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ORDER STRUCTURE ON CERTAIN CLASSES OF IDEALS IN GROUP ALGEBRAS AND AMENABILITY

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Let G be a separable, locally compact group and let $\mathcal{J}_d(G)$ be the set of all closed left ideals in $L^1(G)$ which have the form $J_\mu = \{f - f * \mu : f \in L^1(G)\}^-$ for some discrete probability measure μ. It is shown that if $\mathcal{J}_d(G)$ has a unique maximal element with respect to the order structure by set inclusion, then G is amenable. This answers a problem of G.A. Willis. We also examine cardinal numbers of the sets of maximal elements in $\mathcal{J}_d(G)$ for nonamenable groups.

Let G be a locally compact group, and let $L^1(G)$ and $M(G)$ be the group and the measure algebras on G, respectively. As usual, we shall identify $L^1(G)$ with the ideal of $M(G)$ consisting of measures which are absolutely continuous with respect to left Haar measure. Let $PM(G)$ denote the set of probability measures in $M(G)$. For each $\mu \in PM(G)$, define

$$J_\mu = \{f - f * \mu : f \in L^1(G)\}^-$$

and

$$\mathcal{J}(G) = \{J_\mu : \mu \in PM(G)\}.$$

Then J_μ is a closed left ideal in $L^1(G)$ and $\mathcal{J}(G)$ is partially ordered by set inclusion. We also consider two classes of ordered subsets $\mathcal{J}_a(G)$ and $\mathcal{J}_d(G)$ of $\mathcal{J}(G)$ defined by

$$\mathcal{J}_a(G) = \{J_\mu : \mu \in PM_a(G)\}$$

and

$$\mathcal{J}_d(G) = \{J_\mu : \mu \in PM_d(G)\},$$

where $PM_a(G)$ and $PM_d(G)$ denote respectively the sets of absolutely continuous and discrete probability measures in $M(G)$.

Willis [**9**, Theorem 1.2 (b)] characterised the amenability of G in terms of the order structure of $\mathcal{J}(G)$ or $\mathcal{J}_a(G)$ as follows: For a separable(that is, second countable), locally compact group G, $\mathcal{J}_a(G)$ (or $\mathcal{J}(G)$) has a unique maximal element if and only if G is amenable. It was also shown that $\mathcal{J}_d(G)$ has a unique maximal element whenever G is a

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separable, amenable, locally compact group ([**9**, Proposition 1.3]). However the following problem remained open ([**9**, Problem 6.1]: If G is a separable, locally compact group and if $\mathcal{J}_d(G)$ has a unique maximal element, then is G amenable? In this note we resolve this problem affirmatively. It is also shown that $\mathcal{J}_d(G)$ has uncountably many maximal elements for some class of nonamenable groups.

Let $L_0^1(G)$ be the augmentation ideal of $L^1(G)$, that is

$$L_0^1(G) = \{f \in L^1(G) : \int_G f = 0\}.$$

It is well known that $L_0^1(G)$ coincides with the closed linear span of $\{f - f * \delta_x : f \in L^1(G), x \in G\}$, where δ_x denotes the point mass at $x \in G$ ([**6**, Section 1, Proposition 1] or [**7**, Proposition 3.6.11]). In the following, this fact will be used repeatedly.

The following theorem shows that the answer to Willis' problem is affirmative. Our proof of the theorem also reproduces a result of Willis [**9**, Theorem 1.2 (b)].

Theorem 1. *Let G be a separable, locally compact group and let X be one of $\mathcal{J}(G)$, $\mathcal{J}_a(G)$, and $\mathcal{J}_d(G)$. If X has a unique maximal element, then G is amenable.*

Proof: Assume that X has a unique maximal element, J_μ say. We claim that $f - f * \delta_x \in J_\mu$ for each $x \in G$ and $f \in L^1(G)$. This is valid when $X = \mathcal{J}(G)$ or $X = \mathcal{J}_d(G)$. In fact, since $f - f * \delta_x \in J_{\delta_x}$ and every element of X is contained in a maximal one ([**9**, Theorem 1.2(a)]), we have $f - f * \delta_x \in J_{\delta_x} \subseteq J_\mu$. Now suppose that $X = \mathcal{J}_a(G)$, and let $\{u_\lambda\}_{\lambda \in \Lambda}$ be a bounded approximate identity for $L^1(G)$ such that $u_\lambda \in PM(G)$ for each $\lambda \in \Lambda$. (We may take a sequential bounded approximate identity because of the separability of G.) Since $f - f * u_\lambda * \delta_x \in J_{u_\lambda * \delta_x}$ and $u_\lambda * \delta_x \in L^1(G)$, we also have

$$f - f * u_\lambda * \delta_x \in J_{u_\lambda * \delta_x} \subseteq J_\mu$$

for each $\lambda \in \Lambda$. Hence it follows from the closedness of J_μ that

$$f - f * \delta_x = \lim_\lambda (f - f * u_\lambda * \delta_x) \in J_\mu,$$

as desired. Since $L_0^1(G)$ is equal to the closed linear subspace generated by $\{f - f * \delta_x : f \in L^1(G), x \in G\}$, our claim implies that $L_0^1(G) \subseteq J_\mu$. But the converse inclusion relation is clear, and so we have $L_0^1(G) = J_\mu$. Notice now that J_μ has a bounded right approximate identity. Indeed it is easy to verify that $\left\{u_\lambda - u_\lambda * \left(\sum_{i=1}^n \mu^i\right)/n\right\}_{(\lambda,n) \in \Lambda \times \mathbf{N}}$ is a bounded right approximate identity for the left ideal J_μ whenever $\{u_\lambda\}_{\lambda \in \Lambda}$ be a bounded approximate identity for $L^1(G)$. Thus $L_0^1(G)(= J_\mu)$ has a bounded right approximate identity. Therefore we may apply a result of Reiter [5] to conclude that G is amenable. ∎

Remarks 1. (1) Following [8] we say that $\mu \in PM(G)$ is ergodic by convolutions if $\lim_{n \to \infty} \left\| f * \left(\sum_{i=1}^n \mu^i\right)/n \right\|_1 = 0$ for all $f \in L_0^1(G)$. Rosenblatt [**8**, Proposition 1.9] showed

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that if there exists a probability measure on a locally compact group G which is ergodic by convolutions, then G is σ-compact and amenable. It is obvious that $\mu \in PM(G)$ is ergodic by convolutions if and only if $L_0^1(G) = J_\mu$. Recall also that if G is a *σ-compact* locally compact group, then every element in $\mathcal{J}(G)$ (or $\mathcal{J}_a(G)$) is contained in a maximal one (see [**9**]). These facts may be combined with [**8**, Proposition 1.9 and Theorem 1.10] to show that the following conditions (i), (ii), and (iii) are equivalent whenever G is a σ-compact locally compact group:

(i) G is amenable;

(ii) there exists an absolutely continuous probability measure on G which is ergodic by convolutions;

(iii) $\mathcal{J}(G)$ (or $\mathcal{J}_a(G)$) has a unique maximal element.

(2) The proof of Theorem 1 yields that if G is a separable, locally compact group, then $\mathcal{J}_d(G)$ has a unique maximal element if and only if $L_0^1(G) \in \mathcal{J}_d(G)$. Thus it is still true that if G is a separable, locally compact group, then the amenability of G is equivalent to each of the following conditions (ii)′ and (iii)′:

(ii)′ there exists a discrete probability measure on G which is ergodic by convolutions;

(iii)′ $\mathcal{J}_d(G)$ has a unique maximal element.

The statement that $\mathcal{J}_d(G)$ has a unique maximal element does not necessarily imply the separability of G. Indeed, let G be the Bohr compactification of $\boldsymbol{Z}$. It is then easily shown that G is a compact Abelian group which is not separable. Since G has a countable dense subset (for example, the image of $\boldsymbol{Z}$ under the canonical continuous homomorphism of $\boldsymbol{Z}$ to G), it follows from [**8**, Corollary 1.14] that $L_0^1(G) \in \mathcal{J}_d(G)$. We can also obtain some examples of σ-compact locally compact groups for which $L_0^1(G) \notin \mathcal{J}_d(G)$.

The argument used in Theorem 1 may be applied to show that there exist many maximal elements in $\mathcal{J}(G), \mathcal{J}_a(G)$, and $\mathcal{J}_d(G)$ for some nonamenable groups. The following result gives a partial improvement of Theorem 1.

THEOREM 2. *Let G be a separable, connected, locally compact group which is nonamenable. Then $\mathcal{J}_d(G)$ (or $\mathcal{J}(G)$) has uncountably many maximal elements.*

PROOF: We shall give the proof for $\mathcal{J}_d(G)$ only. The proof for $\mathcal{J}(G)$ is exactly the same. Suppose that the set of maximal elements in $\mathcal{J}_d(G)$ has at most countably many elements $\{J_{\mu_n}\}_{n\geqslant 1}$. We shall prove that G is amenable. Now define

$$H_n = \{x \in G : J_{\delta_x} \subseteq J_{\mu_n}\}$$

for each n. Then H_n is a closed subgroup of G. Indeed, H_n is closed because the mapping $a \to f * \delta_a$ of G into $L^1(G)$ is continuous for each $f \in L^1(G)$. That H_n is a subgroup of G follows immediately from the relations

$$f - f * \delta_{xy} = f - f * \delta_x + \left\{(f * \delta_x) - (f * \delta_x) * \delta_y\right\}$$

and

$$f - f * \delta_{x^{-1}} = -\{(f * \delta_{x^{-1}}) - (f * \delta_{x^{-1}}) * \delta_x\},$$

where $f \in L^1(G)$ and $x, y \in G$. Since every ideal in $\mathcal{J}_d(G)$ is contained in a maximal element ([**9**, Theorem 1.2(a)]), we also have $G = \bigcup_{n\geqslant 1} H_n$. Thus Baire category theorem implies that H_{n_0} is open for some n_0. Noting now that G is connected, we have

$$G = H_{n_0} = \{x \in G : J_{\delta_x} \subseteq J_{\mu_{n_0}}\}.$$

But then $L_0^1(G) = J_{\mu_{n_0}}$, and so G is amenable, as desired. ∎

We can also prove the following result on cardinal numbers of the set of maximal elements in $\mathcal{J}_a(G)$.

THEOREM 3. *Let G be a separable, connected, nonamenable locally compact group. Then $\mathcal{J}_a(G)$ has infinitely many maximal elements.*

PROOF: Assume that the set of maximal elements in $\mathcal{J}_a(G)$ consists of finitely many elements $\{J_{\mu_i}\}_{i=1}^m$. It will be shown that G is amenable. Choose and fix $x \in G$. Let $\{u_n\}_{n\in \boldsymbol{N}}$ be a *sequential* bounded approximate identity for $L^1(G)$ such that $u_n \in PM(G)$ for each $n \in \boldsymbol{N}$. Since $u_n * \delta_x \in L^1(G)$ for every n, $J_{u_n*\delta_x} \subseteq J_{\mu_{i_n}}$ for some i_n $(1 \leqslant i_n \leqslant m)$ ([**9**, Theorem 1.2(a)]), and so we have

$$\boldsymbol{N} = \bigcup_{i=1}^{m} \{n \in \boldsymbol{N} : J_{u_n*\delta_x} \subseteq J_{\mu_i}\}.$$

Hence there exist some i $(1 \leqslant i \leqslant m)$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $J_{u_{n_k}*\delta_x} \subseteq J_{\mu_i}$ for all k. Since $\lim_{k\to\infty}(f - f * u_{n_k} * \delta_x) = f - f * \delta_x$ for each $f \in L^1(G)$, it follows from the closedness of J_{μ_i} that $J_{\delta_x} \subseteq J_{\mu_i}$. Thus we conclude that

$$G = \bigcup_{i=1}^{m} \{x \in G : J_{\delta_x} \subseteq J_{\mu_i}\}.$$

Now the argument used in the proof of Theorem 2 may be applied to show that $L_0^1(G) = J_{\mu_{i_0}}$ and hence G is amenable. ∎

REMARKS 2. (1) It is well known that $SL(n, \boldsymbol{R})$ $(n \geqslant 2)$ is a connected Lie group which contains $\mathbf{F}_2$ (the free group on two generators) as a discrete subgroup ([**3**, Proposition 3.2] or [**4**, Corollary 14.6]). Thus $SL(n, \boldsymbol{R})$ is nonamenable, and so Theorem 2 implies that $\mathcal{J}(SL(n, \boldsymbol{R}))$ and $\mathcal{J}_d(SL(n, \boldsymbol{R}))$ have uncountably many maximal elements. It also follows from Theorem 3 that $\mathcal{J}_a(SL(n, \boldsymbol{R}))$ has infinitely many maximal elements. More generally Theorem 2 and Theorem 3 may be applied to every noncompact, connected, semisimple Lie group with finite centre (see [**7**, Theorem 8.7.6]).

(2) Both Theorem 2 for $\mathcal{J}(G)$ and Theorem 3 are also valid for *all* connected, nonamenable locally compact groups. These may be shown by applying the theorem of Kakutani and Kodaira which asserts that if G is σ-compact, then it has a compact normal subgroup K such that G/K is separable (see [**2**] or [**1**, Theorem 8.7]).

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