PULSE EMISSION FROM THE LIGHT CYLINDER

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1. INTRODUCTION

The light cylinder is the second most popular location for pulsar emission. The fact that it has a sizable following, and yet is not the most popular location, is due to many aspects of light cylinder models. They have several advantages in explaining the pulsar phenomenon, but they have been little developed. This is mainly because they deal with the light cylinder, where our general ignorance of magnetospheric structure is greatest. Although light cylinder models have had considerable success in explaining the optical radiation from the Crab and Vela pulsars[1], I will restrict this paper to a discussion of radio pulsars.

2. COROTATION

As everyone must know, the light cylinder is the place where the magnetospheric plasma would be travelling at the speed of light if it rotated rigidly with the central star. In any astrophysical plasma the magnetic field lines and the plasma will be locked together (in the "frozen-in" field condition) because of the high conductivity of the plasma. If the magnetic field energy density is greater than that of the plasma, it will carry the plasma with it, and vice versa. In a pulsar magnetosphere without any evacuated regions, the strong magnetic field, locked into the rotation of the star, will carry the surrounding plasma around in corotation unless

$$\gamma \rho c^2 > B^2/8\pi \quad , \text{ where} \tag{2.1}$$

$$\gamma \equiv (1 - v^2/c^2)^{-0.5} \ . \tag{2.2}$$

γ is the special relativistic factor, ρ is the mass density of the plasma, v is the corotational velocity, c is the speed of light, and B is the magnetic field strength. In the non-relativistic limit, this implies $v > v_A$, where v_A is the Alfvén speed.

W. Sieber and R. Wielebinski (eds.), Pulsars, 141–152.

If the pulsar magnetosphere is charge separated, with only electrons or positrons contributing to the mass density near the light cylinder, corotation can extend to very near the light cylinder. In the Goldreich and Julian (1969) model for instance, where

$$n_e \cong 7 \times 10^{-2} \, (B/P) \ , \tag{2.3}$$

corotation could extend up to

$$\gamma \rho c^2 = \gamma n_e m_e c^2 = B^2/8\pi \ , \text{ or} \tag{2.4}$$

$$\gamma \cong 7 \times 10^5 \, BP \ ,$$

where n_e is the electron (or positron) number density, m_e is the particle mass, and P is the pulsar period. For a magnetic field of 1 Gauss at the light cylinder, and pulsar periods near one second, this could result in ultra-relativistic corotation[2].

3. RELATIVISTIC EFFECTS

Most of the advantages of light cylinder models for pulsars derive from relativistic effects. There are three effects which influence the observed character of radiation emitted there. First, there is the aberration of light. If a rapidly moving source emits photons isotropically in its rest frame, in our frame most of those photons are emitted in the forward direction because of the addition of momenta. This effect concentrates the light into a "headlight" beam and also causes the source to appear rotated in the direction of its motion. We see it by photons which it emitted more "backward" than they are in our reference frame[3]. See Figure 1.

The second major relativistic effect is the Doppler shift, which makes photons emitted in the forward direction more energetic than those emitted in the backward direction. This combines with the aberration to produce a sharply peaked beam in the forward direction.

The third major effect is that of light travel time. A source moving rapidly toward us is lessening the light travel time ("catching up with its own wavefronts") so that light emitted over a long time interval may be seen by us as a short flash.

Now consider a source corotating with a neutron star near the light cylinder. If we are close to the star's equator, we see the source strongest, and we see it strongest when it is coming most directly toward us, for then its light is aberrated in our direction and Doppler shifted to a higher energy; and emission taking place over a long time interval will arrive at us in a short time. Refer to Figure 1. Mathematically (Ferguson, 1976a),

$$I'(\nu') = I(\nu) \, / \, [\gamma \, (1 - \beta \cos \delta)]^3 \ , \tag{3.1}$$

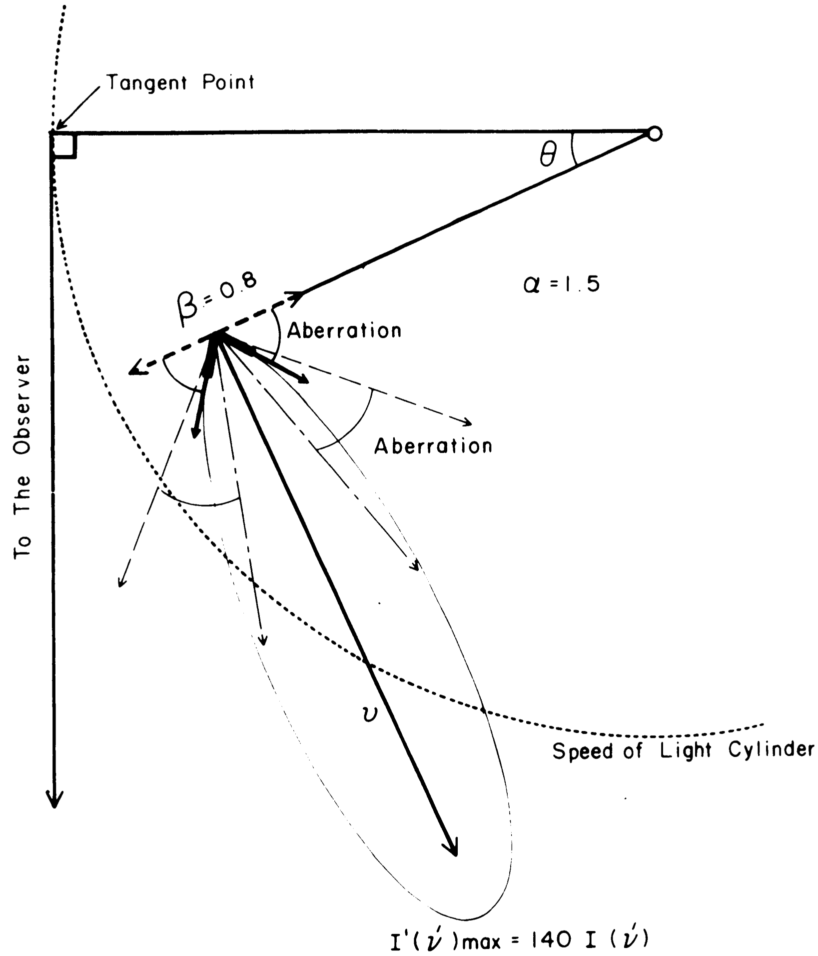

Figure 1: Relativistic effects in light cylinder models. The drawing is to scale for β = 0.8 and α = 1.5.

$$\nu' = \nu / \gamma (1 - \beta \cos \delta) \quad \text{, and} \tag{3.2}$$

$$dt'_{obs} / dt'_{em} = (1 - \beta \cos \delta) \quad , \tag{3.3}$$

where I is the specific intensity at frequency ν, t is time, β is the rotation v/c, and δ is the angle between the velocity vector and the vector toward us (our line of sight). Primed quantities are observed; unprimed are in the rest frame of the emitter.

If we assume an emitter in a circular orbit (which may or may not be strictly corotating), radiating with a spectral index α given by $I \propto \nu^{-\alpha}$, then

$$I'(\nu') = I(\nu') \, / \, [\gamma\,(1-\beta\,\cos\delta)]^{3+\alpha} \;. \qquad (3.4)$$

If we further assume that the source is corotating with the central star, that we are T radians out of the equator, and the source is θ radians past the "tangential point" when its velocity is closest to our line of sight, then

$$I'(\nu') = I(\nu') \, / \, [\gamma\,(1-\beta\cos\theta\,\cos T)]^{3+\alpha} \;, \text{ and} \qquad (3.5)$$

$$t'_{obs}/P = (\theta - \beta\,\sin\theta\,\cos T)/2\pi \;. \qquad (3.6)$$

See Smith (1970, 1971), McCrea (1972) and Zheleznyakov (1971). If strict corotation does not hold[4], the P in the last formula must be replaced by the instantaneous period of revolution.

For even quite modest values of β, the relativistic effects can be quite large. Let us take $\beta = 0.9$, $T = 10°$, and $\alpha = 2$. Then,

$$I'(\nu') = I(\nu') \times 829 \text{ at maximum,}$$

$$I'(\nu') = I(\nu') \, / \, 1517 \text{ at minimum, and}$$

$$t_{1/2}/P = 0.0071 \cong 2.^{\circ}5 \text{ of rotation} \;.$$

Here $t_{1/2}$ is the full duration at half maximum of the beaming effect. Furthermore, at maximum intensity the source appears to be rotated by 31.°7 toward its velocity vector.

From equation (3.4) above, we can find that the beam width of an isotropically emitting corotating source is $\cos\delta = \beta$, so from equation (3.5) we can see most pulsars of $\sin T < 1/\gamma$, with pulse widths that are compressed to

$$t_{obs}/P \cong 1/4\pi\,\gamma^{3} \;. \qquad (3.7)$$

If there is a preferred direction of polarization in the source, the aberration and time compression result in its apparently changing its direction very rapidly, also.

Thus the observed narrow pulses of pulsars, the observed high ratio of "on" to "off" pulse intensities, the observed near independence of pulse width with frequency, and the observed rapid changes of position angle through the pulses are explained in light cylinder models without drastically reducing the number of pulsars seen.

4. INTERPRETATIONS

While everyone agrees upon the foregoing, light cylinder interpretations of the pulsar phenomenon are numerous and diverse. In his original work, Smith (1971, 1973) interpreted the subpulse as the basic pulsar beam and saw microstructure as a modulation of the emission from the subpulse region. Gold (1968, 1969) and others[5] on the other hand, saw the subpulse as something built up out of microstructure beams with very high corotational γs. In what follows, I will present what I feel is an interpretation most consistent with the observational facts.

4.1 Type of Radiation

First of all, the radiation is likely to be of the cyclotron or low energy synchrotron type. If we assume a dipole magnetic field and a neutron star radius of 10^6 cm, the magnetic field at the light cylinder becomes

$$B_{LC} \cong 2 \times 10^8 \, (\dot{P})^{0.5} / P^{2.5} \; . \tag{4.1}$$

The gyration frequency (or cyclotron frequency, where cyclotron here implies radiation at the fundamental, regardless of electron energy) is given by (Pacholczyk, 1970)

$$\nu_g = e \, B_{LC} / \gamma_e m_e c \;\; , \tag{4.2}$$

where e is the quantum of charge and γ_e is the particle energy factor $E/m_e c^2$, and the frequency of the synchrotron maximum is given by

$$\nu_s \cong 0.45 \, \gamma_e{}^3 \, \nu_g \; . \tag{4.3}$$

For pulsars of $P \sim 1$ s we find the values of B_{LC}, ν_g and ν_s in the table below.

	$\dot{P} = 10^{-17}$ (low)	$\dot{P} = 6 \times 10^{14}$ (high)
B_{LC}	0.63 Gauss	49.0 Gauss
ν_g	$1.1 \times 10^7/\gamma_e$ Hz	$8.6 \times 10^8/\gamma_e$ Hz
ν_s	$5.5 \times 10^6 \, \gamma_e{}^2$ Hz	$3.9 \times 10^8 \, \gamma_e{}^2$ Hz

Since pulsars have spectra which peak at about 100 MHz, for the pulsar of low B_{LC} we need $\gamma_e \cong 4.5$ for the synchrotron maximum to be so high. This is not a very relativistic particle. For the pulsar of high B_{LC} γ_e of $\cong 9$ will bring the fundamental down to about 100 MHz. In any case, particle γs of one to a few are sufficient for producing the peaks in pulsar spectra. What this means is, of course, that the magnetic field at the light cylinder is about the right strength for cyclotron and low energy synchrotron radiation to be efficiently produced[6]. We can expect the observed frequencies to be affected also by the corotation Doppler shift.

4.2 Coherence

In any model assuming cyclotron or low energy synchrotron radiation from the light cylinder, the emission must be highly coherent. This is because the high brightness temperatures observed are in conflict with the maximum brightness temperature self absorption imposed on an incoherent source.

The simplest method to obtain broad band coherence is to have the electrons (or positrons) bunched together at high densities. The bunch will then radiate any wavelength more than about 4 times its size with nearly perfect coherence. When we limit the size of the bunches in this way, and limit the density by the corotation criterion, we find that very many bunches are required; even though the individual bunches radiate at N^2 times the incoherent rate, where N is the number of particles per bunch, the radiation from the entire ensemble of bunches will add in an incoherent manner[7].

In any event, the necessary large number of incoherently adding bunches is desirable, whether the radiation is cyclotron or synchrotron. In the cyclotron case, the emission from any one bunch will be at a single frequency, the fundamental. In order to cover the observed broad band spectrum of a pulsar pulse, we need many bunches with particles of different energies. For synchrotron radiation, harmonics higher than the fundamental will be beamed along the pitch angle of the particles in the bunch. In order to maintain consistency with the observed pulse widths in pulsars, we would like the total radiation field to be nearly isotropic in the corotating reference frame. By using many synchrotron bunches of different pitch angle we can achieve this goal. In either case, we will now have an emission region radiating over a broad bandwidth in a nearly isotropic manner in the rest frame of corotation.

4.3 Polarization

The polarization behavior in the two cases discussed above will be somewhat different. For any radiating particle or coherent bunch the circular polarization will change sign and the linear polarization will be complete when the pitch angle of the radiating particle passes the line of sight. A large range of pitch angles is required in the synchrotron case, so we expect the circular polarization to be almost completely washed out, and the linear polarization to be lowered, by the incoherent addition of radiation from many bunches. In the cyclotron case, large amounts of both linear and circular polarization will arise if the range of pitch angles is limited.

The observed polarization in the individual pulses of many pulsars fits in well with light cylinder models. Assuming a constant field line direction in the emission region, with the polarization angle and amount dependent on the projections of the field line across and along the line of sight, the models can explain the following aspects of the observed polarization:

1. Linear and circular polarizations approaching 100%.
2. Changes of linear and circular percentages on the time scale of the pulse width.

3. Rapid position angle swings of ∿ 180° when the field line crosses the (aberrated) line of sight.
4. No rapid change of polarization when the field line does not come close to the line of sight.
5. Polarization changes following the individual pulses, not the pulsar longitude.

See Ferguson (1976b) and Ferguson and Seiradakis (1978) for details[8]. By assuming a distribution of emission regions near the light cylinder the integrated polarization of pulsars is neatly explained (Ferguson, 1978 and 1980). The form of the solution can be nearly identical to that predicted by a dipole at a polar cap!

4.4 Microstructure and Subpulses

So far I have largely avoided talking of subpulses or micropulses, talking instead about individual pulses. This is because much that I have said can apply to either subpulses or micropulses, depending on interpretation. Manchester et al. (1973) showed that it is difficult for microstructure to be a simple time modulation of the subpulse. Their argument is based on the energy densities required if the subpulse emission regions are of a size which light can traverse on a micropulse time scale. For the record, subpulse corotation γs are about 2 or 3 based on their widths. It has become fashionable (e.g. Cordes and Hankins, 1977, and Endean, 1980) to think that subpulses are constructed of microstructure, which in the relativistic beaming theory would imply γs of 5 to 10 or more. I think this is wrong, for several reasons:

1) When a micropulse sits atop a subpulse, it is often polarized in a different direction and amount than the subpulse underneath. This implies, in the context of relativistic beaming theory, a different apparent magnetic field direction, either due to a different amount of aberration (a different γ) or a different location of emission or both (see Ferguson and Seiradakis, 1978).
2) Isolated micropulses often show gross polarization changes within a micropulse and with about the same time scale.
3) Isolated subpulses show polarization changes not on the micropulse time scale but on the subpulse time scale.
4) Subpulses are almost always very broadband, and may not follow the dispersion law exactly, as in dual component pulsars where the component separation narrows with frequency.
5) Micropulses always follow the dispersion law exactly, and for PSR 1133+16 at least may have bandwidths much less than the subpulse bandwidth (Boriakoff and Ferguson, 1980).

This leads me to the following picture of pulsar emission. The micropulses are emitted at γs of 5 to 10 or more. Each small micropulse emission region contains many coherent bunches of all different energies. Each micropulse is seen at all emitted frequencies when its emission region passes the tangential point. The bandwidth is limited by energy density considerations, by an upper limit to its particle energies, or by a limit on the bunch size. Quasi-periodic microstructure comes from many periodically spaced emission regions, perhaps occupying excited locations in a standing plasma wave of very long wavelength.

Subpulses, on the other hand, come from farther down in the magnetosphere, at γs of 2 or 3. The change of subpulse separation with frequency implies a location vs frequency mapping. This is almost certainly due to a change of particle energy along a field line, because at subpulse γs, the change of observed frequency with a change in co-rotation γ is almost completely counteracted by the change of emission frequency with field strength falloff.

Here, the low frequency subpulse radiation comes from farther down along a field line (perhaps the last closed field line). As we go out along the field, progressively higher frequencies are emitted, the component separation decreases, and the increased value of γ decreases the subpulse beamwidth and compresses the polarization changes into shorter time spans. All of this happens within a narrow range of γ. Probably the lower limit of radiation is at a shock wave in the magnetosphere, which may also produce the necessary bunching. Here γ_e must be highest, to produce the lowest fundamental frequencies. As we go up out of the shock, the electrons lose energy, emitting higher frequencies, until finally they have no energy left. There the component separation stops decreasing, coincident with a change in emission spectrum, as has been observed (Sieber et al., 1975 and Backer, 1972). In all regions, the frequency mapping is loose.

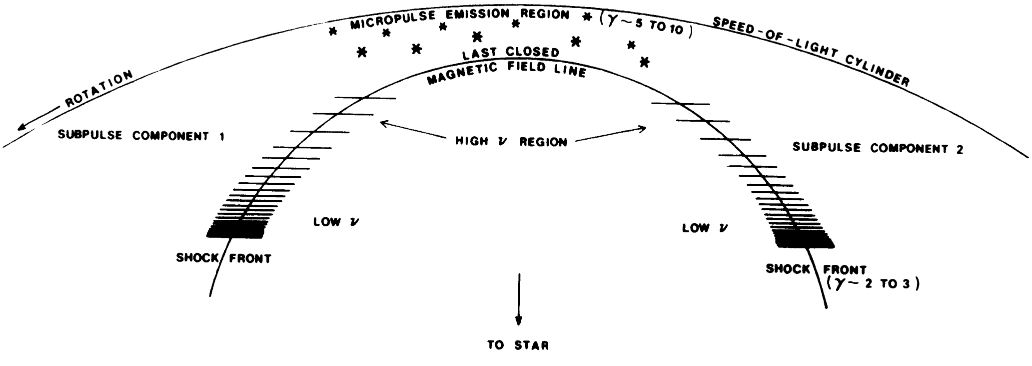

Figure 2: A model for subpulses and micropulses

I don't yet know how to produce the symmetrical shock wave necessary in this picture. It can't be the Alfvén shock Ardavan (1976) envisages, because his is much nearer the light cylinder. Having built up such a splendid light cylinder edifice as I have in the past few paragraphs, I hesitate to accept his result on the location of the shock.

4.5 Drifting Subpulses

Drifting subpulses can be seen in this picture as collective motions of the emitting bunches after passing through the shock (a sort of current along the field lines) or as a wave crest which the bunches pass through and get compressed into smaller regions of space. I prefer the second interpretation, for it explains the lack of symmetry in drift directions, whereas particle motions should be symmetric in this picture. In either case the polarization behavior will follow the emitting regions, to be determined by the local apparent field line direction. Thus, the polarization behavior will drift with the subpulses, as observed[9]!

4.6 Beaming in the Emission Frame

Incidentally, there is good evidence that the assumption we have been making of isotropic emission in the corotating frame is not strictly valid. From polarization fits to individual pulses (as in Ferguson, 1976b and Ferguson and Seiradakis, 1978), it is often found that the pulse peak does not come exactly at the tangential point. For micropulses, this can hardly change things much, for the relativistic beaming is so strong. For subpulses, however, the amount and direction of the intrinsic beaming can produce asymmetric pulses, change the slope of drifting subpulse bands, and give misleading values of γ for the emission regions. Polarization model fits may be better for defining these things than intensity measurements.

5. TESTS OF THE MODELS

There are many quantitative tests of the ideas in this paper (perhaps too many)! For instance, if subpulse and micropulse widths depend mainly on their corotation γs, we will expect the number of pulsars showing microstructure to be less than the number showing subpulses by about the ratio of γs. (We see either type of emission only when $\sin T \cong T < 1/\gamma$). For PSR 1133+16, the subpulse γ is about 2.4 and the micropulse γ is about 6.0. If this is typical, we would expect $2.4/6.0 \cong 40\%$ of pulsars to have microstructure. We must wait for the statistics to come in.

The prediction that the subpulse polarization changes will become narrower at higher frequencies awaits testing by multifrequency polarization observations. Since the polarization is only affected by aberration, whereas the pulse intensity involves both aberration and the Doppler shift, we may expect to see the pulse width contract more rapidly with frequency than the polarization.

I am happy to say that one prediction of the model has already been confirmed in a number of pulsars. This is that for a given pulsar, subpulses of higher intensity are narrower, on the average (Bartel et al., 1980). The same should hold true for micropulses, until the point is reached where $1/\gamma$ becomes less than sin T. It would be interesting to have plots of microstructure width versus intensity for a sample of pulsars.

A method of determining true pulsar radial velocities would be most welcome. In the relativistic beaming theory, we are within about 25° of the rotation equators of most pulsars, implying that if they are moving along their rotation axes (Tademaru, 1977 and Morris et al., 1979) their radial velocities should be much less than their tangential velocities.

If we could resolve the pulsar magnetosphere we could "see" the motion of pulse emission regions as super-light velocities perpendicular to the line of sight (Lyne, 1971). This is because the same criterion holds for apparent super-light velocities as for seeing the pulses themselves - the velocity vector must be within $1/\gamma$ of the line of sight.

The imaginative observer, having understood this paper, will no doubt think of many more observational tests. We beg that he interprets his results in the same imaginative frame of mind, and makes an honest attempt to see how they can fit into a relativistic beaming theory which explains much, but is probably also greatly in need of modification.

NOTES

1. For the Crab pulsar, see Zheleznyakov and Shaposhnikov (1972), Ferguson et al. (1974), Cocke and Ferguson (1974) and Benford (1975). The Vela pulsar is discussed by Pacini (1971), Wallace et al. (1977) and Davila et al. (1980).

2. See also Endean (1972 and 1976), Henriksen and Norton (1975), and Kuo-Petravic et al. (1975).

3. This was first realized by Terrel (1959!).

4. The model of Mestel et al. (1979), for instance, has emission taking place just outside the light cylinder where corotation may only be partial.

5. Notable is a paper by Endean (1980), carrying corotation to its logical extreme.

6. I believe that the magnetic field strengths may be uncertain by an order of magnitude. See Malov and Malofeev (1977) for more details.

7. This is well discussed in Smith (1973 and 1977).

8. It is intriguing that a model fit to subpulse widths gives nearly identical values of T and β as are found from polarization model fits for the pulsars for which we have obtained relativistic beaming model fits. This shows that the parts of the model dealing with intensity and with polarization are consistent.

9. For a different light cylinder interpretation, see Zheleznyakov (1971) as supported by Krishnamohan (1980).

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DISCUSSION

HEWISH: In light cylinder models having emitting sources at different longitudes it is possible that intensity modulation is simultaneous in the rest frame of the sources. Non-simultaneity in the observer's frame might then lead to observable time-asymmetry effects. The mean pulse profile computed for pulses that follow nulls might show significant differences from the overall mean profile.

FERGUSON: I agree that such effects should be looked for. However, I am not so sure that the intensity modulation would be simultaneous in the rest frame of the sources if they were at different longitudes.

BARTEL: You probably can explain the occurrance of subpulses in the binary pulsar PSR 1913+16 despite the large angle of 45° between the line of sight and the orbital plane. Do you predict that microstructure does not exist in PSR 1913+16 emission?

FERGUSON: That prediction may be premature. If the pulsar rotation axis is precessing perpendicular to our line of sight at the moment, the lack of change in the pulse width puts no restriction on the angle the rotation axis makes with the line of sight. If we are almost in the pulsar equator, microstructure may exist in PSR 1913+16.

FOWLER: How does your light cylinder model account for the apparent memory mechanism present during radio nulls?

FERGUSON: I believe the location of emission at a given time may be due to the location of nodes or maxima in a plasma wave near the light cylinder, especially in pulsars with quasi-periodic microstructure or marching subpulses. Presumably then, when a null occurs, the wave may become a standing wave or only slowly moving wave because of a change in particle density. A close look at plasma waves possible at the light cylinder would be extremely useful in further developing the model.