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NORMAL PARTITIONS OF IDEMPOTENTS OF REGULAR SEMIGROUPS

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Abstract

A characterization is provided here for any normal partition of the set of idempotents of a regular semigroup S. As a by-product of the method used, a new characterization of the greatest congruence on S corresponding to a given normal partition of its idempotents is obtained.

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Introduction

Any congruence on a regular semigroup S induces a partition, called a *normal partition*, of its set of idempotents E_S. In her doctoral dissertation, Feigenbaum (1975) provided a characterization for any congruence on S and for any normal partition of E_S. The aim here is to make use of the sandwich sets of Nambooripad (1974) to obtain a simpler characterization of a normal partition of E_S. As a by-product of the method used, an alternative characterization of the greatest congruence on S corresponding to a given normal partition of E_S is obtained.

1. Definitions and preliminary results

In all that follows let S denote a regular semigroup and let E_S be its set of idempotents. For $e,f \in E_S$ the *sandwich set* of e,f is

$$S(e,f) = \{g \in E_S;\ ge = g = fg,\ egf = ef\}.$$

For $a \in S$ let $V(a)$ denote the set of inverses of a. The following lemma is taken from Nambooripad (1974) and Clifford (1974).

LEMMA 1.1. *Suppose* $e,f,h,k \in E_S$, $a,b \in S$, $a' \in V(a)$ *and* $b' \in V(b)$. *Then*

(i) $S(e,f) \neq \square$;

(ii) *if* $e \mathscr{L} h$ *and* $f \mathscr{R} k$ *then* $S(e,f) = S(h,k)$;

(iii) *if* $g \in S(a'a, bb')$ *then* $agb = ab$ *and* $b'ga' \in V(ab)$.

In the light of (ii) and (iii) of the lemma, define $S(a,b) = S(a'a, bb')$ for any $a' \in V(a)$ and $b' \in V(b)$.

LEMMA 1.2. *Let τ be a congruence on S, $a, b \in S$, $a' \in V(a)$ and $b' \in V(b)$.*
(i) *If $a\tau \in E_{S/\tau}$ then $S(a,a) \subseteq a\tau$.*
(ii) *If $a \mathscr{H} b$ and $b^* = a'ab'aa'$ then $a' \mathscr{H} b^*$ and $b^* \in V(b)$.*

PROOF. By Lemma 1.1(iii) $a'S(a,a)a' \subseteq V(a^2)$. If $a\tau = (a^2)\tau$ then

$$S(a,a) = aa'S(a,a)a'a \subseteq aV(a^2)a \subseteq a\tau.$$

Thus (i) is verified. (ii) follows directly from the observation that $a \mathscr{R} b$ if and only if $aa'bb' = bb'$, $bb'aa' = aa'$ and $a \mathscr{L} b$ if and only if $a'ab'b = a'a$, $b'ba'a = b'b$.

Let π denote an equivalence relation on E_S and let $P_\pi = \{e\pi; e \in E_S\}$. P_π is a normal partition of E_S if and only if there is a congruence τ on S so that the restriction of τ to E_S is π. Then $P_\pi = \{e\tau \cap E_S; e \in E_S\}$.

Feigenbaum (1975) proved that a partition P_π of E_S is a normal partition if and only if for any $e_i \in E_S$, $x_i, y_i \in S^1$ where $i = 1, \ldots, n$, so that $x_0(e_0\pi)y_0 \cap E_S \neq \square$, $x_n(e_n\pi)y_n \cap E_S \neq \square$ and $x_j(e_j\pi)y_j \cap x_{j+1}(e_{j+1}\pi)y_{j+1} \neq \square$ for $j = 0, \ldots, n-1$, then there exists $e \in E_S$ so that $(x_0(e_0\pi)y_0 \cup x_n(e_n\pi)y_n) \cap E_S \subseteq e\pi$.

We will consider a partition P_π of E_S to be a partial groupoid under the partial binary operation $*$ defined by $e\pi * f\pi = g\pi$, $e, f, g \in E_S$, if and only if

$$\square \neq (e\pi)(f\pi) \cap E_S \subseteq g\pi.$$

Define a partial operation by S on P by $e\pi^{c'} = g\pi$, $e, g \in E_S$, $c \in S$, $c' \in V(c)$, if and only if $\square \neq c'(e\pi)c \cap E_S \subseteq g\pi$.

A partition P_π of E_S is an *N-partition* if and only if for each $e, f \in E_S$, $c \in S$ and $c' \in V(c)$ then

(1) $e\pi * f\pi \supseteq S(ef, ef)$ or $(e\pi)(f\pi) \cap E_S = \square$ and
(2) $e\pi^{c'} \supseteq S(c'ec, c'ec)$ or $c'(e\pi)c \cap E_S = \square$.

By Lemma 1.2(i) it can be seen that a normal partition of E_S is an N-partition. The converse will be proved in the next section.

Given a partition P_π of E_S define relations $\mathscr{R}_\pi$ and $\mathscr{L}_\pi$ on S by

$$\mathscr{R}_\pi = \{(a,b) \in S \times S;\ (aa')\pi * (bb')\pi = (bb')\pi,$$
$$(bb')\pi * (aa')\pi = (aa')\pi \text{ for some } a' \in V(a) \text{ and } b' \in V(b)\}$$

and

$$\mathscr{L}_\pi = \{(a,b) \in S \times S;\ (a'a)\pi * (b'b)\pi = (a'a)\pi,$$
$$(b'b)\pi * (a'a)\pi = (b'b)\pi \text{ for some } a' \in V(a) \text{ and } b' \in V(b)\}.$$

Note that $a \mathscr{R}_\pi aa' \mathscr{R}_\pi aa^*$ and $a \mathscr{L}_\pi a'a \mathscr{L}_\pi a^*a$ for any $a', a^* \in V(a)$.

LEMMA 1.3. *Let P_π be an N-partition of E_S and $e, f \in E_S$. Then*
(i) *$e \mathscr{R}_\pi f$ $[e \mathscr{L}_\pi f]$ if and only if $S(e,f) \subseteq e\pi$ $[f\pi]$ and $S(f,e) \subseteq f\pi$ $[e\pi]$;*
(ii) *$\mathscr{R}_\pi$ and $\mathscr{L}_\pi$ are transitive relations.*

PROOF. Suppose $S(e,f) \subseteq e\pi$ and $S(f,e) \subseteq f\pi$. Since $fS(e,f) = S(e,f)$ and $eS(f,e) = S(f,e)$ then $f\pi * e\pi = e\pi$ and $e\pi * f\pi = f\pi$ so $e \mathscr{R}_\pi f$.

Conversely suppose $e \mathscr{R}_\pi f$ and $p \in S(e,f)$. Choose $q \in S(ef, ef)$, $r \in S(q,p)$ and $t \in S(qp,qp)$. Note that by Lemma 1.1(iii), $p \in V(ef)$ and $r \in V(qp)$. By the definition of sandwich sets we have $pe = p = fp$ so $p(ef) = pf$, $(ef)p = ep$ and similarly $r(qp) = rp$, $(qp)r = qr$. So $S(ef,ef) = S(pf,ep)$ and $S(qp,qp) = S(rp,qr)$. Hence $qpf = q = epq$ so $eq = q$, and then $t = qrt = eqrt = et$. Note that $epqe$ and ep are idempotents. Since $e \mathscr{R}_\pi f$ we have by condition (1) that $q\pi = e\pi * f\pi = f\pi$ and then $p\pi = (fp)\pi = f\pi * p\pi = q\pi * p\pi = t\pi$. Hence $(ep)\pi = e\pi * p\pi = e\pi * t\pi = t\pi = p\pi$. So $(epqe)\pi = (qe)\pi = q\pi * e\pi = f\pi * e\pi = e\pi$. But then

$$(epqe)\pi = (ep)\pi * (qe)\pi = p\pi * e\pi = (pe)\pi = p\pi.$$

Thus $p\pi = e\pi$. Similarly $S(f,e) \subseteq f\pi$. Thus (i) is proved.

Assume $a \mathscr{R}_\pi b \mathscr{R}_\pi c$. Then $aa' \mathscr{R}_\pi bb' \mathscr{R}_\pi bb^* \mathscr{R}_\pi cc'$ for some $a' \in V(a)$, $b', b^* \in V(b)$ and $c' \in V(c)$. Choose $p \in S(bb^*, cc')$, $q \in S(bb', p)$ and $r \in S(aa', q)$. By (i) then $p\pi = (bb^*)\pi$, $q\pi = (bb')\pi$ and $r\pi = (aa')\pi$. Also $cc'p = p$, $pq = q$ and $qr = r$ so $cc'r = r$. Hence $(cc')\pi * (aa')\pi = (aa')\pi$ and similarly $(aa')\pi * (cc')\pi = (cc')\pi$. Dually $\mathscr{L}_\pi$ is transitive.

For a partition P_π of E_S define a relation $\mathscr{H}_\pi$ on S by $\mathscr{H}_\pi = \mathscr{R}_\pi \cap \mathscr{L}_\pi$.

Note that if P_π is a normal partition of E_S induced by a congruence τ on S then $a \mathscr{R}_\pi b$, or $a \mathscr{L}_\pi b$, or $a \mathscr{H}_\pi b$ if and only if $a\tau \mathscr{R} b\tau$, or $a\tau \mathscr{L} b\tau$, or $a\tau \mathscr{H} b\tau$ respectively.

LEMMA 1.4. *Let P_π be an N-partition of E_S, $(e,f) \in \pi$ and $c \in S$. Then $ec \mathscr{H}_\pi fc$ and $ce \mathscr{H}_\pi cf$.*

PROOF. For some $c' \in V(c)$ let $g = cc'$, $h \in S(e,g)$ and $k \in S(f,g)$. By Lemma 1.1(iii) we may write $(ec)' = c'he \in V(ec)$ and $(fc)' = c'kf \in V(fc)$. So $ec(ec)' = eh$, $fc(fc)' = fk$, $(ec)'ec = c'hc$ and $(fc)'(fc) = c'kc$. We will prove that $eh \mathscr{R}_\pi fk$ and $c'hc \mathscr{L}_\pi c'kc$. The proof that $ce \mathscr{H}_\pi cf$ is similar.

By condition (1), $(eh)\pi = r\pi$ where $r \in S(fh,fh)$. But $gh = h$ and $fg = fkg$ so $fkgh(fh)'r = r = fkr$ for some $(fh)' \in V(fh)$. Hence

$$(fk)\pi * (eh)\pi = (fk)\pi * r\pi = (eh)\pi.$$

Likewise $(eh)\pi * (fk)\pi = (fk)\pi$ so $eh \mathscr{R}_\pi fk$.

Again by condition (1) $(he)\pi = h\pi * e\pi = h\pi * f\pi = s\pi$ where $s \in S(hf,hf)$. So $(hg)\pi = (heg)\pi = (he)\pi * g\pi = s\pi * g\pi = t\pi$ where $t \in S(sg,sg)$. For $(hf)' \in V(hf)$ and $(sg)' \in V(sg)$ we have $s(hf)'hf = s = sf$ so

$$t = t(sg)'sg = t(sg')sfg = t(sg)'sfkg = tkg.$$

Thus $(hg)\pi = t\pi*(kg)\pi = (hg)\pi*(kg)\pi$ and similarly $(kg)\pi*(hg)\pi = (kg)\pi$. We therefore have $hg \mathscr{L}_\pi kg$. Now choose $p \in S(hg, kg)$. By Lemma 1.3(i), $p \in (kg)\pi$. Recalling that $gh = h$ and $g = cc'$ it can be readily checked that $c'pc \in S(c'hc, c'kc)$. Using condition (2)

$$(c'kc)\pi = (c'kgc)\pi = (c'pc)\pi = (c'phgc)\pi = (c'p(gh)gc)\pi$$

$$= (c'pc)\pi*(c'hc)\pi = (c'kc)\pi*(c'hc)\pi.$$

Likewise $(c'hc)\pi*(c'kc)\pi = (c'hc)\pi$. Hence $(c'hc)\mathscr{L}_\pi(c'kc)$.

2. Normal partitions

In this section normal partitions of E_S will be characterized and the greatest congruences associated with these partitions will be determined.

THEOREM 2.1. *Let S be a regular semigroup and P_π be an N-partition of E_S. Then P_π is a normal partition and $\rho_\pi = \{(a,b) \in \mathscr{H}_\pi;$ for each $c \in S$, $ca \mathscr{H}_\pi cb$ and $ac \mathscr{H}_\pi bc\}$ is the greatest congruence on S that induces P_π.*

PROOF. Clearly ρ_π is symmetric, reflexive and compatible and by Lemma 1.3(ii) it is transitive. So ρ_π is a congruence on S. If $(e,f) \in \pi$ then clearly $e \mathscr{H}_\pi f$ and by Lemma 1.4 $(e,f) \in \rho_\pi$. Conversely, if $(e,f) \in \rho_\pi \cap E_S \times E_S$ then $e \mathscr{H}_\pi f$. But then by Lemma 1.3(i), $S(e,f) \subseteq e\pi \cap f\pi$ so $e\pi = f\pi$. Thus P_π is the normal partition of E_S induced by ρ_π. Let τ be a congruence on S that induces P_π. If $(a,b) \in \tau$ then $a\tau \mathscr{H} b\tau$ in S/τ and as noted before Lemma 1.4 then $a \mathscr{H}_\pi b$. Therefore $\rho_\pi \supseteq \tau$.

Since a normal partition of E_S is an *N-partition* then:

COROLLARY 2.2. *P_π is an N-partition of E_S if and only if it is a normal partition of E_S.*

We can refine the description of ρ_π by using the characterization of Hall (1973) of the greatest idempotent separating congruence μ on S. Using Lemma 1.2(ii), Hall's definition translates to the following:

$$\mu = \{(a,b) \in \mathscr{H};\ \text{for some [any] } a' \in V(a),\ b' \in V(b) \text{ and each}$$
$$\text{idempotent } e \leqslant aa' \text{ then } a'ea = a'ab'aa'eb\}.$$

Note that if τ is a congruence on S, and $e,f \in E_S$ then $e\tau \leqslant f\tau$ in S/τ if and only if $(fe)\tau \mathscr{R} e\tau \mathscr{L} (ef)\tau$ in S/τ. Hence for a normal partition P_π, define $e\pi \leqslant f\pi$ if and only if $(fe)\mathscr{R}_\pi e \mathscr{L}_\pi (ef)$.

THEOREM 2.3. *Let S be a regular semigroup and P_π be a normal partition of E_S. Then*

$$\rho_\pi = \{(a,b) \in \mathscr{H}_\pi;\ \textit{for some [any] } a' \in V(a),\ b' \in V(b) \textit{ and each idempotent } e \textit{ so that } e\pi \leqslant (aa')\pi \textit{ then } i\pi = j\pi \textit{ where } i \in S(a'ea, a'ea) \textit{ and } j \in S(a'ab'aa'eb, a'ab'aa'eb)\}.$$

PROOF. Let σ_π be the least congruence on S inducing P_π and let μ_π be the greatest idempotent separating congruence on S/σ_π. Then $(a,b) \in \rho_\pi$ if and only if $(a\sigma_\pi, b\sigma_\pi) \in \mu_\pi$. By the note preceding Lemma 1.4 we have $a\mathscr{H}_\pi b$ if and only if $a\sigma_\pi \mathscr{H} b\sigma_\pi$. For $e \in E_S$, $a \in S$ and $a' \in V(a)$ we have $e\pi \leqslant (aa')\pi$ if and only if $e\sigma_\pi \leqslant (aa')\sigma_\pi$. Now suppose $e \in E_S$, $a, b \in S$, $a\mathscr{H}_\pi b$, $a' \in V(a)$, $b' \in V(b)$ and $e\pi \leqslant (aa')\pi$. Then $(a'ea)\sigma_\pi \in E_{S/\sigma_\pi}$. Also, since $a\sigma_\pi \mathscr{H} b\sigma_\pi$ it can be easily checked that $(a'ab'aa'eb)\sigma_\pi \in E_{S/\sigma_\pi}$. Hence by Lemma 1.2(i), $(a'ea)\sigma_\pi = (a'ab'aa'eb)\sigma_\pi$ if and only if $S(a'ea, a'ea)$ and $S(a'ab'aa'eb, a'ab'aa'eb)$ are in the same π-class. The result follows from the definition of μ_π.

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