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# THE $\aleph_1$-PRODUCT OF DG-INJECTIVE COMPLEXES

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*Abstract* Given a left Noetherian ring $R$, we give a necessary and sufficient condition in order that a complex of $R$-modules be DG-injective. Using this result we prove that if $(K_i)_{i\in I}$ is a family of DG-injective complexes of left $R$-modules and $K$ is the $\aleph_1$-product of $(K_i)_{i\in I}$ (i.e. $K \subset \prod_{i\in I} K_i$ is such that, for each $n$, $K^n \subset \prod_{i\in I} K_i^n$ consists of all $(x_i)_{i\in I}$ such that $\{i \mid x_i \neq 0\}$ is at most countable), then $K$ is DG-injective.

*Keywords:* DG-injective complexes; $\aleph$-products; exact precover

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## 1. Introduction

A ring $R$ is left Noetherian if and only if the direct sum of any family of injective left $R$-modules is injective. This result of Bass [**2**] led to a series of similar closure questions concerning classes of modules and complexes of modules.

Perhaps surprisingly, we were able to prove that over a left Noetherian ring $R$ it is not true that the direct sum of DG-injective complexes is DG-injective [**7**]. This means that the class of DG-injective complexes is not closed under $\aleph_0$-products.

In this paper we prove that over such a left Noetherian ring $R$ the class of DG-injective complexes is closed under $\aleph_1$-products.

We start by giving necessary and sufficient conditions for a complex to be DG-injective. We prove first (Proposition 2.10) that over any ring $R$ a complex $K$ is DG-injective if and only if $\mathrm{Ext}^1(M, K) = 0$ for every bounded-above exact complex $M$.

We use this result to show (Proposition 2.11) that if $R$ is a left Noetherian ring, then a complex $K$ is DG-injective if and only if $\mathrm{Ext}^1(M, K) = 0$ for any bounded-above exact complex $M$ with each module $M_n$ finitely generated.

Using the previous result we show that, if $R$ is a left Noetherian ring, $(K_i)_{i\in I}$ is a family of DG-injective complexes of left $R$-modules and $K$ is the $\aleph_1$-product of $(K_i)_{i\in I}$, then $K$ is DG-injective.

## 2. Preliminaries

Let $R$ be any ring. A (chain) complex $C$ of $R$-modules is a sequence

$$C = \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \to \cdots$$

of $R$-modules and $R$-homomorphisms such that $\partial_{n-1} \circ \partial_n = 0$ for all $n \in \mathbb{Z}$.

A chain complex of the form

$$C = \cdots C^{-2} \xrightarrow{\partial^{-2}} C^{-1} \xrightarrow{\partial^{-1}} C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} C^2 \to \cdots$$

is called a cochain complex. In this case, $\partial^{n+1} \circ \partial^n = 0$ for all $n \in \mathbb{Z}$. We note that a cochain complex is simply a chain complex with $C^i$ replaced by $C_{-i}$ and $\partial^i$ by $\partial_{-i}$. So, for example, it is more convenient to write a complex

$$0 \to C_0 \to C_{-1} \to C_{-2} \to \cdots$$

as

$$0 \to C^0 \to C^1 \to C^2 \to \cdots .$$

Throughout the paper we use both the subscript notation for complexes and the superscript notation.

When we use superscripts for a complex we will use subscripts to distinguish complexes: for example, if $(K_i)_{i \in I}$ is a family of complexes, then $K_i^n$ denotes the degree-$n$ term of the complex $K_i$.

If $X$ and $Y$ are both complexes of left $R$-modules, then $\mathcal{H}\mathsf{om}(X, Y)$ denotes the complex with

$$\mathcal{H}\mathsf{om}(X, Y)_n = \prod_{q=p+n} \operatorname{Hom}_R(X_p, Y_q)$$

and with differential given by $\partial(f) = \partial \circ f - (-1)^n f \circ \partial$, for $f \in \mathcal{H}\mathsf{om}(X, Y)_n$.

We use the terminology of [**1**] and say that a complex $I$ is DG-injective if each $I^n$ is injective and if $\mathcal{H}\mathsf{om}(E, I)$ is exact for any exact complex $E$.

Throughout the paper, $\operatorname{Hom}(X, Y)$ denotes the set of morphisms from $X$ to $Y$ in the category of complexes, and the $\operatorname{Ext}^i(X, Y)$ are the right-derived functors of Hom.

**Proposition 2.1 (see Proposition 3.4 in [5]).** *A complex $I$ is DG-injective if and only if $\operatorname{Ext}^1(E, I) = 0$ for any exact complex $E$.*

**Definition 2.2 (see p. 35 in [5]).** A DG-injective complex

$$I = \cdots \to I^{n-1} \xrightarrow{g_{n-1}} I^n \xrightarrow{g_n} I^{n+1} \to \cdots$$

is said to be minimal DG-injective if, for each $n$, $\operatorname{Ker} g_n$ is essential in $I^n$.

**Proposition 2.3 (see Proposition 3.16 in [5]).** *A DG-injective complex is the direct sum of an injective complex and a minimal DG-injective complex. This direct sum decomposition is unique up to isomorphism.*

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**Definition 2.4 (see [5]).** A morphism of complexes $\Phi : E \to X$ is an exact precover of $X$ if $E$ is exact and if $\mathrm{Hom}(F, E) \to \mathrm{Hom}(F, X)$ is surjective for any exact complex $F$.

If, moreover, any $f : E \to E$ such that $\Phi = \Phi \circ f$ is an automorphism of $E$, then $\Phi : E \to X$ is called an exact cover of $X$.

**Theorem 2.5 (see Theorem 3.18 in [5]).** *Every complex $X$ has an exact cover $E \to X$. A morphism $E \to X$ of complexes is an exact cover of $X$ if and only if $E$ is exact, $E \to X$ is surjective and $\mathrm{Ker}(E \to X)$ is a minimal DG-injective complex. If $E \to X$ is an exact cover, $E$ is injective if and only if $X$ is DG-injective.*

We recall that, for any $n$ and for any complex $X$, $X[n]$ denotes the complex such that $X[n]^m = X^{n+m}$ and whose boundary operators are $(-1)^n \partial^{n+m}$.

If $f : X \to Y$ is a morphism of complexes, then there is an exact sequence $0 \to Y \to M(f) \to X[1] \to 0$ with $M(f)$ the associated mapping cone ($M(f)^n = X^{n+1} \oplus Y^n$ and $\partial(x, y) = (-\partial x, f(x) + \partial y)$ for $(x, y) \in X^{n+1} \oplus Y^n$).

**Lemma 2.6 (see Lemma 3.21 in [5]).** *Let $I$ be a DG-injective complex and let $\mathrm{Id} : I \to I$ give the exact sequence $0 \to I \to M(\mathrm{Id}) \to I[1] \to 0$. Then $M(\mathrm{Id})$ is injective and $M(\mathrm{Id}) \to I[1]$ is an exact precover. If $I$ is minimal, then $I \to M(\mathrm{Id})$ is an injective envelope and $M(\mathrm{Id}) \to I[1]$ is an exact cover.*

**Lemma 2.7.** *If $(K_i)_{i \in I}$ is a family of DG-injective complexes, then $\prod_{i \in I} K_i$ is DG-injective.*

**Proof.** Let $M$ be an exact complex.

Each $K_i$ is DG-injective, so by Proposition 2.1 we have $\mathrm{Ext}^1(M, K_i) = 0$ for any $i \in I$. Since

$$\mathrm{Ext}^1\left(M, \prod_{i \in I} K_i\right) \simeq \prod_{i \in I} \mathrm{Ext}^1(M, K_i) = 0$$

for any exact complex $M$, it follows (by Proposition 2.1) that $\prod_{i \in I} K_i$ is DG-injective. $\square$

**Definition 2.8.** Given an ordinal number $\lambda$ and a family $(M_\alpha)_{\alpha < \lambda}$ of subcomplexes of a complex $M$, we say that the family is a continuous chain of subcomplexes if $M_\alpha \subset M_\beta$ whenever $\alpha \leqslant \beta < \lambda$ and if $M_\beta = \bigcup_{\alpha < \beta} M_\alpha$ whenever $\beta < \lambda$ is a limit ordinal. A family $(M_\alpha)_{\alpha \leqslant \lambda}$ is called a continuous chain if $(M_\alpha)_{\alpha < \lambda + 1}$ is one such.

A similar argument to that in the proof of Theorem 7.3.4 in [**4**] gives the following useful result.

**Theorem 2.9.** *Let $M$ and $N$ be complexes of left $R$-modules and suppose that $M$ is the union of a continuous chain of complexes $(M_\alpha)_{\alpha < \lambda}$. Then, if $\mathrm{Ext}^1(M_0, N) = 0$ and $\mathrm{Ext}^1(M_{\alpha+1}/M_\alpha, N) = 0$ whenever $\alpha + 1 < \lambda$, then $\mathrm{Ext}^1(M, N) = 0$.*

Our first result is the following proposition.

**Proposition 2.10.** *For any ring $R$, a complex $K$ is DG-injective if and only if $\mathrm{Ext}^1(F, K) = 0$ for every exact complex $F = \cdots \to F_2 \to F_1 \to F_0 \to 0$.*

**Proof.** ($\Leftarrow$.) Let

$$F = \cdots \to F_2 \xrightarrow{h_2} F_1 \xrightarrow{h_1} F_0 \xrightarrow{h_0} F^1 \xrightarrow{\lambda_1} F^2 \xrightarrow{\lambda_2} \cdots$$

be any exact complex. For any $n \geqslant 0$ let

$$\bar{F}_n = \cdots \to F_1 \xrightarrow{h_1} F_0 \xrightarrow{h_0} \cdots \to F^n \xrightarrow{\lambda_n} M_n \to 0,$$

where $M_n = \operatorname{Ker} \lambda_{n+1}$.

We have $\bar{F}_n \subset \bar{F}_{n+1}$, and

$$\bar{F}_{n+1}/\bar{F}_n \simeq 0 \to M_{n+1} \xrightarrow{\mathrm{Id}} M_{n+1} \to 0, \quad \text{for any } n \geqslant 0.$$

$F$ is the union of the continuous chain of complexes $(\bar{F}_n)_{n \geqslant 0}$.

By hypothesis, $\operatorname{Ext}^1(\bar{F}_0, K) = 0$ and $\operatorname{Ext}^1(\bar{F}_{n+1}/\bar{F}_n, K) = 0$ for any $n \geqslant 0$. By Theorem 2.9, we have that $\operatorname{Ext}^1(F, K) = 0$.

Since $\operatorname{Ext}^1(F, K) = 0$ for any exact complex $F$, it follows that $K$ is DG-injective (Proposition 2.1).

($\Rightarrow$.) Since $K$ is DG-injective, we have $\operatorname{Ext}^1(F, K) = 0$ for any exact complex $F$. $\square$

Using this result we can prove the following.

**Proposition 2.11.** *Let $R$ be a left Noetherian ring. A complex $K$ is DG-injective if and only if $\operatorname{Ext}^1(M, K) = 0$ for any exact complex*

$$M = \cdots \to M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 \to 0,$$

*with each $M_n$ finitely generated.*

**Proof.** ($\Leftarrow$.) Let

$$M = \cdots \to M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 \to 0$$

be any exact complex. Let $n \geqslant 0$ and let $x \in M_n$. We can easily see that there is an exact subcomplex $S \subset M$ such that $x \in S_n$ and such that $S_i$ is finitely generated for each $i \geqslant 0$. For let $S_n = Rx$ and let $S_{n-1} = f_n(Rx)$ if $n \geqslant 1$ and of course $S_{n-1} = S_{-1} = 0$ if $n = 0$. Then let $S_i = 0$ for $i \leqslant n-2$. Then choose $S_{n+1} \subset M_{n+1}$ finitely generated and such that $f_{n+1}(S_{n+1}) = \operatorname{Ker} f_n \cap S_n$. Then, in a similar manner, let $S_{n+2} \subset M_{n+2}$ be finitely generated and such that $f_{n+2}(S_{n+2}) = \operatorname{Ker} f_{n+1} \cap S_{n+1}$. Proceeding in this manner we get the desired exact subcomplex $S = \cdots \to S_2 \to S_1 \to S_0 \to 0$.

Now let $y \in M_m$ for some $m \geqslant 0$. Consider the quotient complex $M/S$ (which is also exact) and the element $y + S_m \in (M/S)_m = M_m/S_m$. By the argument above there is an exact subcomplex $T/S \subset M/S$ with $y + S_m \in (T/S)_m = T_m/S_m$ and such that each $(T/S)_i$ is finitely generated. But then each $T_i$ is also finitely generated. So we have $S \subset T$ with both $S$ and $T$ bounded-above exact complexes of finitely generated modules and such that $x \in T_n$, $y \in T_m$.

Using this procedure we see that we can write $M$ as the union of a continuous chain $(M^\alpha)_{\alpha<\lambda}$ of complexes for some ordinal $\lambda$ such that each of $M^0$ and $M^{\alpha+1}/M^\alpha$ for

$\alpha+1 < \lambda$ is a bounded-above exact complex of finitely generated modules. By hypothesis, $\mathrm{Ext}^1(M^0, K) = 0$ and $\mathrm{Ext}^1(M^{\alpha+1}/M^{\alpha}, K) = 0$ whenever $\alpha + 1 < \lambda$. By Theorem 2.9, we have that $\mathrm{Ext}^1(M, K) = 0$.

Since $\mathrm{Ext}^1(M, K) = 0$ for any bounded-above exact complex $M$, it follows that $K$ is DG-injective (Proposition 2.10).

($\Rightarrow$.) Since $K$ is DG-injective, we have $\mathrm{Ext}^1(M, K) = 0$ for any exact complex $M$. $\square$

## 3. The $\aleph_1$-product of a family of DG-injective complexes over a left Noetherian ring is DG-injective

Let $(K_i)_{i\in I}$ be a family of complexes of $R$-modules ($R$ any ring). The $\aleph_1$-product of $(K_i)_{i\in I}$ is the complex $K$ such that, for each $n$, $K^n \subset \prod_{i\in I} K_i^n$ consists of all $(x_i)_{i\in I}$ such that $\mathrm{Card}\{i \mid x_i \neq 0\} < \aleph_1$, that is, $\{i \mid x_i \neq 0\}$ is at most countable. We will denote this product by $\prod_{\aleph_1} K_i$.

We use the following result to prove that the $\aleph_1$-product of a family of DG-injective complexes is still DG-injective.

**Lemma 3.1.** *If $M$ is a finitely generated left $R$-module, then* $\mathrm{Hom}(M, \prod_{\aleph_1} K_i) \simeq \prod_{\aleph_1} \mathrm{Hom}(M, K_i)$ *for any family $(K_i)_{i\in I}$ of left $R$-modules.*

**Proof.** Let

$$T : \mathrm{Hom}\left(M, \prod_{\aleph_1} K_i\right) \to \prod_{i\in I} \mathrm{Hom}(M, K_i).$$

$T(\alpha) = (\alpha_i)_{i\in I}$ with $\alpha_i = p_i \circ \alpha$, where $p_j : \prod_{i\in I} K_i \to K_j$, $p_j((x_i)_i) = x_j$.

Let $\{x^1, \dots, x^n\}$ be a set of generators of $M$. Let $\alpha(x^l) = (x_i^l)_{i\in I}$.

Since $\{j \mid x_j^l \neq 0\}$ is at most countable for each $l \in \{1, 2, \dots, n\}$, it follows that

$$\bigcup_{l=1}^{n} \{j \mid x_j^l \neq 0\}$$

is at most countable. This implies that $\{j \mid \alpha_j \neq 0\}$ is at most countable.

So $T(\alpha) = (\alpha_i)_i \in \prod_{\aleph_1} \mathrm{Hom}(M, K_i)$.

(i) If $T(\alpha) = 0$, then $\alpha_i = 0$ for any $i \in I$, so $\alpha = 0$. Hence $T$ is injective.

(ii) Let $(\alpha_i)_{i\in I} \in \prod_{\aleph_1} \mathrm{Hom}(M, K_i)$ and let $\alpha = (\alpha_i)_{i\in I}$.

Since $\{i \mid \alpha_i \neq 0\}$ is at most countable, we have that $\alpha(x) = (\alpha_i(x))_i \in \prod_{\aleph_1} K_i$, for any $x \in M$.

So $\alpha \in \mathrm{Hom}(M, \prod_{\aleph_1} K_i)$ and $T(\alpha) = (\alpha_i)_{i\in I}$. Hence $T$ is surjective.

(iii) Since

$$T(\alpha + \beta) = (p_i \circ (\alpha + \beta))_i = (p_i \circ \alpha)_i + (p_i \circ \beta)_i = T(\alpha) + T(\beta),$$

for any $\alpha, \beta \in \mathrm{Hom}(M, \prod_{\aleph_1} K_i)$, it follows that $T$ is a homomorphism.

So $\mathrm{Hom}(M, \prod_{\aleph_1} K_i) \simeq \prod_{\aleph_1} \mathrm{Hom}(M, K_i)$. $\square$

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**Proposition 3.2.** *Let $M = \cdots \to M_2 \to M_1 \to M_0 \to 0$ be an exact complex with each $M_n$ a finitely generated left $R$-module.*

*We have $\operatorname{Hom}(M, \prod_{\aleph_1} K_i) \simeq \prod_{\aleph_1} \operatorname{Hom}(M, K_i)$ for any family $(K_i)_{i\in I}$ of complexes of left $R$-modules.*

**Proof.** Let $p_i : \prod_{\aleph_1} K_j \to K_i$, $p_i^n((x_j)_j) = x_i$, $\forall (x_j)_j \in \prod_{\aleph_1}(K_j^n)$. Let $T : \operatorname{Hom}(M, \prod_{\aleph_1} K_i) \to \prod_{i\in I} \operatorname{Hom}(M, K_i)$ be defined by $T(\alpha) = (\alpha_i)_i$ with $\alpha_i = p_i \circ \alpha$, for each $i \in I$. Then $\alpha_i^n = p_i^n \circ \alpha^n$ for any $i \in I$ and and $n \geqslant 0$:

$$
\begin{tikzcd}
M = \cdots \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\alpha^2"] & M_1 \arrow[r] \arrow[d, "\alpha^1"] & M_0 \arrow[r] \arrow[d, "\alpha^0"] & 0 \arrow[d] & \\
\prod_{\aleph_1} K_j = \cdots \arrow[r] \arrow[d, "p_i"] & \prod_{\aleph_1} K_j^2 \arrow[r] \arrow[d, "p_i^2"] & \prod_{\aleph_1} K_j^1 \arrow[r] \arrow[d, "p_i^1"] & \prod_{\aleph_1} K_j^0 \arrow[r] \arrow[d, "p_i^0"] & \prod_{\aleph_1} K_j^{-1} \arrow[r] \arrow[d, "p_i^{-1}"] & \cdots \\
K_i = \cdots \arrow[r] & K_i^2 \arrow[r] & K_i^1 \arrow[r] & K_i^0 \arrow[r] & K_i^{-1} \arrow[r] & \cdots
\end{tikzcd}
$$

Since each $M_n$ is finitely generated, we have

$$\operatorname{Hom}\left(M_n, \prod_{\aleph_1} K_i^n\right) \simeq \prod_{\aleph_1} \operatorname{Hom}(M_n, K_i^n) \quad \text{(by Lemma 3.1)}.$$

So for each $n \geqslant 0$ we have that the set $\{i \mid \alpha_i^n \neq 0\}$ is at most countable.

Then, since $\alpha_i = (\alpha_i^n)_n$, we have that

$$\{i \mid \alpha_i \neq 0\} = \bigcup_{n\geqslant 0} \{i \mid \alpha_i^n \neq 0\}.$$

Since a countable union of countable sets is still countable, it follows that $\{i \mid \alpha_i \neq 0\}$ is at most countable.

Hence $(\alpha_i)_i \in \prod_{\aleph_1} \operatorname{Hom}(M, K_i)$.

Thus $T : \operatorname{Hom}(M, \prod_{\aleph_1} K_i) \to \prod_{\aleph_1} \operatorname{Hom}(M, K_i)$.

(i) If $(\alpha_i)_i \in \prod_{\aleph_1} \operatorname{Hom}(M, K_i)$, then $\alpha = (\alpha_i)_i \in \operatorname{Hom}(M, \prod_{\aleph_1} K_i)$ and $T(\alpha) = (\alpha_i)_i$.

(ii) $T(\alpha) = 0 \Leftrightarrow \alpha_i = 0$, $\forall i \in I \Leftrightarrow \alpha = 0$.

(iii) $T(\alpha + \beta) = (p_i \circ (\alpha + \beta))_i = (p_i \circ \alpha)_i + (p_i \circ \beta)_i = T(\alpha) + T(\beta)$ for any $\alpha, \beta \in \operatorname{Hom}(M, \prod_{\aleph_1} K_i)$.

So $T$ is an isomorphism. □

**Lemma 3.3.** *If*

$$0 \to A_i \xrightarrow{h_i} B_i \xrightarrow{l_i} C_i \to 0$$

*is an exact sequence of left $R$-modules for each $i \in I$, then the sequence*

$$0 \to \prod_{\aleph_1} A_i \xrightarrow{h} \prod_{\aleph_1} B_i \xrightarrow{l} \prod_{\aleph_1} C_i \to 0$$

*is exact (with $h = \prod_{i\in I} h_i|_{\prod_{\aleph_1} A_i}$ and $l = \prod_{i\in I} l_i|_{\prod_{\aleph_1} B_i}$).*

**Proof.** (i) Since $l \circ h((x_i)_i) = (l_i(h_i(x_i)))_i = 0$ for any $x = (x_i)_{i\in I} \in \prod_{\aleph_1} A_i$, it follows that $\operatorname{Im} h \subset \operatorname{Ker} l$.

(ii) Let $x = (x_i)_{i\in I} \in \operatorname{Ker} l \subset \prod_{\aleph_1} B_i$. For each $i \in I$ we have $x_i \in \operatorname{Ker} l_i = \operatorname{Im} h_i$, so $x_i = h_i(y_i)$ for some $y_i \in A_i$. Since $y_i \neq 0$ implies $x_i \neq 0$ (because $h_i$ is an injection) and $\{i \mid x_i \neq 0\}$ is at most countable, it follows that $\{i \mid y_i \neq 0\}$ is at most countable. Thus $x = (h_i(y_i))_i = h(y)$ with $y = (y_i)_i \in \prod_{\aleph_1} A_i$. Hence $\operatorname{Ker} l \subset \operatorname{Im} h \subset \operatorname{Ker} l$.

(iii) Since each $h_i$ is an injection, it follows that $h$ is an injection.

(iv) Let $y = (y_i)_{i\in I} \in \prod_{\aleph_1} C_i$. By hypothesis, for each $i \in I$ there is $x_i \in B_i$ such that $y_i = l_i(x_i)$. Let $z = (z_i)_{i\in I} \in \prod_{i\in I} B_i$ with $z_i = x_i$ if $y_i \neq 0$ and $z_i = 0$ if $y_i = 0$. Then $l(z) = y$. Since $\{i \mid y_i \neq 0\}$ is at most countable, we have that $\{i \mid z_i \neq 0\}$ is at most countable. So $z \in \prod_{\aleph_1} B_i$ and $l(z) = y$. Thus $l$ is surjective. □

**Corollary 3.4.** *If $(D_i)_{i\in I}$ is a family of exact complexes of left $R$-modules, then $\prod_{\aleph_1} D_i$ is an exact complex.*

**Corollary 3.5.** *Let $R$ be a left Noetherian ring. If $(E_i)_{i\in I}$ is a family of injective left $R$-modules, then $\prod_{\aleph_1} E_i$ is an injective left $R$-module.*

**Proof.** Let $J$ be a left ideal of $R$. The sequence $0 \to J \to R \to R/J \to 0$ is exact and each $E_i$ is injective, so the sequence

$$0 \to \operatorname{Hom}(R/J, E_i) \to \operatorname{Hom}(R, E_i) \to \operatorname{Hom}(J, E_i) \to 0$$

is exact for any $i \in I$. By Lemma 3.3, the sequence

$$0 \to \prod_{\aleph_1} \operatorname{Hom}(R/J, E_i) \to \prod_{\aleph_1} \operatorname{Hom}(R, E_i) \to \prod_{\aleph_1} \operatorname{Hom}(J, E_i) \to 0$$

is exact. Since $R$ is left Noetherian, both $J$ and $R/J$ are finitely generated left $R$-modules. Then by Lemma 3.1 we have that

$$\prod_{\aleph_1} \operatorname{Hom}(R/J, E_i) \simeq \operatorname{Hom}\Bigl(R/J, \prod_{\aleph_1} E_i\Bigr), \qquad \prod_{\aleph_1} \operatorname{Hom}(R, E_i) \simeq \operatorname{Hom}\Bigl(R, \prod_{\aleph_1} E_i\Bigr),$$
$$\prod_{\aleph_1} \operatorname{Hom}(J, E_i) \simeq \operatorname{Hom}\Bigl(J, \prod_{\aleph_1} E_i\Bigr).$$

Hence the sequence

$$0 \to \operatorname{Hom}\Bigl(R/J, \prod_{\aleph_1} E_i\Bigr) \to \operatorname{Hom}\Bigl(R, \prod_{\aleph_1} E_i\Bigr) \to \operatorname{Hom}\Bigl(J, \prod_{\aleph_1} E_i\Bigr) \to 0$$

is exact for any left ideal $J$ of $R$. By [**4**, Theorem 3.1.3], the module $\prod_{\aleph_1} E_i$ is injective. □

We can now prove that over a left Noetherian ring $R$ an $\aleph_1$-product of DG-injective complexes is still DG-injective.

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**Proposition 3.6.** *If $R$ is a left Noetherian ring, $(K_i)_{i\in I}$ is a family of DG-injective complexes and $K = \prod_{\aleph_1} K_i$, then $K$ is a DG-injective complex.*

**Proof.** Since $K_i$ is DG-injective, there is an exact sequence $0 \to D_i \to E_i \to K_i \to 0$ with $E_i$ injective, $D_i$ DG-injective and such that $\mathrm{Hom}(M, -)$ leaves the sequence exact for any exact complex $M$ (Lemma 2.6).

By Proposition 2.11, it suffices to show that $\mathrm{Ext}^1(M, K) = 0$ for a bounded-above exact complex of finitely generated modules $M = \cdots \to M_1 \to M_0 \to 0$.

Let $M$ be such an exact complex.

By Lemma 2.6, the sequence $0 \to \mathrm{Hom}(M, D_i) \to \mathrm{Hom}(M, E_i) \to \mathrm{Hom}(M, K_i) \to 0$ is exact for each $i \in I$. By Lemma 3.3, we have an exact sequence:

$$0 \to \prod_{\aleph_1} \mathrm{Hom}(M, D_i) \to \prod_{\aleph_1} \mathrm{Hom}(M, E_i) \to \prod_{\aleph_1} \mathrm{Hom}(M, K_i) \to 0.$$

Since each $M_n$ is finitely generated ($n \geqslant 0$) we have

$$\prod_{\aleph_1} \mathrm{Hom}(M, T_i) \simeq \mathrm{Hom}\left(M, \prod_{\aleph_1} T_i\right)$$

for any family of complexes $(T_i)_{i\in I}$ (by Proposition 3.2).

Thus we have an exact sequence

$$0 \to \mathrm{Hom}\left(M, \prod_{\aleph_1} D_i\right) \to \mathrm{Hom}\left(M, \prod_{\aleph_1} E_i\right) \to \mathrm{Hom}\left(M, \prod_{\aleph_1} K_i\right) \to 0. \tag{3.1}$$

But the fact that $0 \to D_i \to E_i \to K_i \to 0$ is exact for each $i \in I$ implies that

$$0 \to \prod_{\aleph_1} D_i \to \prod_{\aleph_1} E_i \to \prod_{\aleph_1} K_i \to 0$$

is an exact sequence (Lemma 3.3). Consequently, we have an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}\left(M, \prod_{\aleph_1} D_i\right) &\to \mathrm{Hom}\left(M, \prod_{\aleph_1} E_i\right) \to \mathrm{Hom}\left(M, \prod_{\aleph_1} K_i\right) \\ &\to \mathrm{Ext}^1\left(M, \prod_{\aleph_1} D_i\right) \to \mathrm{Ext}^1\left(M, \prod_{\aleph_1} E_i\right) = 0, \end{aligned} \tag{3.2}$$

since $\prod_{\aleph_1} E_i$ is injective (see Corollaries 3.4 and 3.5 and Theorem 3.1.3 in [**6**]).

By (3.1) and (3.2), we have $\mathrm{Ext}^1(M, \prod_{\aleph_1} D_i) = 0$ for any exact bounded-above complex $M$ with each $M_n$ finitely generated. Consequently, $\prod_{\aleph_1} D_i$ is a DG-injective complex (by Proposition 2.11). Since

$$0 \to \prod_{\aleph_1} D_i \to \prod_{\aleph_1} E_i \to \prod_{\aleph_1} K_i \to 0$$

is exact and $\prod_{\aleph_1} D_i$, $\prod_{\aleph_1} E_i$ are DG-injective, it follows that $\prod_{\aleph_1} K_i$ is DG-injective (see the remark on p. 31 of [**5**]). □

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**Corollary 3.7.** *Let $R$ be a left Noetherian ring and let $(X_i)_{i\in I}$ be a family of complexes of left $R$-modules. If $E_i \to X_i$ is an exact cover for each $i \in I$, then $\prod_{\aleph_1} E_i \to \prod_{\aleph_1} X_i$ is an exact precover.*

**Proof.** By Theorem 2.5, $K_i = \mathrm{Ker}(E_i \to X_i)$ is a DG-injective complex for each $i \in I$. Since $0 \to K_i \to E_i \to X_i \to 0$ is exact for each $i \in I$, it follows that

$$0 \to \prod_{\aleph_1} K_i \to \prod_{\aleph_1} E_i \to \prod_{\aleph_1} X_i \to 0$$

is an exact sequence of complexes (Lemma 3.3).

By Corollary 3.4, $\prod_{\aleph_1} E_i$ is an exact complex.

Let $D$ be an exact complex. Since the sequence

$$0 \to \prod_{\aleph_1} K_i \to \prod_{\aleph_1} E_i \to \prod_{\aleph_1} X_i \to 0$$

is exact, we have an associated exact sequence

$$0 \to \mathrm{Hom}\left(D, \prod_{\aleph_1} K_i\right) \to \mathrm{Hom}\left(D, \prod_{\aleph_1} E_i\right) \to \mathrm{Hom}\left(D, \prod_{\aleph_1} X_i\right) \to \mathrm{Ext}^1\left(D, \prod_{\aleph_1} K_i\right) = 0$$

(since, by Proposition 3.6, $\prod_{\aleph_1} K_i$ is DG-injective).

Thus $\prod_{\aleph_1} E_i \to \prod_{\aleph_1} X_i$ is an exact precover. □

**Remark 3.8.** With the hypotheses of Corollary 3.7, $\prod_{\aleph_1} E_i \to \prod_{\aleph_1} X_i$ is not necessarily an exact cover.

For example, let $R = \mathbb{Z}$ and let

$$X = \cdots \to 0 \to \overset{0\text{th}}{\mathbb{Z}} \to 0 \to \cdots .$$

Since $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ is a minimal injective resolution of $\mathbb{Z}$, it follows that the complex

$$E = \cdots 0 \to 0 \to \overset{0\text{th}}{\mathbb{Z}} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \cdots$$

with the map $E \to X$, which is the identity on $\mathbb{Z}$, is an exact cover of $X$ (see the example on p. 37 of [**5**]).

Let $X_n = X$ for any $n \in \mathbb{N}$ and let $E_n = E$ for any $n \in \mathbb{N}$. Thus $E_n \to X_n$ is an exact cover for any $n \in \mathbb{N}$.

Suppose $\prod_{\aleph_1} E_n \to \prod_{\aleph_1} X_n$ is an exact cover, i.e. $\prod_{n=1}^{\infty} E_n \to \prod_{n=1}^{\infty} X_n$ is an exact cover. Then $\mathrm{Ker}(\prod_{n=1}^{\infty} E_n \to \prod_{n=1}^{\infty} X_n)$ is a minimal DG-injective complex, i.e. the complex

$$0 \to \prod_{n=1}^{\infty} \mathbb{Q}_n \to \prod_{n=1}^{\infty} (\mathbb{Q}/\mathbb{Z})_n \to 0$$

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is minimal DG-injective (with $\mathbb{Q}_n = \mathbb{Q}$ and $(\mathbb{Q}/\mathbb{Z})_n = \mathbb{Q}/\mathbb{Z}$ for any $n \in \mathbb{N}$). Consequently,

$$\prod_{n=1}^{\infty} \mathbb{Z}_n = \operatorname{Ker}\left(\prod_{n=1}^{\infty} \mathbb{Q}_n \to \prod_{n=1}^{\infty} (\mathbb{Q}/\mathbb{Z})_n\right)$$

is essential in $\prod_{n=1}^{\infty} \mathbb{Q}_n$ (with $\mathbb{Z}_n = \mathbb{Z}$ for any $n \in \mathbb{N}$).

Let $(x_n)_n = (1/p_n)_n$, where $p_n$ is prime, $p_n \neq p_k$ for $n \neq k$. Since $(x_n)_n \in \prod_{n=1}^{\infty} \mathbb{Q}_n$, $(x_n)_n \neq 0$ and $\prod_{n=1}^{\infty} \mathbb{Z}_n$ is essential in $\prod_{n=1}^{\infty} \mathbb{Q}_n$, there exists $r \in \mathbb{Z}$ such that $r(1/p_n) \in \mathbb{Z}$ for any $n \geqslant 1$ [**9**, Exercise 3.25], which means that $p_n \mid r$ for any prime number $p_n$. But $r \in \mathbb{Z}$ is the product of a finite number of primes. Contradiction.

**Remark 3.9.** Using a few elementary facts about cardinal arithmetic it is not hard to modify the arguments in this paper to prove that the $\aleph_\beta$-product of DG-injective complexes is DG-injective if $\beta > 0$ is not a limit ordinal (so $\beta = \alpha + 1$ for some $\alpha$) or if $\beta$ is a limit ordinal which is not cofinal with $\omega$. For the sake of simplicity we restricted ourselves to the $\aleph_1$ case.

**Remark 3.10.** For other results about $\aleph$-products see [**3**] and [**8**].

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