and **5**. A sufficient list of equations for this is

(1) $d(x, y, y) = x = e(x, x, x)$,
(2) $d(x, x, y) = d(x, y, x) = e(x, x, y)$,
(3) $e(x, y, z) = e(x, z, y) = e(y, z, x)$, and
(4) $e\big(e(x, y, z), y, z\big) = e(x, y, z)$.

To model these equations in a nontrivial module, the fact that e is idempotent (equation (1)) and totally symmetric (equation (3)) forces $e(x, y, z) = bx+by+bz$ for some ring element b satisfying $3b = 1$. Using this representation in equation (4) leads to $b^2 = b = 0$. Hence $1 = 3b = 0$, forcing triviality of the module. To model these equations in a 2-element semilattice, note that the total symmetry of e (equation (3)) forces $e(x, y, z) = x \wedge y \wedge z$. By equation (1), d cannot depend on its second two variables in the 2-element semilattice, so it must be that $d(x, y, z) = x$. Now equation (2) fails. Thus, $\mathcal{V}$ omits types **1**, **2**, and **5**.

6.2 Example

In Corollary 5.5 we showed that if $\mathcal{V}$ is finitely generated and omits type **1**, then the maximum of the chief factor sizes of the generating algebras bounds the sizes of the chief factors of type **2** in the variety. It is natural to ask whether one can bound the sizes of the chief factors of type **2** in the variety by the sizes of the chief factors of type **2** in the generators. The answer is no, as this example shows. We describe algebras with no chief factors of type **2** that generate a variety omitting types **1** and **5**, but a chief factor of type **2** appears in the variety.

We again define two algebras **A** and **B**, this time both have underlying set $\{0, 1\}$. Define the operations in the following way on **A**:

$$d_1(x, y, z) = x, \quad d_2(x, y, z) = x, \quad p(x, y, z) = x \oplus y \oplus z,$$
$$e_0(x, y, z) = z, \quad e_1(x, y, z) = z, \quad g(x, y) = x \wedge y,$$

and in the following way on **B**:

$$d_1(x, y, z) = x \vee (y' \wedge z), \quad d_2(x, y, z) = x \vee z, \quad p(x, y, z) = x \vee y \vee z,$$
$$e_0(x, y, z) = x \vee z, \quad e_1(x, y, z) = (x \wedge y') \vee z, \quad g(x, y) = 1.$$

Then **A** and **B** are simple algebras of type **3**. The variety they generate omits types **1** and **5**, as one can verify by checking that, with $d_0(x, y, z) = x$ and $e_2(x, y, z) = z$, the operations $d_0, d_1, d_2, e_0, e_1, e_2, p$ satisfy the equations listed in [4, Theorem 9.8]. The congruence lattice of $\mathbf{A} \times \mathbf{B}$, pictured in Figure 3, is isomorphic to $\mathbf{N}_5$, and its critical quotient has type **2**.

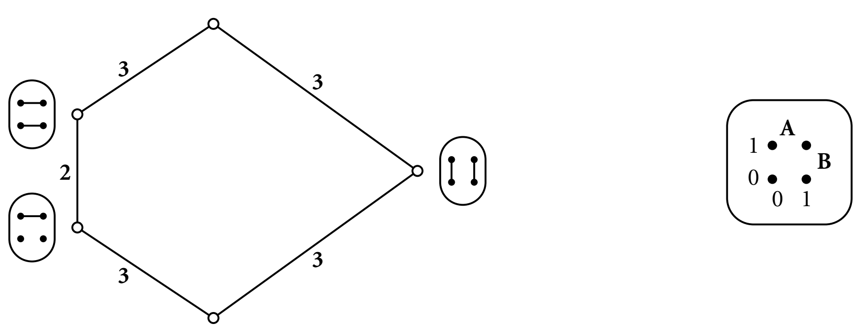

Figure 3: The congruence lattice of $\mathbf{A} \times \mathbf{B}$.

https://doi.org/10.4153/CJM-2002-028-x Published online by Cambridge University Press

6.3 Example

In relation to Problem 1.5, we exhibit a finite simple algebra $\mathbf{A}$ that has size (and hence chief factor size) $n+1$, but in the variety generated by $\mathbf{A}$ there exists a chief factor of type $\mathbf{2}$ whose size is $1+(n^2+n)/2$.

Let $\mathbf{K} = \mathbb{Z}_2$ be the two-element field, and let A be the set of all vectors in $\mathbf{K}^n$ that contain at most one coordinate that is 1 (and all other coordinates are zero). Then $|A| = n+1$, and A is the union of n one-dimensional subspaces. Any two elements of A lie in a coset modulo some one-dimensional subspace.

The basic operations of $\mathbf{A}$ are the polynomials of the module $\mathbf{K}_n$ whose range is in a 2-element subset of A. With this choice $\mathbf{A}$ is a finite simple algebra of type $\mathbf{2}$ whose traces are exactly the two-element subsets.

Let Δ be the restriction of the diagonal congruence of $\mathbf{K}_n^2$ to $\mathbf{A}^2$ (so $(a,b)\ \Delta\ (c,d)$ if and only if $a-b=c-d$, where $-$ is the subtraction operation of $\mathbf{K}_n$). If $a \neq b$ and $c \neq d$, then $(a,b)\ \Delta\ (c,d)$ if and only if the traces $\{a,b\}$ and $\{c,d\}$ have the same direction. But $\mathbf{A}$ does not have a pair of distinct traces with the same direction, hence if $a \neq b$ and $(a,b)\ \Delta\ (c,d)$ then $\{a,b\} = \{c,d\}$. This shows that the nondiagonal blocks of Δ have the form $\{(a,b),(b,a)\}$. In particular, $|\mathbf{A}^2/\Delta| = \frac{1}{2}(|A|^2-|A|)+1 = n(n+1)/2+1$.

To see that $\mathbf{A}^2/\Delta$ is simple, choose elements $(a,b)/\Delta \neq (c,d)/\Delta$ arbitrarily. Assume first that $a \neq b$ and $c \neq d$. Then $\{a,b\}$ and $\{c,d\}$ are traces of $\mathbf{A}$ that have different directions, so $\mathbf{A}$ has an idempotent polynomial e that has range $\{a,b\}$ and is constant on $\{c,d\}$ (by the second part of Lemma 4.1). The function that is e acting coordinatewise modulo Δ is a polynomial mapping the pair $\big((a,b)/\Delta,(c,d)/\Delta\big)$ to $\big((a,b)/\Delta,(a,a)/\Delta\big)$. It follows that any nontrivial congruence on $\mathbf{A}^2/\Delta$ must relate some nondiagonal block to the diagonal block. Conversely, if $a \neq b$, then any congruence of $\mathbf{A}^2/\Delta$ which relates $(a,b)/\Delta$ to $(a,a)/\Delta$ relates every nondiagonal block $(c,d)/\Delta$ to the diagonal block. To see this, pick a polynomial isomorphism p from the trace $\{a,b\}$ to the trace $\{c,d\}$. Then p acting coordinatewise modulo Δ is a polynomial mapping the pair $\big((a,b)/\Delta,(a,a)/\Delta\big)$ to $\big((c,d)/\Delta,(c,c)/\Delta\big)$. This shows that $\mathbf{A}^2/\Delta$ is simple. That it has type $\mathbf{2}$ is clear from the fact that it is a simple member of a variety generated by a finite abelian algebra, so it is abelian, but it cannot be strongly abelian since Δ does not collapse all traces of type $\mathbf{2}$ of $\mathbf{A}^2$.

6.4 Example

Theorem 1.4 shows that there exists a finite bound for all chief factors of type $\mathbf{2}$ in any finitely generated variety. We show that the hypothesis "finitely generated" cannot be weakened to "locally finite".

We define an algebra $\mathbf{A}$ with two binary operations $\cdot$ and $+$ as follows. For $n \in \mathbb{N}$ define $f_n\colon \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ by

$$f_n(i) = \begin{cases} n & \text{if } |n-i| \leq 1 \\ \infty & \text{otherwise.} \end{cases}$$

Let $A = \mathbb{N} \cup \{\infty\} \cup \{f_n \mid n \in \mathbb{N}\}$. Also let $\oplus$ be the group operation on $\{\infty, 0\}$

https://doi.org/10.4153/CJM-2002-028-x Published online by Cambridge University Press

having ∞ as the identity element, and let $e\colon A \to \{\infty, 0\}$ be defined by

$$e(x) = \begin{cases} 0 & \text{if } x \in \{0, 1\} \\ \infty & \text{otherwise.} \end{cases}$$

Finally, define

$$x \cdot y = \begin{cases} x(y) & \text{if } x \in \{f_n \mid n \in \mathbb{N}\} \text{ and } y \in \mathbb{N} \\ \infty & \text{otherwise} \end{cases}$$

$$x + y = e(x) \oplus e(y).$$

Let $\mu = (\mathbb{N} \cup \{\infty\})^2 \cup 0_A$.

Claim 6.4.1 *The algebra* $\mathbf{A}$ *is subdirectly irreducible with monolith* μ. *The congruence* μ *is abelian but not strongly abelian.* $\mathrm{V}(\mathbf{A})$ *is locally finite (but not finitely generated).*

Proof The relation μ is a congruence of $\mathbf{A}$ because the fundamental operations of $\mathbf{A}$ are constant modulo μ. The reduct $\langle A, \cdot \rangle$ is a graph $*$-algebra in the sense of [7], and is already subdirectly irreducible with monolith μ because of the nature of the underlying graph.

Since $\{\infty, 0\}$ is a $\mathbf{1}$-snag in μ, it is not strongly abelian. To see that μ is abelian, note that any failure of the term condition can be localized in a finitely generated subalgebra of $\mathbf{A}$, which in turn is contained in $\mathbf{A}_n$ for some n, where $\mathbf{A}_n$ is the subalgebra of $\mathbf{A}$ having universe $\{0, 1, \ldots, n\} \cup \{\infty\} \cup \{f_0, f_1, \ldots, f_n\}$. The restriction $\mu|_{A_n}$ is a minimal congruence of $\mathbf{A}_n$ (in fact, it is the monolith), and obviously $\{\infty, 0\}$ is a $\langle 0_{A_n}, \mu|_{A_n} \rangle$-minimal set. A routine analysis of the polynomials of $\mathbf{A}$ shows that $\mathbf{A}|_{\{\infty,0\}}$ is abelian.

One can see by inspection that $\mathbf{A}$ is uniformly locally finite; in fact, each n-generated subalgebra of $\mathbf{A}$ has at most $2n + 2$ elements. Since $\mathbf{A}$ has finite signature, there are only finitely many isomorphism types among the n-generated subalgebras of $\mathbf{A}$, hence, $\mathrm{V}(\mathbf{A})$ is locally finite. It is not finitely generated by Theorem 1.4. ■

6.5 Example

We show that the chief factor size of a subalgebra of an algebra $\mathbf{B}$ is not necessarily bounded by the chief factor size of $\mathbf{B}$. For any algebra $\mathbf{A}$ we define a new algebra $\mathbf{C}$ in the following way. Add a new element 1 to the universe of $\mathbf{A}$, extend all the operations of $\mathbf{A}$ arbitrarily, and define a 4-ary operation d such that $d(x, y, z, 1)$ is the ternary discriminator, while $d(x, y, z, a) = x$ if $a \in A$. The new algebra $\mathbf{C}$ has a discriminator polynomial, and therefore $\mathbf{B} = \mathbf{C}^n$ has only the 2^n obvious congruences. Hence the chief factor size of $\mathbf{C}^n$ is $|C| = |A| + 1$. On the other hand, $\mathbf{A}^n$ is a subalgebra of $\mathbf{C}^n$, and its subalgebras can have arbitrarily large chief factors (if $\mathbf{A}$ is chosen to be one of the examples referred to in the introduction).

References

[1] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*. Springer-Verlag, 1981.

[2] R. Freese and R. McKenzie, *Residually small varieties with modular congruence lattices.* Trans. Amer. Math. Soc. **264**(1981), 419–430.

[3] ______, *Commutator Theory for Congruence Modular Varieties*. London Math. Soc. Lecture Note Ser. **125**, Cambridge University Press, Cambridge-New York, 1987.

[4] D. Hobby and R. McKenzie, *The Structure of Finite Algebras*. Contemp. Math. **76**, Amer. Math. Soc., 1988.

[5] K. A. Kearnes, E. W. Kiss, and M. A. Valeriote, *Minimal sets and varieties*. Trans. Amer. Math. Soc. **350**(1998), 1–41.

[6] ______, *A geometric consequence of residual smallness.* Ann. Pure Appl. Logic. **99**(1999), 137–169.

[7] K. A. Kearnes and R. D. Willard, *Inherently nonfinitely based solvable algebras*. Canad. Math. Bull. **37**(1994), 514–521.

[8] E. W. Kiss, *An easy way to minimal algebras*. Internat. J. Algebra Comput. **7**(1997), 55–75.

[9] R. McKenzie, *The residual bounds of finite algebras.* Internat. J. Algebra Comput. **6**(1996), 1–28.

[10] R. W. Quackenbush, *Equational classes generated by finite algebras*. Algebra Universalis **1**(1971), 265–266.

[11] C. Shallon, *Non-finitely based algebras derived from lattices*. Ph.D. thesis, UCLA, 1978.

[12] W. Taylor, *Subdirectly irreducible algebras in regular permutable varieties*. Proc. Amer. Math. Soc. **75**(1979), 196–200.

Department of Mathematics
University of Colorado
Boulder, CO 80309-0395
USA
email: kearnes@euclid.colorado.edu

Eötvös University
Department of Algebra and Number Theory
Múzeum krt. 6–8
1088 Budapest
Hungary
email: ewkiss@cs.elte.hu

Bolyai Institute
Aradi vértanúk tere 1
H-6720 Szeged
Hungary
email: A.Szendrei@math.u-szeged.hu

Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario
N2L 3G1
email: rdwillar@uwaterloo.ca

https://doi.org/10.4153/CJM-2002-028-x Published online by Cambridge University Press