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APPROXIMATING THE STIELTJES INTEGRAL FOR (φ, Φ)-LIPSCHITZIAN INTEGRATORS

S. S. DRAGOMIR

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Abstract

Approximations for the Stieltjes integral with (φ, Φ)-Lipschitzian integrators are given. Applications for the Riemann integral of a product and for the generalized trapezoid and Ostrowski inequalities are also provided.

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1. Introduction

One can approximate the *Stieltjes integral* $\int_a^b f(t)\,du(t)$ with the following simpler quantities:

$$\frac{1}{b-a}[u(b)-u(a)] \cdot \int_a^b f(t)\,dt \quad \text{(see [16, 17])}, \tag{1.1}$$

$$f(x)\,[u(b)-u(a)] \quad \text{(see [9, 10])}, \tag{1.2}$$

or with

$$[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a) \quad \text{(see [15])}, \tag{1.3}$$

where $x \in [a, b]$.

In order to provide *a priori* sharp bounds for the *approximation error*, consider the functionals

$$D(f, u; a, b) := \int_a^b f(t)\,du(t) - \frac{1}{b-a}[u(b)-u(a)] \cdot \int_a^b f(t)\,dt,$$

$$\Theta(f, u; a, b, x) := \int_a^b f(t)\,du(t) - f(x)\,[u(b)-u(a)],$$

and

$$T(f, u; a, b, x) := \int_a^b f(t)\, du(t) - [u(b) - u(x)] f(b) - [u(x) - u(a)] f(a).$$

If the *integrand* f is *Riemann integrable* on $[a, b]$ and the *integrator* $u : [a, b] \to \mathbb{R}$ is L-*Lipschitzian*, that is,

$$|u(t) - u(s)| \le L|t - s| \quad \text{for each } t, s \in [a, b], \tag{1.4}$$

then the Stieltjes integral $\int_a^b f(t)\, du(t)$ exists and, as pointed out in [16],

$$|D(f, u; a, b)| \le L \int_a^b \left| f(t) - \int_a^b \frac{1}{b-a} f(s)\, ds \right| dt. \tag{1.5}$$

The inequality (1.5) is sharp in the sense that the multiplicative constant $C = 1$ in front of L cannot be replaced by a smaller quantity. Moreover, if there exists constants m, $M \in \mathbb{R}$ such that $m \le f(t) \le M$ for almost all $t \in [a, b]$, then [16]

$$|D(f, u; a, b)| \le \tfrac{1}{2} L(M - m)(b - a). \tag{1.6}$$

The constant $1/2$ is the best possible in (1.6).

A different approach in the case of integrands of bounded variation was considered by the same authors in 2001, [17], where they showed that

$$|D(f, u; a, b)| \le \max_{t \in [a,b]} \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| \bigvee_a^b (u), \tag{1.7}$$

provided that f is continuous and u is of bounded variation. Here $\bigvee_a^b (u)$ denotes the total variation of u on $[a, b]$. The inequality (1.7) is sharp.

If we assume that f is K-Lipschitzian, then [17]

$$|D(f, u; a, b)| \le \frac{1}{2} K(b - a) \bigvee_a^b (u), \tag{1.8}$$

with $1/2$ the best possible constant in (1.8).

For various bounds on the error functional $D(f, u; a, b)$ where f and u belong to different classes of function for which the Stieltjes integral exists, see [7, 12–14] and the references therein.

For the functional $\theta(f, u; a, b, x)$ we have the bound [9]

$$
\begin{aligned}
&|\theta(f, u; a, b, x)| \\
&\quad \le H\left[(x-a)^r \bigvee_a^x(f) + (b-x)^r \bigvee_x^b(f)\right] \\
&\quad \le H \times \begin{cases}
[(x-a)^r + (b-x)^r]\left[\frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^x(f) - \bigvee_x^b(f)\right|\right], \\
[(x-a)^{qr} + (b-x)^{qr}]^{1/q}\left[\left(\bigvee_a^x(f)\right)^p + \left(\bigvee_x^b(f)\right)^p\right]^{1/p} \\
\qquad\qquad \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\
\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \bigvee_a^b(f),
\end{cases} \qquad (1.9)
\end{aligned}
$$

provided f is of bounded variation and u is of $r - H$-*Hölder type*, that is,

$$
|u(t) - u(s)| \le H|t-s|^r \quad \text{for each } t, s \in [a, b], \qquad (1.10)
$$

with given $H > 0$ and $r \in (0, 1]$.

If f is of $q - K$-Hölder type and u is of bounded variation, then [10]

$$
|\theta(f, u; a, b, x)| \le K\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^q \bigvee_a^b(u), \qquad (1.11)
$$

for any $x \in [a, b]$.

If u is monotonic non-decreasing and f of $q - K$-Hölder type, then the following refinement of (1.11) also holds [7]:

$$
\begin{aligned}
&|\theta(f, u; a, b, x)| \\
&\quad \le K\left[(b-x)^q u(b) - (x-a)^q u(a) + q\left\{\int_a^x \frac{u(t)\,dt}{(x-t)^{1-q}} - \int_x^b \frac{u(t)\,dt}{(t-x)^{1-q}}\right\}\right] \\
&\quad \le K[(b-x)^q[u(b) - u(x)] + (x-a)^q[u(x) - u(a)]] \\
&\quad \le K\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^q [u(b) - u(a)], \qquad (1.12)
\end{aligned}
$$

for any $x \in [a, b]$.

If f is monotonic non-decreasing and u is of $r - H$-Hölder type, then [7]

$$
\begin{aligned}
&|\theta(f, u; a, b, x)| \\
&\quad \le H\left[[(x-a)^r - (b-x)^r]f(x) + r\left\{\int_a^x \frac{f(t)\,dt}{(b-t)^{1-r}} - \int_x^b \frac{f(t)\,dt}{(t-r)^{1-r}}\right\}\right] \\
&\quad \le H\{(b-x)^r[f(b) - f(x)] + (x-a)^r[f(x) - f(a)]\} \\
&\quad \le H\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r [f(b) - f(a)], \qquad (1.13)
\end{aligned}
$$

for any $x \in [a, b]$.

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The error functional $T(f, u; a, b, x)$ satisfies similar bounds, see [1, 2, 7, 15] and the details are omitted.

The main aim of this paper is to provide a different approximation of the Stieltjes integral $\int_a^b f(t)\,du(t)$ in terms of the simpler quantity

$$\frac{\varphi+\Phi}{2}\int_a^b f(t)\,dt,$$

provided that the integrator u is (φ, Φ)-Lipschitzian on $[a, b]$.

Applications for the Riemann integral of a product of two functions and for the generalized trapezoid and Ostrowski inequalities are also provided.

2. (φ, Φ)-Lipschitzian functions

We say that the function $v : [a, b] \to \mathbb{R}$ is *L-Lipschitzian* on $[a, b]$ if

$$|v(t) - v(s)| \le L|t - s| \quad \text{for any } t, s \in [a, b], \tag{2.1}$$

where $L > 0$ is a given constant.

The following lemma may be stated.

LEMMA 2.1. *Let $u : [a, b] \to \mathbb{R}$ and $\varphi, \Phi \in \mathbb{R}$ with $\Phi > \varphi$. The following statements are equivalent:*

(i) *the function $u - (\varphi + \Phi)/2 \cdot e$, where $e(t) = t$, $t \in [a, b]$, is $(\Phi - \varphi)/2$-Lipschitzian;*

(ii) *we have the inequality*

$$\varphi \le \frac{u(t) - u(s)}{t - s} \le \Phi \quad \textit{for each } t, s \in [a, b] \textit{ with } t \ne s; \tag{2.2}$$

(iii) *we have the inequality*

$$\varphi(t - s) \le u(t) - u(s) \le \Phi(t - s) \quad \textit{for each } t, s \in [a, b] \textit{ with } t > s. \tag{2.3}$$

Following [18], we can introduce the following concept.

DEFINITION 1. The function $u : [a, b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i)–(iii) is said to be (φ, Φ)-Lipschitzian on $[a, b]$.

Notice that, in [18], the definition was introduced on utilizing the statement (iii) and only the equivalence (i) $\leftrightarrow$ (iii) was considered.

Utilizing *Lagrange's mean value theorem*, we can state the following result that provides practical examples of (φ, Φ)-Lipschitzian functions.

PROPOSITION 2.2. *Let $u : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If*

$$-\infty < \gamma := \inf_{t \in (a,b)} u'(t), \qquad \sup_{t \in (a,b)} u'(t) =: \Gamma < \infty \tag{2.4}$$

then u is (γ, Γ)-Lipschitzian on $[a, b]$.

3. Inequalities for Stieltjes integrals

The following result may be stated.

THEOREM 3.1. *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$, $\varphi, \Phi \in \mathbb{R}$ with $\Phi > \varphi$ and $u : [a, b] \to \mathbb{R}$ a (φ, Φ)-Lipschitzian function on $[a, b]$. Then the Stieltjes integral $\int_a^b f(t)\, du(t)$ exists and defining the functional*

$$\Sigma(f, u, \varphi, \Phi; a, b) := \int_a^b f(t)\, du(t) - \frac{\varphi + \Phi}{2} \cdot \int_a^b f(t)\, dt,$$

we have

$$|\Sigma(f, u, \varphi, \Phi; a, b)| \le \frac{1}{2}(\Phi - \varphi) \int_a^b |f(t)|\, dt. \tag{3.1}$$

The constant $1/2$ is the best possible in (3.1).

PROOF. It is known that if $p : [a, b] \to \mathbb{R}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{R}$ is L-Lipschitzian, then the Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and

$$\left| \int_a^b p(t)\, dv(t) \right| \le L \int_a^b |p(t)|\, dt. \tag{3.2}$$

Since φ, Φ are finite, we can find a positive L such that $-L < \varphi < \Phi < L$ and by (2.2) we deduce that u is L-Lipschitzian. Therefore the Stieltjes integral exists and, by (3.2),

$$\left| \int_a^b f(t)\, d\left(u(t) - \frac{\varphi + \Phi}{2} \cdot t \right) \right| \le \frac{1}{2}(\Phi - \varphi) \int_a^b |f(t)|\, dt. \tag{3.3}$$

Since

$$\int_a^b f(t)\, d\left(u(t) - \frac{\varphi + \Phi}{2} \cdot t \right) = \int_a^b f(t)\, du(t) - \frac{\varphi + \Phi}{2} \int_a^b f(t)\, dt,$$

by (3.3) we deduce (3.1).

To prove the sharpness of the constant $1/2$, assume that the inequality (3.1) holds with a constant $C > 0$, that is,

$$|\Sigma(f, u, \varphi, \Phi; a, b)| \le C(\Phi - \varphi) \int_a^b |f(t)|\, dt, \tag{3.4}$$

provided f is Riemann integrable on $[a, b]$ and u is (φ, Φ)-Lipschitzian.

Consider the function $u(t) := |t - (a+b)/2|$. By the triangle inequality,

$$|u(t) - u(s)| = \left|\left|t - \frac{a+b}{2}\right| - \left|\frac{a+b}{2} - s\right|\right| \le |t - s| \quad \text{for each } t, s \in [a, b],$$

which shows that u is L-Lipschitzian with $L = 1$ or (φ, Φ)-Lipschitzian with $\varphi = -1$, $\Phi = 1$.

For a Riemann integrable function $f : [a, b] \to \mathbb{R}$ we then have

$$\begin{aligned}
\int_a^b f(t)\, du(t) &= \int_a^{(a+b)/2} f(t)\, d\left(\frac{a+b}{2} - t\right) + \int_{(a+b)/2}^b f(t)\, d\left(t - \frac{a+b}{2}\right) \\
&= -\int_a^{(a+b)/2} f(t)\, dt + \int_{(a+b)/2}^b f(t)\, dt \\
&= \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) f(t)\, dt.
\end{aligned}$$

If $g : [a, b] \to \mathbb{R}$ is Riemann integrable and non-negative almost everywhere on $[a, b]$ and if we choose $f(t) = \operatorname{sgn}(t - (a+b)/2) g(t)$, $t \in [a, b]$, then

$$\begin{aligned}
\int_a^b f(t)\, du(t) &= \int_a^b g(t)\, dt > 0, \\
\int_a^b |f(t)|\, dt &= \int_a^b g(t)\, dt,
\end{aligned}$$

and by (3.4) we deduce that

$$\int_a^b g(t)\, dt \le 2C \int_a^b g(t)\, dt,$$

which implies that $C \ge 1/2$. □

COROLLARY 3.2. *Let $g : [a, b] \to \mathbb{R}$ be a Riemann integrable function on $[a, b]$ and $u : [a, b] \to \mathbb{R}$ a (φ, Φ)-Lipschitzian function on $[a, b]$. Then*

$$|D(f, u; a, b)| \le \frac{1}{2}(\Phi - \varphi) \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right| dt. \tag{3.5}$$

The constant $1/2$ is the best possible in (3.5).

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REMARK 1. The inequality (3.5) has been obtained by Liu in [18], from which, in the case of usual Lipschitzian functions, one recaptures the result of Dragomir and Fedotov from [16]:

$$|D(f, u; a, b)| \le L \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| dt. \tag{3.6}$$

The following particular case of Theorem 3.1 is also of interest.

COROLLARY 3.3. *Let $g : [a, b] \to \mathbb{R}$ be a Riemann integrable function on $[a, b]$ such that*

$$-\infty < m \le g(t) \le M < \infty \quad \text{for almost every } t \in [a, b]. \tag{3.7}$$

If $u : [a, b] \to \mathbb{R}$ is (φ, Φ)-Lipschitzian on $[a, b]$, then

$$\begin{aligned} &\left| \int_a^b g(t)\, du(t) - \frac{\varphi + \Phi}{2} \int_a^b g(t)\, dt \right. \\ &\quad \left. - \frac{m+M}{2}[u(b) - u(a)] + \frac{(\varphi + \Phi)\,(m+M)}{4}(b-a) \right| \\ &\quad \le \frac{1}{2}(\Phi - \varphi) \int_a^b \left| g(t) - \frac{m+M}{2} \right| dt \le \frac{1}{4}(M-m)\,(\Phi - \varphi)\,(b-a). \end{aligned} \tag{3.8}$$

The constants $1/2$ and $1/4$ are the best possible in (3.8).

PROOF. The first inequality in (3.8) follows directly from Theorem 3.1 on choosing $f(t) = g(t) - (m+M)/2$, $t \in [a, b]$.

The second inequality in (3.8) is obvious by the fact that

$$\left| g(t) - \frac{m+M}{2} \right| \le \frac{1}{2}(M-m) \quad \text{for almost every } t \in [a, b].$$

Now, for the sharpness on the constants, if we choose $u(t) = |t - (a+b)/2|$, $t \in [a, b]$, then u is $(-1, 1)$-Lipschitzian on $[a, b]$, $u(a) = u(b) = (b-a)/2$ and the left-hand side of (3.8) reduces to

$$\left| \int_a^b g(t)\, du(t) \right| = \left| \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) g(t)\, dt \right|.$$

If we choose $g(t) = \operatorname{sgn}(t - (a+b)/2) h(t)$ with $h : [a, b] \to \mathbb{R}$ a Riemann integrable function with the properties

$$0 \le h(t) \le 1 \quad \text{for almost every } t \in [a, b] \quad \text{and} \quad \int_a^b h(t)\, dt = b - a,$$

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(for instance $h(t) = 1$, $t \in [a, b]$), then g is bounded above by $M = 1$ and below by $m = -1$,

$$\int_a^b g(t)\,du(t) = \int_a^b h(t)\,dt = b - a,$$
$$\int_a^b \left| g(t) - \frac{m+M}{2} \right| dt = \int_a^b h(t)\,dt = b - a,$$

and in both sides of (3.8) we get the same quantity $b - a$. □

The following result of Ostrowski type can be stated as well.

COROLLARY 3.4. *Let $g : [a, b] \to \mathbb{R}$ be a Riemann integrable function and $u : [a, b] \to \mathbb{R}$ a (φ, Φ)-Lipschitzian function on $[a, b]$. Then, for each $x \in [a, b]$, we have the inequality*

$$\left| \int_a^b g(t)\,du(t) - \frac{\varphi+\Phi}{2} \int_a^b g(t)\,dt - g(x)\left[u(b) - u(a) - \frac{\varphi+\Phi}{2}(b-a) \right] \right| \leq \frac{1}{2}(\Phi - \varphi) \int_a^b |g(t) - g(x)|\,dt. \tag{3.9}$$

The constant $1/2$ is the best possible in (3.9).

PROOF. The inequality follows from (1.7) on choosing $f(t) = g(t) - g(x)$. For $x \in (a, b)$, define $u(t) = |t - x|$, $t \in [a, b]$. Then u is $(-1, 1)$-Lipschitzian and

$$\int_a^b g(t)\,du(t) = \int_a^x g(t)\,d(x-t) + \int_x^b g(t)\,d(t-x) = \int_a^b \operatorname{sgn}(t-x) g(t)\,dt.$$

Now, if we choose $g(t) = \operatorname{sgn}(t-x)h(t)$ with $h : [a, b] \to [0, \infty)$ a Riemann integrable function, then the left-hand side of (3.9) reduces to

$$\left| \int_a^b g(t)\,du(t) \right| = \left| \int_a^b \operatorname{sgn}(t-x) g(t)\,dt \right| = \int_a^b h(t)\,dt.$$

Since

$$\int_a^b |g(t) - g(x)|\,dt = \int_a^b h(t)\,dt,$$

on both sides of (3.9) we have the same quantity $\int_a^b h(t)\,dt$. □

REMARK 2. If we define the function $B : [a, b] \to \mathbb{R}$ by

$$B(x) := \int_a^b |g(t) - g(x)|\,dt,$$

then we can provide various bounds for B depending on the classes of functions g considered.

For instance, if $g : [a, b] \to \mathbb{R}$ is of $r - H$-Hölder type, where $H > 0$ and $r \in (0, 1]$ are given, then

$$B(x) \le H \int_a^b |t - x|^r \, dt = \frac{H}{r+1}[(b-x)^{r+1} + (x-a)^{r+1}]. \tag{3.10}$$

If g is absolutely continuous, then $g(t) - g(x) = \int_x^t g'(s)\, ds$ and since

$$|g(t) - g(x)| = \left| \int_x^t g'(s)\, ds \right| \le \begin{cases} |t-x| \|g'\|_\infty & \text{if } g' \in L_\infty[a, b], \\ |t-x|^{1/q} \|g'\|_p & \text{if } g' \in L_p[a, b], \\ & p > 1,\ \dfrac{1}{p} + \dfrac{1}{q} = 1, \\ \|g'\|_1, & \end{cases}$$

where

$$\|g'\|_\infty := \operatorname*{ess\,sup}_{s \in [a,b]} |g'(s)|, \qquad \|g'\|_p := \left(\int_a^b |g'(s)|^p \, ds \right)^{1/p}, \qquad p \ge 1,$$

then

$$B(x) \le \begin{cases} \frac{1}{2} \|g'\|_\infty [(x-a)^2 + (b-x)^2] & \text{if } g' \in L_\infty[a, b], \\ \dfrac{q}{q+1} \|g'\|_p [(b-x)^{(q+1)/q} + (x-a)^{(q+1)/q}] & \text{if } g' \in L_p[a, b], \\ & p > 1,\ \dfrac{1}{p} + \dfrac{1}{q} = 1, \\ \|g'\|_1 (b-a). & \end{cases} \tag{3.11}$$

If g is monotonic non-decreasing, then

$$\begin{aligned} B(x) &= \int_a^x (g(x) - g(t))\, dt + \int_x^b (g(t) - g(x))\, dt \\ &= (x-a)g(x) - (b-x)g(x) + \int_x^b g(t)\, dt - \int_a^x g(t)\, dt \\ &= [2x - (a+b)]g(x) + \int_a^b \operatorname{sgn}(t-x) g(t)\, dt. \end{aligned} \tag{3.12}$$

Also, by the monotonicity of g on $[a, b]$,

$$\int_x^b g(t)\, dt \le g(b)\,(b-x) \quad \text{and} \quad -\int_a^x g(t)\, dt \le -g(a)\,(x-a)$$

for each $x \in [a, b]$, implying that

$$\begin{aligned} B(x) &\leq (x-a)g(x) - (b-x)g(x) + g(b)\,(b-x) - g(a)\,(x-a) \\ &= (x-a)\,[g(x) - g(a)] + (b-x)\,[g(b) - g(x)] \\ &\leq \max(x-a, b-x)\,[g(b) - g(a)] \\ &= \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right][g(b) - g(a)]. \end{aligned} \tag{3.13}$$

Utilizing the result incorporated in the Equations (3.10)–(3.13), we can provide the following proposition that provides upper bounds for the absolute value of the functional

$$\begin{aligned} \Psi(g, u; a, b, x) := \int_a^b g(t)\,du(t) - \frac{\varphi + \Phi}{2}\int_a^b g(t)\,dt \\ - g(x)\left[u(b) - u(a) - \frac{\varphi + \Phi}{2}(b-a)\right] \end{aligned}$$

that are coarser than the one in (3.9) but, perhaps, more useful in applications.

PROPOSITION 3.5. *Let $u : [a, b] \to \mathbb{R}$ be a (φ, Φ)-Lipschitzian function and g a Riemann integrable function on $[a, b]$.*

(i) *If $g : [a, b] \to \mathbb{R}$ is of $r - H$-Hölder type (K-Lipschitzian) then*

$$\begin{aligned} |\Psi(g, u; a, b, x)| &\leq \frac{1}{2}(\Phi - \varphi)\frac{H}{r+1}[(b-x)^{r+1} + (x-a)^{r+1}] \\ &\left(\leq \frac{1}{2}(\Phi - \varphi)K\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right]\right), \end{aligned} \tag{3.14}$$

for any $x \in [a, b]$.

(ii) *If $g : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$, then*

$$\begin{aligned} |\Psi(g, u; a, b, x)| &\leq \frac{1}{2}(\Phi - \varphi) \\ &\times \begin{cases} \|g'\|_\infty\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] & \text{if } g' \in L_\infty[a, b], \\ \frac{q}{q+1}\|g'\|_p[(b-x)^{(q+1)/q} + (x-a)^{(q+1)/q}] & \text{if } g' \in L_p[a, b], \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ (b-a)\|g'\|_1, \end{cases} \end{aligned} \tag{3.15}$$

for any $x \in [a, b]$.

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(iii) *If* $g : [a, b] \to \mathbb{R}$ *is monotonic non-decreasing on* $[a, b]$, *then*

$$\begin{aligned}|\Psi(g, u; a, b, x)| &\le \frac{1}{2}(\Phi - \varphi)\left\{[2x - (a + b)] + \int_a^b \operatorname{sgn}(t - x) g(t)\, dt\right\}\\ &\le \frac{1}{2}(\Phi - \varphi)\left\{(x - a)\,[g(x) - g(a)] + (b - x)\,[g(b) - g(x)]\right\}\\ &\le \frac{1}{2}(\Phi - \varphi)\left[\frac{1}{2}(b - a) + \left|x - \frac{a + b}{2}\right|\right][g(b) - g(a)], \end{aligned} \tag{3.16}$$

for any $x \in [a, b]$.

In practical applications dealing with the approximation of the Stieltjes integral $\int_a^b g(t)\, du(t)$, the case $x = (a + b)/2$ is of special interest.

If we introduce the functional

$$\begin{aligned}M(g, u; a, b) := \int_a^b g(t)\, du(t) - \frac{\varphi + \Phi}{2}\int_a^b g(t)\, dt\\ - g\left(\frac{a + b}{2}\right)\left[u(b) - u(a) - \frac{\varphi + \Phi}{2}(b - a)\right],\end{aligned}$$

then the following particular case of Proposition 3.5 can be stated.

COROLLARY 3.6. *Assume that* g *and* u *are as in Proposition 3.5.*

(i) *If* g *is of* $r - H$*-Hölder type (*K*-Lipschitzian), then*

$$|M(g, u; a, b)| \le \frac{H(\Phi - \varphi)}{2^{r+1}(r + 1)}(b - a)^{r+1}\left(\le \frac{1}{8}(\Phi - \varphi)K(b - a)^2\right). \tag{3.17}$$

(ii) *If* g *is absolutely continuous on* $[a, b]$, *then*

$$|M(g, u; a, b)| \le \begin{cases} \frac{1}{8}(\Phi - \varphi)(b - a)^2\|g'\|_\infty & \text{if } g' \in L_\infty[a, b],\\ \dfrac{q(\Phi - \varphi)}{(q + 1)2^{(q+1)/q}}\|g'\|_p(b - a)^{(q+1)/q} & \text{if } g' \in L_p[a, b],\\ & p > 1,\ \dfrac{1}{p} + \dfrac{1}{q} = 1,\\ \frac{1}{2}(\Phi - \varphi)\|g'\|_1(b - a). \end{cases} \tag{3.18}$$

(iii) *If* g *is monotonic non-decreasing on* $[a, b]$, *then*

$$\begin{aligned}|M(g, u; a, b)| &\le \frac{1}{2}(\Phi - \varphi)\int_a^b \operatorname{sgn}\left(t - \frac{a + b}{2}\right) g(t)\, dt\\ &\le \frac{1}{4}(\Phi - \varphi)\,[g(b) - g(a)].\end{aligned} \tag{3.19}$$

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4. Inequalities for the weighted Riemann integral

If $h:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$, then $u(t):=\int_a^t h(s)\,ds$ is absolutely continuous on $[a,b]$ and for a Riemann integrable function $f:[a,b]\to\mathbb{R}$ we have

$$\int_a^b f(t)\,du(t)=\int_a^b f(t)h(t)\,dt. \tag{4.1}$$

If n, N are real numbers with $N>n$ and

$$n\le h(t)\le N \quad \text{for almost every } t\in[a,b], \tag{4.2}$$

then

$$n\le \frac{u(t)-u(s)}{t-s}=\frac{\int_s^t h(z)\,dz}{t-s}\le N,$$

for any $t>s$, showing that $u(t)=\int_a^t h(z)\,dz$ is (n,N)-Lipschitzian on $[a,b]$.

Utilizing Theorem 3.1, we can state the following result for weighted integrals.

PROPOSITION 4.1. *Let $f, h:[a,b]\to\mathbb{R}$ be two Riemann integrable functions such that h satisfies (4.1). Then*

$$\left|\int_a^b f(t)h(t)\,dt-\frac{n+N}{2}\int_a^b f(t)\,dt\right|\le\frac{1}{2}(N-n)\int_a^b |f(t)|\,dt. \tag{4.3}$$

The constant $1/2$ is the best possible.

PROOF. The inequality follows from (3.1) for $u(t)=\int_a^t h(s)\,ds$.

For the best constant, we choose $f(t)=t-(a+b)/2$ and $h(t)=\operatorname{sgn}(t-(a+b)/2)$. Then $n=-1$, $N=1$ and

$$\begin{aligned}\int_a^b f(t)h(t)\,dt&=\int_a^b\left(t-\frac{a+b}{2}\right)\operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt\\&=\int_a^b\left|t-\frac{a+b}{2}\right|dt=\frac{(b-a)^2}{4},\end{aligned}$$

$$\int_a^b f(t)\,dt=0 \quad\text{and}\quad \int_a^b |f(t)|\,dt=\frac{(b-a)^2}{4},$$

which produces the same quantity on both sides of (4.3). □

COROLLARY 4.2. *Let g and h be Riemann integrable on $[a,b]$ and let h satisfy the condition (4.2). Then*

$$\begin{aligned}&\left|\int_a^b g(t)h(t)\,dt-\frac{1}{b-a}\int_a^b h(t)\,dt\cdot\int_a^b g(t)\,dt\right|\\&\quad\le\frac{1}{2}(N-n)\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt. \end{aligned}\tag{4.4}$$

The constant $1/2$ is the best possible.

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REMARK 3. This result has been obtained by Cheng and Sun in [6]. The natural extension to abstract Lebesgue integrals and the sharpness of the constant have been established by Cerone and Dragomir in [4].

COROLLARY 4.3. *Let g and h be Riemann integrable functions satisfying the boundedness conditions (3.7) and (4.2). Then*

$$\begin{aligned}\Bigg| \int_a^b g(t)h(t)\,dt - \frac{n+N}{2}\int_a^b g(t)\,dt \\ - \frac{m+M}{2}\int_a^b h(t)\,dt + \frac{(n+N)\,(m+M)}{4}(b-a)\Bigg| \\ \leq \frac{1}{2}(N-n)\int_a^b \left| g(t) - \frac{m+M}{2}\right| dt \leq \frac{1}{4}(M-m)\,(N-n)\,(b-a). \qquad (4.5)\end{aligned}$$

The constants $1/2$ and $1/4$ are the best possible in (4.5).

REMARK 4. The inequality between the first and the last terms in (4.5) has been obtained in [11]. A generalization for the abstract Lebesgue integral has been given as well.

COROLLARY 4.4. *Let g, h be Riemann integrable functions and let h satisfy the boundedness condition (4.2). Then*

$$\begin{aligned}\left| \int_a^b g(t)h(t)\,dt - \frac{n+N}{2}\int_a^b g(t)\,dt - g(x)\left[\int_a^b h(t)\,dt - \frac{n+N}{2}(b-a)\right]\right| \\ \leq \frac{1}{2}(N-n)\int_a^b |g(t)-g(x)|\,dt, \qquad (4.6)\end{aligned}$$

for any $x \in [a, b]$.

The constant $1/2$ is the best possible in (4.6).

If we introduce the operator

$$\begin{aligned}\widetilde{\Psi}(g, h; a, b, x) &:= \Psi\left(g, \int_a^{\cdot} h(s)\,ds; a, b, x\right) \\ &= \int_a^b g(t)h(t)\,dt - \frac{n+N}{2}\int_a^b g(t)\,dt - g(x) \\ &\quad \times \left[\int_a^b h(t)\,dt - \frac{n+N}{2}(b-a)\right], \qquad (4.7)\end{aligned}$$

then the following may be stated as well.

PROPOSITION 4.5. *Let $g, h : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$ and let h satisfy the boundedness condition (4.2).*

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(i) *If $g : [a, b] \to \mathbb{R}$ is of $r - H$-Hölder type (K-Lipschitzian), then $\widetilde{\Psi}(g, h; a, b, x)$ satisfies the inequality*

$$|\widetilde{\Psi}(g, h; a, b, x)| \leq \frac{1}{2}(N-n) \cdot \frac{H}{r+1}[(b-x)^{r+1} + (x-a)^{r+1}] \\ \left(\leq \frac{1}{2}(N-n)K\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right]\right) \tag{4.8}$$

for any $x \in [a, b]$.

(ii) *If g is absolutely continuous on $[a, b]$, then*

$$|\widetilde{\Psi}(g, h; a, b, x)| \leq \frac{1}{2}(N-n) \\ \times \begin{cases} \|g'\|_\infty \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] & \text{if } g' \in L_\infty[a, b], \\ \frac{q}{q+1}\|g'\|_p[(b-x)^{(q+1)/q} + (x-a)^{(q+1)/q}] & \text{if } g' \in L_p[a, b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ (b-a)\|g'\|_1, \end{cases} \tag{4.9}$$

for any $x \in [a, b]$.

(iii) *If $g : [a, b] \to \mathbb{R}$ is monotonic non-decreasing on $[a, b]$, then*

$$|\widetilde{\Psi}(g, h; a, b, x)| \\ \leq \frac{1}{2}(N-n)\left\{[2x - (a+b)] + \int_a^b \operatorname{sgn}(t-x)g(t)\,dt\right\} \\ \leq \frac{1}{2}(N-n)\{(x-a)[g(x) - g(a)] + (b-x)[g(b) - g(x)]\} \\ \leq \frac{1}{2}(N-n)\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right][g(b) - g(a)], \tag{4.10}$$

for any $x \in [a, b]$.

Finally, on defining

$$\widetilde{M}(g, h; a, b) = M\left(g, \int_a^{\cdot} h(s)\,ds; a, b\right) \\ = \int_a^b g(t)\,du(t) - \frac{n+N}{2}\int_a^b g(t)\,dt \\ - g\left(\frac{a+b}{2}\right)\left[\int_a^b h(t)\,dt - \frac{n+N}{2}(b-a)\right], \tag{4.11}$$

then $\widetilde{M}(g, h; a, b)$ satisfies the inequalities (3.17)–(3.19) with n and N replacing φ and Φ.

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5. Applications for the generalized trapezoid formula

The following natural application for the generalized trapezoid formula can be stated.

PROPOSITION 5.1. *Let $f : [a, b] \to \mathbb{R}$ be a (φ, Φ)-Lipschitzian function. Then*

$$\left|\int_a^b f(t)\,dt - \left[f(b)(b-x) + f(a)(x-a) + \frac{\varphi+\Phi}{2}(b-a)\left(x - \frac{a+b}{2}\right)\right]\right| \le \frac{1}{2}(\Phi-\varphi)\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right], \tag{5.1}$$

for each $x \in [a, b]$.

The multiplicative constant $1/2$ is the best possible.

PROOF. For any $x \in [a, b]$ we have the identity (see [5])

$$\int_a^b (t-x)\,df(t) = f(b)\,(b-x) + f(a)\,(x-a) - \int_a^b f(t)\,dt. \tag{5.2}$$

Since f is assumed to be (φ, Φ)-Lipschitzian, then, on applying Theorem 3.1, we have from (5.2) that

$$\left|\int_a^b (t-x)\,df(t) - \frac{\varphi+\Phi}{2}\int_a^b (t-x)\,dt\right| \le \frac{1}{2}(\Phi-\varphi)\int_a^b |t-x|\,dt. \tag{5.3}$$

Since

$$\int_a^b (t-x)\,dt = (b-a)\left(\frac{a+b}{2} - x\right), \quad x \in [a, b]$$

and

$$\int_a^b |t-x|\,dt = \frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2, \quad x \in [a, b],$$

hence (5.3) provides the desired inequality (5.1). □

REMARK 5. For $x = a$, we get the 'right rectangle' inequality

$$\left|\int_a^b f(t)\,dt - f(b)\,(b-a) + \frac{\varphi+\Phi}{4}(b-a)^2\right| \le \frac{1}{4}(\Phi-\varphi)\,(b-a)^2, \tag{5.4}$$

while for $x = b$ we obtain the 'left rectangle' inequality

$$\left|\int_a^b f(t)\,dt - f(a)\,(b-a) - \frac{\varphi+\Phi}{4}(b-a)^2\right| \le \frac{1}{4}(\Phi-\varphi)\,(b-a)^2. \tag{5.5}$$

The case $x = (a+b)/2$ provides the best possible inequality in (5.1), the ‘trapezoid inequality’:

$$\left| \int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a) \right| \le \frac{1}{8}(\Phi - \varphi)\,(b-a)^2. \tag{5.6}$$

The constant $1/8$ is the best possible.

This inequality has been obtained by Liu in [18] as a particular case of Corollary 3.2.

REMARK 6. If f is L-Lipschitzian, that is, $\varphi = -L$, $\Phi = L$, then from (5.1) we get the inequality

$$\left| \int_a^b f(t)\,dt - [f(b)\,(b-x) + f(a)\,(x-a)] \right| \le L\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right], \tag{5.7}$$

for any $x \in [a, b]$, which has been obtained in [3].

6. Applications for Ostrowski-type inequalities

The following particular case of Theorem 3.1 in connection with the celebrated Ostrowski inequality [19] can be stated as well.

PROPOSITION 6.1. *Let $f : [a, b] \to \mathbb{R}$ be a (φ, Φ)-Lipschitzian function. Then*

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - f(x)\,(b-a) + \frac{\varphi + \Phi}{2}(b-a)\left(x - \frac{a+b}{2} \right) \right| \\ &\qquad \le \frac{1}{2}(\Phi - \varphi)\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right], \end{aligned} \tag{6.1}$$

for each $x \in [a, b]$.

The multiplicative constant $1/2$ is the best possible.

PROOF. For any $x \in [a, b]$, we have the Montgomery-type identity [8]

$$\int_a^b p(x,t)\,df(t) = f(x)\,(b-a) - \int_a^b f(t)\,dt, \tag{6.2}$$

for any $x \in [a, b]$, where the kernel $p : [a, b]^2 \to \mathbb{R}$ is defined by

$$p(t,x) := \begin{cases} t - a & \text{if } t \in [a, x], \\ t - b & \text{if } t \in (x, b]. \end{cases}$$

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Since f is assumed to be a (φ, Φ)-Lipschitzian function, then, on applying Theorem 3.1, we have

$$\left|\int_a^b p(x,t)\,df(t) - \frac{\varphi+\Phi}{2}\int_a^b p(x,t)\,dt\right| \le \frac{1}{2}(\Phi-\varphi)\int_a^b |p(x,t)|\,dt. \quad (6.3)$$

Since

$$\int_a^b p(x,t)\,dt = \int_a^x (t-a)\,dt + \int_x^b (t-b)\,dt = (b-a)\left(x - \frac{a+b}{2}\right),$$

and

$$\int_a^b |p(x,t)|\,dt = \int_a^x (t-a)\,dt + \int_x^b (b-t)\,dt = \frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2,$$

by (6.2) and (6.3) we get the desired inequality (6.1). □

REMARK 7. The cases $x = a$ and $x = b$ provide the rectangle inequalities stated in the previous section.

The case $x = (a+b)/2$ provides the best possible inequality in (5.1), the 'midpoint' inequality

$$\left|\int_a^b f(t)\,dt - (b-a) f\left(\frac{a+b}{2}\right)\right| \le \frac{1}{8}(\Phi-\varphi)(b-a)^2. \quad (6.4)$$

The constant $1/8$ is the best possible in (6.4).

This inequality has been obtained by Liu in [18] as a particular case of Corollary 3.2.

REMARK 8. If f is L-Lipschitzian, that is, $\varphi = -L$, $\Phi = L$, then from (6.1) we obtain the inequality:

$$\left|\int_a^b f(t)\,dt - f(x)\,(b-a)\right| \le L\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right], \quad (6.5)$$

for any $x \in [a, b]$, which has been obtained in [9].

References

[1] N. S. Barnett, W. S. Cheung, S. S. Dragomir and A. Sofo, 'Ostrowski and trapezoid type inequalities for the Stieltjes integral with Lipschitzian integrands or integrators', *RGMIA Res. Rep. Coll.* **9**(4) (2006) (Article 9).

[2] P. Cerone, W. S. Cheung and S. S. Dragomir, 'On Ostrowski type inequalities for Stieltjes integrals with absolutely continuous integrands and integrators of bounded variation', *RGMIA Res. Rep. Coll.* **9**(2) (2006) (Article 14).

[3] P. Cerone and S. S. Dragomir, 'Trapezoid type rules from an inequalities point of view', in: *Handbook of analytic computational methods in applied mathematics* (ed. G. Anastassiou) (CRC Press, New York, 2000), pp. 65–134.

[4] ———, 'A refinement of the Grüss inequality and applications', *Tamkang J. Math.* **38**(1) (2007), 37–49.

https://doi.org/10.1017/S0004972708000063 Published online by Cambridge University Press

[5] P. Cerone, S. S. Dragomir and C. E. M. Pearce, 'A generalised trapezoid inequality for functions of bounded variation', *Turkish J. Math.* **24** (2000), 147–163.

[6] X. L. Cheng and J. Sun, 'A note on the perturbed trapezoid inequality', *J. Ineq. Pure Appl. Math.* **3**(2) (2002) (Article 29).

[7] W. S. Cheung and S. S. Dragomir, 'Two Ostrowski type inequalities for the Stieltjes integral of monotonic functions', *Bull. Austral. Math. Soc.* **75** (2007), 299–311.

[8] S. S. Dragomir, 'Ostrowski's inequality for monotonous mappings and applications', *J. KSIAM* **3** (1999), 127–135.

[9] ———, 'On the Ostrowski's inequality for Riemann–Stieltjes integral', *Korean J. Comput. Appl. Math.* **7** (2000), 477–485.

[10] ———, 'On the Ostrowski's inequality for Riemann–Stieltjes integral $\int_a^b f(t)\,du(t)$ where f is of Hölder type and u is of bounded variation and applications', *J. KSIAM* **5** (2001), 35–45.

[11] ———, 'A companion of the Grüss inequality and applications', *Appl. Math. Lett.* **17** (2004), 429–435.

[12] ———, 'Inequalities of Grüss type for the Stieltjes integral', *Kragujevac J. Math.* **26** (2004), 89–122.

[13] ———, 'A generalisation of Cerone's identity and applications', *Tamsui Oxford J. Math. Sci.* **23**(1) (2007), 79–90.

[14] ———, 'Inequalities for Stieltjes integrals with convex integrators and applications', *Appl. Math. Lett.* **20** (2007), 123–130.

[15] S. S. Dragomir, C. Buşe, M. V. Boldea and L. Braescu, 'A generalisation of the trapezoidal rule for the Riemann–Stieltjes integral and applications', *Nonlinear Anal. Forum* **6** (2001), 337–351.

[16] S. S. Dragomir and I. Fedotov, 'An inequality of Grüss type for the Riemann–Stieltjes integral and applications for special means', *Tamkang J. Math.* **29** (1998), 287–292.

[17] ———, 'A Grüss type inequality for mappings of bounded variation and applications for numerical analysis', *Nonlinear Funct. Anal. Appl.* **6** (2001), 425–433.

[18] Z. Liu, 'Refinement of an inequality of Grüss type for Riemann–Stieltjes integral', *Soochow J. Math.* **30** (2004), 483–489.

[19] A. Ostrowski, 'Uber die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert', *Comment. Math. Helv.* **10** (1938), 226–227.

School of Computer Science and Mathematics
Victoria University
PO Box 14428
Melbourne City, VIC 8001
Australia
e-mail: sever.dragomir@vu.edu.au