A DISCRETE METHOD FOR THE LOGARITHMIC-KERNEL INTEGRAL EQUATION ON AN OPEN ARC

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Abstract

Here we discuss the stability and convergence of a quadrature method for Symm's integral equation on an open smooth arc. The method is an adaptation of an approach considered by Sloan and Burn for closed curves. Before applying the quadrature scheme, we use a cosine substitution to remove the endpoint singularity of the solution. The family of methods includes schemes with any order $O(h^p)$ of convergence.

1. Introduction

Let Γ be a smooth open arc in the plane. We consider the numerical solution of the integral equation

$$-\frac{1}{\pi} \int_\Gamma v(y) \log|x - y| ds_y = g(x), \qquad x \in \Gamma. \tag{1.1}$$

Equation (1.1) arises for example in electrostatics, fluid dynamics and in simulation of cracks. Computationally efficient quadrature methods for the solution of (1.1) have recently been introduced and analysed in the case of a closed curve. In [13] Sloan and Burn introduced a family of "unconventional" quadrature methods. This approach was further developed and analysed by Saranen and Sloan [10]. A different quadrature method for a closed curve, involving subtraction of the singularity, was analysed by Saranen in [9]. On the other hand, quadrature methods for Cauchy-singular integral equations on open curves were investigated by Rathsfeld [8], by Prössdorf and Rathsfeld

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[5], and by Prössdorf and Sloan [7]. For some applications of (1.1), see e.g. [2].

Here we propose a quadrature method which is an adaptation of the quadrature method introduced by Sloan and Burn [13] for the closed-curve case. To be more precise, the quadrature method is here applied after the well-known cosine substitution. The cosine substitution was introduced already by Multhopp (1938) in [4] for the airfoil equation of Prandtl, see also [6, Section 3.4] and [11],[12],[16]. For Symm's equation (1.1) it was recently used by Yan and Sloan in [17]. It is well known that even for a smooth right-hand side g the solution v of (1.1) may have the singularity $O(|x - c|^{-1/2})$ at the endpoint c of the arc Γ. The cosine substitution removes this bad behaviour of the solution and therefore enables us to construct numerical schemes with good stability and convergence properties. As in the case of a smooth closed curve (see [10],[13]), the family of quadrature methods in this paper includes schemes of arbitrarily high, but fixed order $O(h^p)$ of convergence. In particular, we describe explicitly a simple method (see Example 4.1) which is stable and has the order $O(h^3)$ of convergence; the discretisation parameter h is inversely proportional to the number of the meshpoints on Γ.

The cosine substitution has already been used for numerical solution of (1.1) by Atkinson and Sloan in [1], by Joe and Yan in [3], and by Sloan and Stephan in [15]. Atkinson and Sloan discuss a completely discrete method which is based on a further discretisation of the Galerkin method with trigonometric trial functions. Joe and Yan, and Sloan and Stephan, analyse collocation methods, but only in the case of an open interval instead of a general arc. Other methods were described by Costabel et al. in [2] and by Sloan and Spence in [14]. All in all, from the above articles only the work [1] of Atkinson and Sloan (in addition to our paper) gives a fully discrete system of equations for a general open arc. Therefore we compare our result to that of [1] in some detail in Section 4.

2. Preliminaries

For the convenience of the reader, we use the notations of Yan and Sloan [17]. We assume that the arc Γ has the parametrisation

$$\nu(\xi) = (\nu_1(\xi), \nu_2(\xi)), \qquad -1 \leq \xi \leq 1, \qquad (2.1)$$

such that $\xi \mapsto \nu(\xi)$ is infinitely differentiable with $|\nu'(\xi)| > 0$, $-1 \leq \xi \leq 1$. As the only restriction on generality, we impose the condition that the capacity of the arc Γ differs from one. By this assumption (1.1) is uniquely

solvable. For a detailed discussion of the capacity we refer to Yan and Sloan [17]. We apply the cosine transformation given by $\xi = \cos t$, $0 \le t \le \pi$, and define $a : [0, \pi] \to \Gamma$ by $a(t) = \nu(\cos t)$. Now (1.1) becomes

$$-\frac{1}{\pi} \int_0^\pi u(\tau) \log |a(t) - a(\tau)| d\tau = f(t), \quad 0 \le t \le \pi, \tag{2.2}$$

where

$$\begin{cases} u(t) = v(a(t)) |\nu'(\cos t)| |\sin t|, \\ f(t) = g(a(t)). \end{cases} \tag{2.3}$$

By the formulae above, the functions $a(t)$, $u(t)$ and $f(t)$ may be considered as 2π-periodic even functions defined on the whole real line $\mathbb{R}$. For the following, it is convenient to consider (2.2) as an integral equation for 2π-periodic even functions u and f in the form

$$(L_e u)(t) = f(t), \quad -\pi \le t \le \pi, \tag{2.4}$$

where

$$(L_e u)(t) = -\frac{1}{2\pi} \int_{-\pi}^\pi u(\tau) \log |a(t) - a(\tau)| d\tau, \quad -\pi \le t \le \pi. \tag{2.5}$$

For the equivalence of (2.2) and (2.4), it is sufficient to observe that the kernel function

$$-\frac{1}{\pi} \log |a(t) - a(\tau)|$$

is also 2π-periodic and even with respect to both the arguments t and τ.

In our analysis of the numerical method, we shall make use of the decomposition

$$L_e u = A_e u + B_e u, \tag{2.6}$$

with

$$(A_e u)(t) = -\frac{1}{2\pi} \int_{-\pi}^\pi u(\tau) \log[2e^{-1} |\cos t - \cos \tau|] d\tau \tag{2.7}$$

and

$$(B_e u)(t) = \int_{-\pi}^\pi u(\tau) b_e(t, \tau) d\tau, \tag{2.8}$$

where

$$b_e(t, \tau) = \begin{cases} -\dfrac{1}{2\pi} \log \left| \dfrac{e}{2} \cdot \dfrac{a(t) - a(\tau)}{\cos t - \cos \tau} \right|, & t \ne -\tau, \tau \pmod{2\pi}, \\ -\dfrac{1}{2\pi} \log \left| \dfrac{e}{2} \nu'(\cos t) \right|, & t = -\tau, \tau \pmod{2\pi}. \end{cases} \tag{2.9}$$

We notice that $b_e(t, \tau)$ is a smooth function of t and τ on $\mathbb{R} \times \mathbb{R}$ and is 2π-periodic and even with respect to each variable, i.e. $b_e \in C_{2\pi,e}^\infty(\mathbb{R} \times \mathbb{R})$.

LEMMA 2.1. *If* u *is even then*

$$(A_e u)(t) = -\frac{1}{\pi} \int_{-\pi}^{\pi} u(\tau) \log \left| 2e^{-\frac{1}{2}} \sin \left(\frac{t-\tau}{2} \right) \right| d\tau. \qquad (2.10)$$

PROOF. Since

$$\cos t - \cos \tau = 2 \sin \frac{t+\tau}{2} \sin \frac{\tau - t}{2},$$

it follows from (2.7) that

$$\begin{aligned}
(A_e u)(t) = & -\frac{1}{2\pi} \int_{-\pi}^{\pi} u(\tau) \log \left[4e^{-1} \left| \sin \frac{t+\tau}{2} \sin \frac{t-\tau}{2} \right| \right] d\tau \\
= & -\frac{1}{2\pi} \int_{-\pi}^{\pi} u(\tau) \log \left| 2e^{-1/2} \sin \frac{t+\tau}{2} \right| d\tau \\
& -\frac{1}{2\pi} \int_{-\pi}^{\pi} u(\tau) \log \left| 2e^{-1/2} \sin \frac{t-\tau}{2} \right| d\tau.
\end{aligned}$$

On making the substitution $\tau = -\tau'$ in the first integral and then using the fact that u is even, the result follows immediately.

The operator A_e admits a canonical continuation A to the whole space of L^2-functions on $[-\pi, \pi]$, defined by

$$(Au)(t) = -\frac{1}{\pi} \int_{-\pi}^{\pi} u(\tau) \log \left| 2e^{-\frac{1}{2}} \sin \left(\frac{t-\tau}{2} \right) \right| d\tau. \qquad (2.11)$$

The operator A can be identified with Symm's integral operator on the circle with the radius $\rho = e^{-1/2}$. The aforementioned fact that the operators A_e and A coincide in the space of the 2π-periodic even functions provides the crucial point of our analysis (as indeed it does for most or all of the methods which employ the cosine transformation). More precisely, in the case where the arc Γ is the interval $[-2e^{-1}, 2e^{-1}]$, our method turns out to be equivalent to the method of Sloan and Burn [13] for the circle with the radius $\rho = e^{-1/2}$.

In the following, the general case of the operator L_e will be reduced to the case $L_e = A_e$ by using a perturbation argument.

The equation (2.4) will be considered in Sobolev spaces of 2π-periodic even functions. These spaces H_e^s, $s \in \mathbb{R}$, consist of 2π-periodic even functions u (more precisely, distributions)

$$u(t) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \hat{u}(m) e^{imt} \qquad (2.12)$$

satisfying

$$\|u\|_s := \left(|\hat{u}(0)|^2 + \sum_{m \neq 0} |m|^{2s} |\hat{u}(m)|^2 \right)^{1/2} < \infty, \qquad (2.13)$$

where

$$\hat{u}(m) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} u(t) e^{-imt} dt. \tag{2.14}$$

For $u \in H_e^s$ there is also the cosine type representation

$$u(t) = \sqrt{\frac{2}{\pi}} \left(\sum_{m=1}^{\infty} \hat{u}^c(m) \cos mt + \tfrac{1}{2} \hat{u}^c(0) \right) \tag{2.15}$$

with the Fourier cosine coefficients $\hat{u}^c(m)$ defined by

$$\hat{u}^c(m) = \sqrt{\frac{2}{\pi}} \int_0^{\pi} u(t) \cos mt \, dt. \tag{2.16}$$

It is easily seen that $\hat{u}(m) = \hat{u}^c(m)$ for $u \in H_e^s$ because u is even. The norm in H_e^s can therefore be expressed alternatively as

$$\|u\|_s = \left(|\hat{u}^c(0)|^2 + \sum_{m=1}^{\infty} |m|^{2s} |\hat{u}^c(m)|^2 \right)^{1/2}.$$

The usual Sobolev spaces H^s of 2π-periodic functions on $\mathbb{R}$ and the corresponding norms are defined by the relations (2.12)–(2.14) without the restriction that the distribution u is even. By our assumption on Γ, the operator L_e defines an isomorphism $L_e : H_e^s \to H_e^{s+1}$ for all $s \in \mathbb{R}$, see Yan and Sloan [17].

3. Description of the method

Our numerical method for the approximate solution of the equation

$$(L_e u)(t) = f(t), \qquad -\pi \le t \le \pi \tag{3.1}$$

(with f and u even functions) is closely related to the method proposed by Sloan and Burn in [13]. The method yields a fully discretised system of equations, where the unknowns are approximations of the solution $u(t)$ at the nodal points of a uniform mesh. Our method consists of a suitable discretisation of a certain Petrov-Galerkin approximation.

First we define a partition of the interval $I = [-\pi, \pi]$. For this let $N = 2L + 1$, $L \ge 1$, so that N is an odd integer. We define the uniform mesh $\{t_j\}_{j=0}^N$ by

$$t_j = -\pi + jh, \qquad 0 \le j \le N, \tag{3.2}$$

with the mesh parameter $h = 2\pi/N$.

Next we introduce the trapezoidal approximation $L_{e,h}$ of the operator L_e, defined by

$$(L_{e,h}u)(t) = -\frac{h}{2\pi}\sum_{j=0}^{N}{}'' \log|a(t) - a(t_j)|u(t_j). \tag{3.3}$$

The notation $\sum''$ in (3.3) means

$$\sum_{j=0}^{N}{}'' c_j = \tfrac{1}{2}(c_0 + c_N) + \sum_{j=1}^{N-1} c_j,$$

a widely used convention. In our Petrov-Galerkin method we use cosine functions as trial functions. First let T^h be the finite-dimensional space of all trigonometric polynomials of degree $\leq L$, that is

$$T^h = \left\{ v(t) = \sum_{-L}^{L} c_n e^{int} : c_n \in \mathbb{C} \right\}. \tag{3.4}$$

The space T_e^h of cosine functions is defined by

$$T_e^h = \left\{ v(t) = \sum_{n=0}^{L} a_n \cos nt : a_n \in \mathbb{C} \right\}. \tag{3.5}$$

Note that the space T_e^h can be characterised as the "even part" of T^h, i.e. the function v belongs to T_e^h if and only if v is even and $v \in T^h$. This property is one of the crucial points in our approach. Correspondingly, as test functions we use "even parts" of the usual 2π-periodic spline spaces S^d. To be more precise, we define

$$S^d = \{v : \mathbb{R} \to \mathbb{C}; v \text{ is } 2\pi\text{-periodic}, v|_{(t_j, t_{j+1})} \in \mathscr{P}^d, v \in C^{d-1}\}, \tag{3.6}$$

where $\mathscr{P}^d$ is the space of polynomials of degree d, and decompose (direct sum)

$$S^d = S_e^d \dotplus S_o^d, \tag{3.7}$$

where

$$S_e^d = \{v \in S^d : v \text{ is even}\}, \tag{3.8}$$

$$S_o^d = \{v \in S^d : v \text{ is odd}\}. \tag{3.9}$$

It is easy to see that the decomposition (3.7) really exists. Furthermore, the spaces in (3.7) have the dimensions

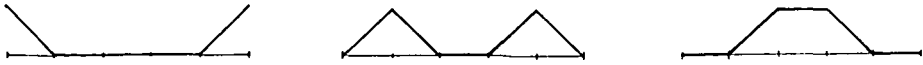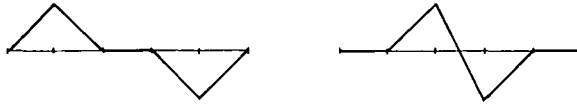$$\dim S^d = N, \quad \dim S_e^d = L + 1, \quad \dim S_o^d = L.$$

FIGURE 1

FIGURE 2

For our testspace we choose S_e^d. (Observe that the spaces T_e^h and S_e^d have the same dimension.) For the practical implementation of the method we need a concrete set of basis functions in S_e^d. Let us illustrate this point in the case of the piecewise linear continuous functions. Let $v_j \in S^1$ be the usual hat-functions (Courant's functions) satisfying

$$v_j(t_k) = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases} \tag{3.10}$$

The space S_e^1 has a basis $\{v_{j,e}\}_0^L$ given by

$$\begin{cases} v_{j,e} = v_j + v_{N-j}, & 1 \leq j \leq L, \\ v_{0,e} = v_0 \end{cases} \tag{3.11}$$

and the space S_o^1 has a basis $\{v_{j,o}\}_1^L$, where

$$v_{j,o} = v_j - v_{N-j}, \quad 1 \leq j \leq L. \tag{3.12}$$

Since $\{v_j\}_0^{N-1}$ is a basis of S^1, the decomposition (3.7) is also proved. For $L = 2$, the graphs of the functions $v_{j,e}$ and $v_{j,o}$ are shown in the Figures 1 and 2.

Our method is defined as a full discretisation of the following Petrov-Galerkin method: find $u_h \in T_e^h$ with

$$(L_e u_h, v) = (f, v), \quad v \in S_e^d. \tag{3.13}$$

To carry out the discretisation, we replace the operator L_e by the operator $L_{e,h}$ and use a discrete inner product instead of the L^2-inner product

$$(u, v) = \int_{-\pi}^{\pi} u(t)\overline{v(t)}dt.$$

Some notations are needed. In the interval $[-\frac{1}{2}, \frac{1}{2}]$, we fix a symmetric quadrature rule

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)dt \simeq \sum_{j=1}^{M} w_j(f(\eta_j) + f(-\eta_j)) =: Q(f), \tag{3.14}$$

with the points η_j such that $0 \le \eta_1 < \cdots < \eta_M < \frac{1}{2}$ and with the weights w_j satisfying

$$\sum_{j=1}^{M} w_j = \tfrac{1}{2}, \quad w_j > 0, \quad 1 \le j \le M. \tag{3.15}$$

The discrete inner product $(u, v)_h$ is given by

$$(u, v)_h = Q_h(u\bar{v}), \tag{3.16}$$

where Q_h is the composite quadrature rule for the interval $[-\pi, \pi]$,

$$Q_h(f) := h \sum_{k=0}^{N-1} \sum_{j=1}^{M} w_j[f((k + \tfrac{1}{2} + \eta_j)h - \pi) + f((k + \tfrac{1}{2} - \eta_j)h - \pi)].$$

Because the basic rule $Q(f)$ in (3.14) is symmetric, we observe that the composite rule $Q_h(f)$ is symmetric with respect to the origin $t = 0$. More precisely, $Q_h(f)$ is of the form (with a certain integer P)

$$Q_h(f) = h \sum_{l=1}^{P} \tilde{w}_l(f(\tau_l) + f(-\tau_l)) \tag{3.17}$$

where $0 \le \tau_l < \pi$, $1 \le l \le P$.

Now, our discretised method is defined as follows: find $u_h \in T_e^h$ such that

$$(L_{e,h}u_h, v)_h = (f, v)_h, \quad v \in S_e^d. \tag{3.18}$$

We briefly describe the matrix equation corresponding to (3.18). Having choosen a basis $v_{k,e}$, $0 \le k \le L$, of S_e^d, we find that (3.18) is equivalent to the $(L + 1) \times (L + 1)$ system of equations

$$\sum_{j=L+1}^{2L+1}{}^{*} a_{kj}u_h(t_j) = f_k, \quad 0 \le k \le L, \tag{3.19}$$

where

$$a_{kj} = -\frac{h}{\pi} \sum_{l=1}^{P} \tilde{w}_l \log|a(\tau_l) - a(t_j)|\overline{v_{k,e}(\tau_l)}, \tag{3.20}$$

$$f_k = \sum_{l=1}^{P} \tilde{w}_l f(\tau_l)\overline{v_{k,e}(\tau_l)}, \tag{3.21}$$

and where the asterisk in (3.19) indicates that the last term of the sum is to be halved. In deriving (3.20), (3.21) we have used the fact that all the functions appearing in (3.18) are even.

REMARK 3.1. It is worth pointing out that (3.18) really is a quadrature method. Namely, consider the quadrature method: find the values u_j, $L + 1 \leq j \leq 2L + 1$ such that

$$\sum_{j=L+1}^{2L+1} a_{kj} u_j = f_k, \quad 0 \leq k \leq L, \tag{3.22}$$

where the unknowns u_j can be considered as approximations $u_j \simeq u(t_j)$. However (3.18) is equivalent to (3.22), since the interpolation problem: for given numbers u_j, $L + 1 \leq j \leq 2L + 1$, find $u_h \in T_e^h$ such that

$$u_h(t_j) = u_j, \quad L + 1 \leq j \leq 2L + 1, \tag{3.23}$$

is uniquely solvable. The solution u_h of (3.23) is given by

$$u_h(t) = \tfrac{1}{2} a_0 + \sum_{n=1}^{L} a_n \cos nt, \quad -\pi \leq t \leq \pi, \tag{3.24}$$

where

$$a_n = \tfrac{4}{N} \sum_{k=1}^{L+1} u_{L+k} \cos \tfrac{n\pi}{N}(2k - 1). \tag{3.25}$$

REMARK 3.2. Our analysis in Sections 4 and 5 will show (under some additional assumptions on the quadrature rule $Q(f)$ that the problem (3.18) is uniquely solvable if the mesh parameter h is small enough. Thus, for small h, we have the unique solvability for the quadrature method (3.22) as well.

REMARK 3.3. The use of the cosine interpolation, described by (3.23)–(3.25), which in effect extends (3.22) to (3.18), is a theoretical tool which, besides the unique solvability, yields also stability results and convergence estimates for the error.

4. A special case

Here we consider the special case where the arc Γ is the interval $[-2e^{-1}, 2e^{-1}]$. For this particular arc we have $L_e = A_e$ and $L_{e,h} = A_{e,h}$, where

$$(A_{e,h} u)(t) = -\frac{h}{2\pi} \sum_{j=0}^{N} {}'' \log[2e^{-1} |\cos t - \cos t_j|] u(t_j). \tag{4.1}$$

Problem (3.18) now becomes

$$u_h \in T_e^h : (A_{e,h} u_h, v)_h = (f, v)_h, \quad v \in S_e^d. \tag{4.2}$$

We are going to show that this problem is equivalent to the problem

$$u_h \in T^h : (A_h u_h, v)_h = (f, v)_h, \quad v \in S^d, \tag{4.3}$$

assuming that f is an even function. In (4.3) the operator A_h is the trapezoidal approximation of the operator A given in (2.11), that is

$$(A_h u)(t) = -\frac{h}{\pi} \sum_{j=0}^{N} {}'' \log \left| 2e^{-\frac{1}{2}} \sin \left(\frac{t - t_j}{2} \right) \right| u(t_j). \tag{4.4}$$

Observe that the function $(A_h u)(t)$ is continuous for $t \neq t_j$. Now, let $C_{2\pi,e}(\mathbb{R})$ and $C_{2\pi,0}(\mathbb{R})$, be the space of all continuous 2π-periodic even and odd functions respectively on $\mathbb{R}$.

We shall need

LEMMA 4.1. *The following assertions are true*:

   (i) $A_h u$ is even, if $u \in C_{2\pi,e}(\mathbb{R})$,
  (ii) $A_h u$ is odd, if $u \in C_{2\pi,0}(\mathbb{R})$,
 (iii) $A_h u = A_{e,h} u$, if $u \in C_{2\pi,e}(\mathbb{R})$.

PROOF. We use the convention

$$\sum_{j=0}^{L} {}' c_j = \tfrac{1}{2} c_0 + \sum_{j=1}^{L} c_j.$$

By the symmetry property $t_{N-j} = -t_j$ of our mesh, we have for even functions u

$$(A_{e,h} u)(t) = -\frac{h}{\pi} \sum_{j=0}^{L} {}' \log[2e^{-1} | \cos t - \cos t_j |] u(t_j). \tag{4.5}$$

For the function $(A_h u)(t)$, we may split the sum in (4.4) into two terms

$$(A_h u)(t) = -\frac{h}{\pi} \sum_{j=0}^{L} {}' \log \left| 2e^{-\frac{1}{2}} \sin \left(\frac{t - t_j}{2} \right) \right| u(t_j)$$

$$-\frac{h}{\pi} \sum_{j=0}^{L} {}' \log \left| 2e^{-\frac{1}{2}} \sin \left(\frac{t + t_j}{2} \right) \right| u(-t_j). \tag{4.6}$$

If $u \in C_{2\pi,e}(\mathbb{R})$, we thus obtain

$$(A_h u)(t) = -\frac{h}{\pi} \sum_{j=0}^{L} {}' \log \left| 4e^{-1} \sin \left(\frac{t - t_j}{2} \right) \sin \left(\frac{t + t_j}{2} \right) \right| u(t_j). \tag{4.7}$$

Assertion (iii) follows from (4.5), (4.7) by the formula

$$\sin a \cdot \sin b = \tfrac{1}{2}[\cos(a - b) - \cos(a + b)].$$

Assertions (i), (ii) can be seen by (4.6).

Now we are able to prove

THEOREM 4.1. *Assume that the problem* (4.3) *is uniquely solvable and that* $f \in C_{2\pi,e}(\mathbb{R})$. *Then the problems* (4.2) *and* (4.3) *are equivalent.*

PROOF. By Lemma 4.1(iii), the problem (4.2) can be written as

$$u_h \in T_e^h : (A_h u_h, v)_h = (f, v)_h, \quad v \in S_e^d. \tag{4.8}$$

Since N is an odd integer, we can write the space T^h (analogously with (3.7)) as the direct sum $T^h = T_e^h \dot{+} T_o^h$, where $T_o^h = \{u \in T^h : u \text{ is odd}\}$. If $u_h \in T^h$, $v \in S^d$, we correspondingly write

$$u_h = u_{h,e} + u_{h,o}, \quad u_{h,e} \in T_e^h, \ u_{h,o} \in T_o^h,$$
$$v = v_e + v_o, \quad v_e \in S_e^d, \ v_o \in S_o^d.$$

If we have an even function $\tilde{u}$ and an odd function $\tilde{v}$, then the symmetry property of the rule Q_h in (3.17) yields

$$(\tilde{u}, \tilde{v})_h = Q_h(\tilde{u}\bar{\tilde{v}}) = 0$$

(and similarly $(\tilde{v}, \tilde{u})_h = 0$). By Lemma 4.1, we thus have

$$(A_h u_{h,e}, v_o)_h = (A_h u_{h,o}, v_e)_h = 0 \tag{4.9}$$

for all $u_h \in T^h$, $v \in S^d$. Hence we obtain

$$(A_h u_h, v)_h = (A_h u_{h,e}, v_e)_h + (A_h u_{h,o}, v_o)_h \tag{4.10}$$

for all $u_h \in T^h$, $v \in S^d$.

Now, if $u_h \in T_e^h$ is a solution of (4.8) then $u_{h,e} = u_h$ and $u_{h,o} = 0$, so by (4.10)

$$(A_h u_h, v)_h = (A_h u_h, v_e)_h = (f, v_e)_h = (f, v)_h$$

for all $v \in S^d$. Thus u_h is also a solution of the problem (4.3).

Conversely, suppose that $u_h \in T^h$ satisfies (4.3). Then we have by (4.10)

$$(A_h u_{h,e}, v_e)_h + (A_h u_{h,o}, v_o)_h = (f, v)_h = (f, v_e)_h \tag{4.11}$$

for all $v \in S^d$. By choosing $v = v_o$, we find

$$(A_h u_{h,o}, v_o)_h = 0. \tag{4.12}$$

Now, combining (4.9) and (4.12) we obtain

$$(A_h u_{h,o}, v)_h = 0 \quad , \quad v \in S^d. \tag{4.13}$$

Since the problem (4.3) is assumed to be uniquely solvable, we conclude that $u_{h,o} = 0$, and therefore $u_h = u_{h,e} \in T_e^h$. In particular, u_h is a solution of (4.2).

Methods of the form (4.3) were studied by Sloan and Burn in [13]. These authors consider the general case of the discrete inner product which is defined by using the composite rule (similar to (3.16)) composed from the basic quadrature rule

$$Q(f) = \sum_{j=1}^{J} \omega_j f(\xi_j) \tag{4.14}$$

for the interval [0,1]. Here $\sum_{j=1}^{J} \omega_j = 1$, $\omega_j > 0$, and $0 < \xi_1 < \cdots < \xi_J < 1$. The quadrature rule (3.14) becomes a special case of (4.14) if we take $J = 2M$ and define

$$\begin{cases} \xi_j = \frac{1}{2} - \eta_{M+1-j}, & \omega_j = w_{M+1-j}, \\ \xi_{M+j} = \frac{1}{2} + \eta_j, & \omega_{M+j} = w_j, \end{cases} \tag{4.15}$$

for the indices $1 \le j \le M$. (Note that $\xi_M = \xi_{M+1}$ if $\eta_1 = 0$.)

Now, referring to the terminology of [13], we say that the method (4.2) is stable and of order p if and only if the related method (4.3) has these properties. By the convergence result of [13] for the circle and by the equivalence of the methods (4.2) and (4.3), we obtain

THEOREM 4.2 (Interval). *Assume that the arc* Γ *is the interval* $\Gamma = [-2e^{-1}, 2e^{-1}]$ *and let* $u \in H_e^s$ *be the solution of* (2.4) *(with* $L_e = A_e$*). Moreover, suppose that the method* (4.2) *is stable and of order* p*. Then, for all* $f \in C_{2\pi,e}(\mathbb{R})$*, there exists a unique solution* $u_h \in T_e^h$ *of the problem* (4.2) *and we have the asymptotic error estimate*

$$\|u - u_h\|_t \le ch^{s-t}\|u\|_s, \tag{4.16}$$

if

$$-1 \le t \le s \le t + p, \quad s > -\tfrac{1}{2}. \tag{4.17}$$

By Sobolev's embedding theorem we obtain for the maximum norm

COROLLARY 4.1. *Assume that* $u \in H_e^s$ *with* $s \ge \frac{1}{2} + \epsilon$*, where* $\epsilon > 0$*. Then we have the pointwise estimate*

$$|u - u_h|_\infty \le ch^{\min\{s-(\frac{1}{2}+\epsilon),p\}}\|u\|_{\min\{s,(\frac{1}{2}+\epsilon)+p\}}. \tag{4.18}$$

In approximating linear functionals of u, error estimates with respect to the Sobolev norms of the negative order are particularly useful. Therefore we formulate separately the following result:

COROLLARY 4.2. *With the assumptions of Theorem* 4.2, *the estimate*

$$\|u - u_h\|_{-1} \le ch^{s+1}\|u\|_s \tag{4.19}$$

holds, provided

$$-\tfrac{1}{2} < s \le p - 1. \tag{4.20}$$

EXAMPLE 4.1. As a concrete example of the quadrature methods in (4.2), we can make the following choice:

$$M = 1, \quad \eta_1 = \tfrac{1}{3}, \quad w_1 = \tfrac{1}{2}. \tag{4.21}$$

In terms of the quadrature rule (4.14) this corresponds to the values

$$J = 2, \quad \xi_1 = \tfrac{1}{6}, \quad \xi_2 = \tfrac{5}{6}, \quad \omega_1 = \omega_2 = \tfrac{1}{2}. \tag{4.22}$$

By [13] it is known that the quadrature method (4.3) with (4.22) is stable and of order $p = 3$.

In Section 5 we are going to show that the above convergence results are true also for the general open arc. Therefore we shall compare our result to that of Atkinson and Sloan [1, Theorem 6] already at this point. Under the assumption $u \in H_e^s$ with $s > \tfrac{1}{2} + \epsilon$, where $\epsilon > 0$, Atkinson and Sloan find an approximation $u_h \in T_e^h$ such that

$$|u - u_h|_\infty \le ch^{s-(\tfrac{1}{2}+\epsilon)}\|u\|_s. \tag{4.23}$$

In particular, the convergence becomes arbitrarily high (in the powers of h) if the solution u is smooth. By (4.18) we have an analogous result, except that the convergence is limited to the order $O(h^p)$. On the other hand, our method requires less smoothness of the exact solution in order to be convergent (if the error is measured by suitable Sobolev norms). In addition we can use the estimates (4.19), for example, in approximating the corresponding potential outside of the arc. Thus if

$$\Phi(x) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} u(\tau)\log|x - a(\tau)|d\tau \tag{4.24}$$

is the potential corresponding to the solution u, we have for the approximate potential

$$\Phi_h(x) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} u_h(\tau)\log|x - a(\tau)|d\tau \tag{4.25}$$

the estimate

$$|\Phi(x) - \Phi_h(x)| \le ch^{s+1}\|u\|_s, \tag{4.26}$$

if $-\frac{1}{2} < s \le p - 1$, with c depending on x.

5. The case of the general open arc

We begin with a lemma relating the integral operator

$$(Ku)(t) = \int_{-\pi}^{\pi} k(t, \tau)u(\tau)d\tau, \quad t \in [-\pi, \pi], \tag{5.1}$$

where $k \in C_{2\pi}(\mathbb{R} \times \mathbb{R})$ (i.e. k is continuous and 2π-periodic with respect to each variable), to its trapezoidal approximation

$$(K_h u)(t) = h \sum_{j=0}^{N} k(t, t_j)u(t_j), \tag{5.2}$$

for the particular case in which u is a trigonometric polynomial in T^h.

LEMMA 5.1. *Suppose* $k \in C_{2\pi}(\mathbb{R} \times \mathbb{R})$ *is m-times continuously differentiable with respect to its second variable, and that the corresponding derivative* $k_\tau^{(m)}(t, \tau)$ *is Hölder continuous with respect to* τ *with exponent* λ, $0 < \lambda < 1$. *Then the estimate*

$$|(K - K_h)u_h|_\infty \le Ch^{m+\lambda}\|u_h\|_0$$

holds for all $u_h \in T^h$.

PROOF (cf. [7, Lemma 1] and [6, Sections 2.3.1–2.3.2]). For $f \in C_{2\pi}$ we may define the trigonometric interpolation operator L_N, with $N = 2L + 1$, by

$$L_N f \in T^h, \quad L_N f(t_j) = f(t_j), \quad 0 \le j \le N.$$

Then it is easy to show that, for $u_h \in T^h$, $K_h u_h$ can be written as

$$(K_h u_h)(t) = \int_{-\pi}^{\pi} L_{N\tau}[k(t, \tau)]u_h(\tau)d\tau, \tag{5.3}$$

where $L_{N\tau}[k(t, \tau)]$ stands for the operator L_N applied to $k(t, \tau)$ with respect to the second variable τ. The result (5.3) follows from the easily verified fact that the trapezoidal rule with spacing $h = 2\pi/N = 2\pi/(2L + 1)$ is exact for all trigonometric polynomials of degree $\le 2L$: since $L_{N\tau}[k(t, \tau)]u_h(\tau)$ is a trigonometric polynomial of degree $\le 2L$ for each

fixed value of t, the integral on the right of (5.3) is equal to its trapezoidal approximation

$$h \sum_{j=0}^{N}{}'' L_{N\tau}[k(t,t_j)]u_h(t_j) = h \sum_{j=0}^{N}{}'' k(t,t_j)u_h(t_j),$$

and so is equal to $K_h u_h(t)$.

From (5.1) and (5.3) we now have

$$(K - K_h)u_h(t) = \int_{-\pi}^{\pi} (k(t,\tau) - L_{N\tau}[k(t,\tau)])u_h(\tau)d\tau.$$

The Cauchy-Schwarz inequality then gives

$$|(K - K_h)u_h|_\infty \leq \max_t \int_{-\pi}^{\pi} |k(t,\tau) - L_{N\tau}[k(t,\tau)]||u_h(\tau)|d\tau$$

$$\leq \varepsilon_h \|u_h\|_0,$$

where

$$\varepsilon_h := \max_t \|k(t,\tau) - L_{N\tau}[k(t,\tau)]\|_0$$

$$\leq (1 + \|L_N\|_{C \to L^2}) \max_t E_h(C; k(t,\cdot)).$$

Here $\|L_N\|_{C \to L^2}$ stands for the norm of the operator $L_N : C \to L^2$, and $E_h(C; k(t,\cdot))$ is the error of best approximation of $k(t,\tau)$ (with fixed t) by trigonometric polynomials from T^h with respect to the uniform norm. Now by Jackson's theorem, $E_h(C; k(t,\cdot)) = O(h^{m+\lambda})$. Since, as is well known, $\sup_N \|L_N\|_{C \to L^2} < \infty$, we have $\varepsilon_h = O(h^{m+\lambda})$, and the proof is complete.

If $k(t,\tau)$ is also differentiable with respect to its first variable t, then similar results may be obtained for the derivatives of $(K - K_h)u_h$. By proceeding in this way we may obtain the following result, which was already found by Saranen and Sloan in [10].

LEMMA 5.2. Let $k \in C_{2\pi,e}^\infty(\mathbb{R} \times \mathbb{R})$. Then, for any $s, t \in \mathbb{R}$ and $\delta > 0$, there exists a constant $C > 0$ (independent of $u_h \in T_e^h$) such that

$$\|(K - K_h)u_h\|_t \leq Ch^\delta \|u_h\|_s \qquad (5.4)$$

holds for all $u_h \in T_e^h$.

The approximation property (5.4) provides the essential tool to discuss the case of the general open arc.

THEOREM 5.1. *Assume that Γ is a smooth open arc such that its capacity differs from* 1. *Let $u \in H_e^s$ be the solution of* (2.4) *and suppose the method* (4.2) *is stable and of order p. Then, for all $f \in C_{2\pi, e}(\mathbb{R})$ and all h sufficiently small, there exists a unique solution $u_h \in T_e^h$ of the problem* (3.18) *and we have the asymptotic error estimate*

$$\|u - u_h\|_t \le ch^{s-t}\|u\|_s \tag{5.5}$$

if $-1 \le t \le s \le t + p$, $s > -\frac{1}{2}$.

PROOF (cf. [10]). Assume initially that (3.18) has a solution u_h. By writing $f = L_e u = (A_e + B_e)u$ and

$$L_{e,h} = A_{e,h} + B_{e,h},$$

(3.18) becomes

$$(A_{e,h}u_h, v)_h = (A_e(u + A_e^{-1}(B_e u - B_{e,h}u_h)), v)_h$$

for all $v \in S_e^d$. By Theorem 4.2 we thus have

$$\|u + A_e^{-1}(B_e u - B_{e,h}u_h) - u_h\|_t \le ch^{s-t}\|u + A_e^{-1}(B_e u - B_{e,h}u_h)\|_s. \tag{5.6}$$

Now, the operator $I + A_e^{-1}B_e : H_e^t \to H_e^t$ is an isomorphism, so we obtain

$$
\begin{aligned}
\|u - u_h\|_t &\le c\|(I + A_e^{-1}B_e)(u - u_h)\|_t \\
&\le c\|u - u_h + A_e^{-1}(B_e u - B_{e,h}u_h)\|_t \\
&\quad + c\|A_e^{-1}(B_e - B_{e,h})u_h\|_t.
\end{aligned} \tag{5.7}
$$

Combining (5.6) and (5.7), we have

$$
\begin{aligned}
\|u - u_h\|_t &\le ch^{s-t}\|u + A_e^{-1}(B_e u - B_{e,h}u_h)\|_s + c\|A_e^{-1}(B_e - B_{e,h})u_h\|_t \\
&\le ch^{s-t}(\|u\|_s + \|A_e^{-1}B_e(u - u_h)\|_s + \|A_e^{-1}(B_e - B_{e,h})u_h\|_s) \\
&\quad + c\|A_e^{-1}(B_e - B_{e,h})u_h\|_t.
\end{aligned} \tag{5.8}
$$

Since the operator B_e is infinitely smoothing, we conclude that

$$\|A_e^{-1}B_e(u - u_h)\|_s \le c\|u - u_h\|_{-1}. \tag{5.9}$$

Applying Lemma 5.2 to the operator B_e with the kernel $b_e \in C_{2\pi, e}^\infty(\mathbb{R} \times \mathbb{R})$, we find for all $\delta > 0$

$$
\begin{aligned}
h^{s-t}\|A_e^{-1}(B_e - B_{e,h})u_h\|_s &+ \|A_e^{-1}(B_e - B_{e,h})u_h\|_t \\
&\le ch^\delta\|u_h\|_{-1} \le ch^\delta(\|u\|_{-1} + \|u - u_h\|_{-1}).
\end{aligned} \tag{5.10}
$$

Choosing $\delta \geq s - t$ and combining (5.8)–(5.10), we obtain

$$\|u - u_h\|_t \leq ch^{s-t}(\|u\|_s + \|u - u_h\|_{-1}) \qquad (5.11)$$

for all $-1 \leq t \leq s \leq t + p$, $s > -\frac{1}{2}$.

Taking $t = -1$, we have by (5.11), for small h

$$\|u - u_h\|_{-1} \leq ch^{s+1}\|u\|_s. \qquad (5.12)$$

Inserting (5.12) back into (5.11), we have the inequality (5.5). Above we have assumed that the solution $u_h \in T_e^h$ exists. But (5.5) with u replaced by zero establishes the uniqueness of the solution for small h, and from this the existence follows immediately.

Acknowledgements

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