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Convex Polynomial Approximation in the Uniform Norm: Conclusion

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Abstract. Estimating the degree of approximation in the uniform norm, of a convex function on a finite interval, by convex algebraic polynomials, has received wide attention over the last twenty years. However, while much progress has been made especially in recent years by, among others, the authors of this article, separately and jointly, there have been left some interesting open questions. In this paper we give final answers to all those open problems. We are able to say, for each r-th differentiable convex function, whether or not its degree of convex polynomial approximation in the uniform norm may be estimated by a Jackson-type estimate involving the weighted Ditzian–Totik kth modulus of smoothness, and how the constants in this estimate behave. It turns out that for some pairs (k, r) we have such estimate with constants depending only on these parameters. For other pairs the estimate is valid, but only with constants that depend on the function being approximated, while there are pairs for which the Jackson-type estimate is, in general, invalid.

1 Introduction

Let $\mathbb{C}[a, b]$ denote the space of continuous functions f on $[a, b]$, equipped with the uniform norm

$$\|f\|_{[a,b]} := \max_{x\in[a,b]} |f(x)|.$$

For $I := [-1, 1]$, we omit the interval from this notation and write $\|\cdot\| := \|\cdot\|_I$. Also, let $\mathbb{P}_n$ be the space of all algebraic polynomials of degree $\le n-1$, and denote by

$$E_n(f) := \inf_{p_n\in\mathbb{P}_n} \|f - p_n\|,$$

the degree of best uniform polynomial approximation of f.

Finally, we denote by Δ^2, the set of convex functions on I, and let

$$E_n^{(2)}(f) := \inf_{p_n\in\mathbb{P}_n\cap\Delta^2} \|f - p_n\|$$

denote the degree of best uniform convex polynomial approximation of $f \in \Delta^2 \cap \mathbb{C}[-1, 1]$.

Throughout this paper we will have parameters k, l, m, r all of which will denote nonnegative integers, with $k + r > 0$.

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With $\varphi(x) := \sqrt{1-x^2}$, we denote by $\mathbb{B}^r$, $r \geq 1$, the space of all functions $f \in \mathbb{C}[-1,1]$ with locally absolutely continuous $(r-1)$-st derivative in $(-1,1)$ such that $\|\varphi^r f^{(r)}\| < \infty$, where for $g \in L^\infty[-1,1]$, we denote

$$\|g\| := \operatorname{ess\,sup}_{x\in[-1,1]} |g(x)|.$$

We use the same notation for the L_∞ norm on I, as there can be no confusion.

The following estimates of the degree of convex polynomial approximation of functions $f \in \mathbb{B}^r \cap \Delta^2$ were proved by Leviatan [11] ($r = 1$ and 2) and by Kopotun [6] ($r = 3$ and $r \geq 5$):

$$E_n^{(2)}(f) \leq \frac{c(r)}{n^r} \|\varphi^r f^{(r)}\|, \quad n \geq r. \tag{1.1}$$

Moreover, Kopotun [6] showed that, in general, (1.1) is invalid for $r = 4$. Namely, for every $A > 0$ and $n \geq 1$, there exists a function $f = f_{n,A} \in \mathbb{B}^4 \cap \Delta^2$, for which

$$E_n^{(2)}(f) > A\|\varphi^4 f^{(4)}\|.$$

Nevertheless, Leviatan and Shevchuk [14] have recently proved that, for $f \in \mathbb{B}^4 \cap \Delta^2$,

$$E_n^{(2)}(f) \leq \frac{c}{n^4}\Big(\|\varphi^4 f^{(4)}\| + \frac{1}{n^2}\|f\|\Big), \quad n \geq 1,$$

with an absolute constant c, which implies (1.1) for $n \geq N(f)$ instead of $n \geq r$.

In fact, Leviatan [11] and Kopotun [8] have obtained estimates refining those in (1.1) and involving, respectively, the Ditzian–Totik (D-T) moduli [3], and the weighted D-T moduli of smoothness (see [16]), defined later in this section. In particular, the following result gives a complete answer, in the case of convex approximation, to a central question in approximation theory, namely, to characterize those (convex) functions with prescribed degree of (convex) polynomial approximation. It follows from [6–8, 10, 11, 14], and finally this paper (the case $\alpha = 4$) that,

Theorem 1.1 *For $f \in \Delta^2$ and any $\alpha > 0$, we have*

$$E_n(f) = O(n^{-\alpha}), \quad n \to \infty \quad \Longleftrightarrow \quad E_n^{(2)}(f) = O(n^{-\alpha}), \quad n \to \infty.$$

Let

$$\varphi_\delta(x) := \sqrt{\big(1 - x - \delta\varphi(x)/2\big)\big(1 + x - \delta\varphi(x)/2\big)} = \sqrt{\big(1 - \delta\varphi(x)/2\big)^2 - x^2},$$

provided the expression under the square-root sign is nonnegative. The weighted D-T modulus of smoothness of a function $f \in \mathbb{C}(-1,1)$, is defined by

$$\omega_{k,r}^\varphi(f,t) := \sup_{0<h\leq t} \big\| \varphi_{kh}^r(\cdot)\Delta_{h\varphi(\cdot)}^k(f,\cdot)\big\|,$$

where

$$\Delta_h^k(f,x) := \begin{cases} \sum_{i=0}^k \binom{k}{i}(-1)^{k-i} f(x-kh/2+ih), & \text{if } |x \pm kh/2| < 1, \\ 0, & \text{otherwise,} \end{cases}$$

is the k-th symmetric difference.

If $r = 0$ and $f \in \mathbb{C}[-1,1]$, then

$$\omega_k^\varphi(f,t) := \omega_{k,0}^\varphi(f,t) = \sup_{0<h\le t} \|\Delta_{h\varphi(\,\cdot\,)}^k(f,\,\cdot\,)\|$$

is the (usual) D-T modulus. Also, if $\varphi(\,\cdot\,)$ in the above definition is replaced by 1, then we get the ordinary k-th modulus of smoothness:

$$\omega_k(f,t) := \sup_{0<h\le t} \|\Delta_h^k(f,\,\cdot\,)\|\,.$$

Since $\varphi_\delta(x) \le \varphi(x) \le 1$, it is clear from the above definitions that, if $f \in \mathbb{C}[-1,1]$, then

$$\omega_{k,r}^\varphi(f,t) \le \omega_k^\varphi(f,t) \le \omega_k(f,t). \tag{1.2}$$

Also, for $f \in \mathbb{C}(-1,1)$ and $k \ge 1$, it follows immediately from [3, Thm. 4.1.3], that

$$\omega_{k+1,r}^\varphi(f,t) \le c\,\omega_{k,r}^\varphi(f,t), \tag{1.3}$$

and

$$\omega_{k,r}^\varphi(f,t) \le c\|\varphi^r f\|. \tag{1.4}$$

Here and in the sequel, we write c for a constant which may depend only on k and r.

Moreover, it immediately follows from the definition that, for any $[a,b] \subset (-1,1)$,

$$\omega_k(f,t,[a,b]) \le C\omega_{k,r}^\varphi(f,t), \tag{1.5}$$

where C depends on k, r, and $\operatorname{dist}\{[a,b],\pm 1\} > 0$, and $\omega_k(f,t,[a,b])$ is the k-th usual modulus of smoothness on $[a,b]$, *i.e.*,

$$\omega_k(f,t,[a,b]) := \sup_{0<h\le t} \|\Delta_h^k(f,\,\cdot\,)\|_{[a+kh/2,\,b-kh/2]}.$$

The modulus $\omega_{k,r}^\varphi$ has many of the properties of the usual and D-T moduli of smoothness. In particular (see [16, (18.14)]), for any $k \in \mathbb{N}$, $r \ge 0$, and $f \in \mathbb{C}(-1,1)$,

$$\omega_{k,r}^\varphi(f,\lambda t) \le c(\lambda+1)^k \omega_{k,r}^\varphi(f,t), \qquad \lambda > 0.$$

This, in turn, implies that if a function f is not a polynomial of degree $\le k-1$, then, for some $C = C(f) > 0$,

$$\omega_{k,r}^\varphi(f,t) \ge Ct^k, \qquad \text{for all } 0 < t \le 1. \tag{1.6}$$

For arbitrary $f \in \mathbb{C}(-1,1)$, the function $\omega^{\varphi}_{k,r}(f,t)$ may be unbounded. However, it was shown in [12, 16] that a necessary and sufficient condition for $\omega^{\varphi}_{k,r}(f,t)$ to be bounded for all $t > 0$ is that $\varphi^r f \in L^{\infty}(-1,1)$. Moreover, if $r \ge 1$, then $\omega^{\varphi}_{k,r}(f,t) \to 0$, as $t \to 0$, if and only if $\lim_{x\to\pm 1} \varphi^r(x) f(x) = 0$. Therefore, we denote $\mathbb{C}^0_{\varphi} := \mathbb{C}[-1,1]$ and, for $r \ge 1$,

$$\mathbb{C}^r_{\varphi} := \{ f \in \mathbb{C}^r(-1,1) \cap \mathbb{C}[-1,1] \mid \lim_{x\to\pm 1} \varphi^r(x) f^{(r)}(x) = 0 \}. \tag{1.7}$$

Clearly $\mathbb{C}^r_{\varphi} \subset \mathbb{B}^r$. If $f \in \mathbb{B}^r$, then $f \in \mathbb{C}^l_{\varphi}$ for all $0 \le l < r$, and

$$\omega^{\varphi}_{r-l,l}(f^{(l)},t) \le c\, t^{r-l} \| \varphi^r f^{(r)} \|, \qquad t > 0. \tag{1.8}$$

Note that for $f \in \mathbb{C}^r_{\varphi}$, and any $0 \le l \le r$ and $k \ge 1$, the following inequalities hold (see [16, Lemma 18.4, (18.19)], and [8, (1.1)]).

$$\omega^{\varphi}_{k+r-l,l}(f^{(l)},t) \le c t^{r-l} \omega^{\varphi}_{k,r}(f^{(r)},t), \quad t > 0, \tag{1.9}$$

in particular, if $l = 0$, then

$$\omega^{\varphi}_{k+r}(f,t) \le c t^r \omega^{\varphi}_{k,r}(f^{(r)},t), \quad t > 0. \tag{1.10}$$

Finally, for $0 \le l < r/2$,

$$\mathbb{C}^r_{\varphi} \subset \mathbb{B}^r \subset \mathbb{C}^l[-1,1], \tag{1.11}$$

and (see [16, (18.18)])

$$\omega^{\varphi}_{k+r-l}(f^{(l)},t) \le c\, t^{r-2l} \omega^{\varphi}_{k,r}(f^{(r)},t), \quad t > 0,\ 0 \le l < r/2. \tag{1.12}$$

In this paper, we are interested in determining for which values of parameters k and r, the statement

> *if* $f \in \Delta^2 \cap \mathbb{C}^r_{\varphi}$, *then*
>
> $$E^{(2)}_n(f) \le \frac{C}{n^r} \omega^{\varphi}_{k,r}\big(f^{(r)}, 1/n\big), \quad n \ge N, \tag{1.13}$$
>
> *where* $C = \text{const} > 0$ *and* $N = \text{const} > 0$,

is valid, and for which it is invalid. Here and later in this paper, for clarity of exposition, we denote $\omega_{0,r}(f,t) := \|\varphi^r f\|$. Hence, in the case $k = 0$, (1.13) becomes:

$$E^{(2)}_n(f) \le \frac{C}{n^r} \|\varphi^r f^{(r)}\|, \quad n \ge N,$$

for $f \in \Delta^2 \cap \mathbb{B}^r$, which is the inequality (1.1).

It turns out that the validity of the above statement depends not only on our choice of k and r but also on whether or not we allow constants appearing in (1.13) to depend on the function f.

For the reader's convenience we describe our results using an array in Figure 1. There, the symbols "$-$", "$\ominus$", and "$+$", have the following meaning.

r								
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋰
6	+	+	+	+	+	+	+	⋯
5	+	+	+	+	+	+	+	⋯
4	⊖	⊖	−	−	−	−	−	⋯
3	+	⊖	⊖	−	−	−	−	⋯
2	+	+	⊖	⊖	−	−	−	⋯
1	+	+	+	⊖	−	−	−	⋯
0		+	+	+	⊖	−	−	⋯
	0	1	2	3	4	5	6	**k**

Figure 1: Convex approximation: validity of $E_n^{(2)}(f) \le Cn^{-r}\omega_{k,r}^{\varphi}\big(f^{(r)}, 1/n\big)$, $n \ge N$

- The symbol "$-$" in the position (k, r) means that there is a function $f \in \Delta^2 \cap \mathbb{C}_\varphi^r$, such that
$$\limsup_{n\to\infty} \frac{n^r E_n^{(2)}(f)}{\omega_{k,r}^{\varphi}(f^{(r)}, 1/n)} = \infty.$$
In other words, the estimate (1.13) is invalid even if we allow constants C and N to depend on the function f.
- The symbol "+" in the position (k, r) means that (1.13) (or (1.1) if $k = 0$) is valid with C depending only on k and r, and $N = k + r$.
- The symbol "$\ominus$" in the position (k, r) means that (1.13) is valid with C depending only on k and r, and N depending on the function f; and there are no constants C and N independent of f, such that (1.13) holds for every function $f \in \Delta^2 \cap \mathbb{C}_\varphi^r$.

These results are obtained in (or can be derived from) the papers listed in Table 1.

The structure of our paper is as follows. In Section 2, we state our main results, which allow us to complete the above figure and table. In Section 3, we collect some auxiliary results, some unexpected properties of the weighted D-T moduli, Lemma 3.3 and Lemma 3.4, which enable us to apply [13, Lemma 4.1] in order to prove Theorem 2.1 in Section 4, hence, filling the gap for $\alpha = 4$ in Theorem 1.1. The negative results are proved in Section 5, by constructing a counterexample which is an essential development of ideas of [2, 6]. Finally, Appendix A is devoted to a short proof of auxiliary inequalities involving Chebyshev polynomials.

2 Main Results

Theorem 2.1 *If $f \in \mathbb{C}_\varphi^2 \cap \Delta^2$, then*

$$E_n^{(2)}(f) \le c\big(n^{-2}\omega_{3,2}^{\varphi}(f'', 1/n) + n^{-6}\|f''\|_{[-1/2,1/2]}\big), \quad n \ge N, \tag{2.1}$$

	Positive results: "+" in position (k, r)	
1986	$(2, 0)$, hence, $\{(k, r) \mid k + r \leq 2\}$	Leviatan [11, Thm. 1]
1994	$(3, 0)$, hence, $\{(k, r) \mid k + r \leq 3\}$	Kopotun [7, Thm. 2] (see also Hu, Leviatan and Yu [5, Thm. 1])
1992	$\{(0, r) \mid r \geq 5\}$	Kopotun [6, Thm. 1]
1995	$\{(k, r) \mid r \geq 5, k \geq 1\}$	Kopotun [8, Thm. 2]
	Positive results: "$\ominus$" in position (k, r)	
2003	$(4, 0)$, hence, $\{(k, r) \mid k + r = 4\}$	Leviatan and Shevchuk [14, Cor. 3.2]
—	$(3, 2)$, hence $(2, 3)$ and $(1, 4)$	**This paper** (see Corollary 2.2)
	Negative results: "+" CANNOT be in position (k, r)	
1981	$(4, 0)$, hence, $\{(k, 0) \mid k \geq 4\}$	Shvedov [17, Thm. 3]
1991	$(3, 1)$, hence, $\{(k, r) \mid 4 - k \leq r \leq 1\}$	Mania (see [16, Theorem 16.1])
1992	$(0, 4)$, hence, $\{(k, r) \mid 4 - k \leq r \leq 4\}$	Kopotun [6, Thm. 2]
	Negative results: "$\ominus$" CANNOT be in position (k, r)	
1992	$(5, 0)$, hence, $\{(k, 0) \mid k \geq 5\}$	Wu and Zhou [18, Thm., p. 206]
2002	$(4, 1)$, hence, $\{(k, r) \mid 5 - k \leq r \leq 1\}$	Nissim and Yushchenko [15, Thm. 2]
—	$(2, 4)$, hence, $\{(k, r) \mid 6 - k \leq r \leq 4\}$	**This paper** (see Corollary 2.4)

Table 1

where c and N are absolute constants. Hence,

$$E_n^{(2)}(f) \leq cn^{-2}\omega_{3,2}^{\varphi}(f'', 1/n), \quad n \geq N(f). \tag{2.2}$$

By virtue of (1.9), an immediate consequence is

Corollary 2.2 *Let $2 \leq r \leq 4$, $1 \leq k \leq 5 - r$. If $f \in \mathbb{C}_\varphi^r \cap \Delta^2$, then*

$$E_n^{(2)}(f) \leq \frac{c}{n^r}\omega_{k,r}^{\varphi}(f^{(r)}, 1/n), \quad n \geq N(f). \tag{2.3}$$

On the other hand, we have the following negative result.

Theorem 2.3 *There is a function $f \in \mathbb{C}_\varphi^4 \cap \Delta^2$, such that*

$$\limsup_{n\to\infty} \frac{n^4 E_n^{(2)}(f)}{\omega_{2,4}^{\varphi}(f^{(4)}, 1/n)} = \infty. \tag{2.4}$$

Corollary 2.4 *Let $0 \leq r \leq 4$ and $k \geq 6 - r$. Then there is a function $f \in \mathbb{C}_\varphi^r \cap \Delta^2$, such that*

$$\limsup_{n\to\infty} \frac{n^r E_n^{(2)}(f)}{\omega_{k,r}^{\varphi}(f^{(r)}, 1/n)} = \infty.$$

3 Auxiliary Results

The following notion of the length of an interval $J := [a,b] \subseteq I$, relative to its position in I, was introduced in [12]:

$$/J/ := \frac{|J|}{\varphi\left(\frac{a+b}{2}\right)},$$

where $|J| := b-a$ is the length of J. The following was proved in [12, (2.20)–(2.22)]:

$$\text{If } J_1 \subseteq J, \text{ then } /J_1/ \leq /J/. \tag{3.1}$$

$$\omega_k(f, |J|, J) \leq \omega_k^{\varphi}(f, /J/). \tag{3.2}$$

$$\omega_k(f, |J|, J) \leq \frac{1}{w^r(a,b)}\omega_{k,r}^{\varphi}(f, /J/), \quad \text{where } w(a,b) := \sqrt{(1+a)(1-b)}. \tag{3.3}$$

Let $x_j := \cos(j\pi/n)$, $0 \leq j \leq n$, be the Chebyshev knots, and denote $I_j := [x_j, x_{j-1}]$, $1 \leq j \leq n$. Then, $|I_j| \sim \frac{\varphi(x_j)}{n} + \frac{1}{n^2}$ (see (5.6) for exact constants in this equivalence, and see [12, (3.1)])

$$\frac{2}{n} \leq /I_j/ \leq \frac{\pi}{n},$$

for all $1 \leq j \leq n$.

Also, for $1 < j < n$ we have

$$|I_j| \leq c w(x_j, x_{j-1}) n^{-1}.$$

Therefore, for $1 < j < n$ and $0 \leq l \leq r$,

$$\begin{aligned} |I_j|^l \omega_{k+r-l}(f^{(l)}, |I_j|, I_j) &\leq c|I_j|^r \omega_k(f^{(r)}, |I_j|, I_j) \\ &\leq c\frac{|I_j|^r}{w^r(x_j, x_{j-1})}\omega_{k,r}^{\varphi}\left(f^{(r)}, /I_j/\right) \\ &\leq cn^{-r}\omega_{k,r}^{\varphi}\left(f^{(r)}, n^{-1}\right). \end{aligned} \tag{3.4}$$

In the cases $j = 1$ and $j = n$, we cannot use the same sequence of estimates since $w(x_j, x_{j-1}) = 0$. However, we have

$$\omega_k(f, t^2) \leq 2\omega_k^{\varphi}(f, t), \quad t > 0, \ k \geq 1,$$

[12, (2.25)] or [3, Corollary 3.1.3]. Hence, since $|I_1|, |I_n| \sim n^{-2}$, we conclude by (1.12) that, for all $0 \leq l < r/2$ and $j = 1$ or n,

$$\begin{aligned} |I_j|^l \omega_{k+r-l}(f^{(l)}, |I_j|, I_j) &\leq cn^{-2l}\omega_{k+r-l}^{\varphi}(f^{(l)}, 1/n) \\ &\leq cn^{-r}\omega_{k,r}^{\varphi}\left(f^{(r)}, 1/n\right). \end{aligned} \tag{3.5}$$

Let $L_{m-1}(g\,; z_0, z_1, \dots, z_{m-1})$ denote the polynomial of degree $\le m-1$ which interpolates a function g at the points $z_0, z_1, \dots, z_{m-1}$. We remind the reader that $[z_0, \dots, z_m\,; g]$ stands for the m-th divided difference of a function g at the knots $z_0, \dots, z_m$ defined by

$$(3.6)\qquad [z_0, z_1, \dots, z_m\,; g] := \frac{g(z_m) - L_{m-1}(g\,; z_0, z_1, \dots, z_{m-1})(z_m)}{(z_m - z_0)(z_m - z_1)\dots(z_m - z_{m-1})}.$$

The following Newton formula for interpolating polynomials is well known:

$$(3.7)\qquad L_{m-1}(g; z_0, z_2, \dots, z_{m-1})(x) = \sum_{i=0}^{m-1}(x - z_0)\cdots(x - z_{i-1})[z_0, \dots, z_i\,; g].$$

Also, assuming that $z_0, z_1, \dots, z_m$ form either a non-increasing or a non-decreasing sequence such that $\min_{0\le i\le m-1}|z_{i+1} - z_i| \sim \max_{0\le i\le m-1}|z_{i+1} - z_i|$, and using Whitney's inequality we have the following estimate:

$$(3.8)\qquad \begin{aligned} &|[z_0, z_1, \dots, z_m\,; g]| \\ &\qquad \le C|z_m - z_0|^{-m}\omega_m\big(g, |z_m - z_0|, [\min\{z_0, z_m\}, \max\{z_0, z_m\}]\big), \end{aligned}$$

where C depends on m and the ratio

$$\min_{0\le i\le m-1}|z_{i+1} - z_i| / \max_{0\le i\le m-1}|z_{i+1} - z_i|.$$

Lemma 3.1 *Let $f \in \mathbb{C}^r_\varphi$, $n \ge k+1$, and let a polynomial p_{k+r} of degree $\le k+r-1$ be such that $p^{(i)}_{k+r}(x_1) = f^{(i)}(x_1)$ for all $i = 0, 1, \dots, r-1$ and $p^{(r)}_{k+r}(x) = L_{k-1}(f^{(r)}\,; x_1, x_2, \dots, x_k)(x)$. Then,*

$$(3.9)\qquad \|f - p_{k+r}\|_{I_1} \le \frac{c}{n^r}\omega^\varphi_{k,r}(f^{(r)}, 1/n).$$

We remark that a similar estimate holds for the interval I_n and a polynomial p_{k+r} defined analogously on I_n.

Proof First, we denote $\mathcal{L}_{k-1} := L_{k-1}(f^{(r)}\,; x_1, x_2, \dots, x_k)$ and note that it follows by Whitney's theorem and (3.3), that for any $\beta \in [x_1, 1)$ and the interval $J := [x_k, \beta]$ we have

$$\begin{aligned} \|\mathcal{L}_{k-1} - f^{(r)}\|_J &\le c\,\omega_k(f^{(r)}, |J|; J) \\ &\le \frac{c}{(1-\beta)^{r/2}}\,\omega^\varphi_{k,r}(f^{(r)}, /J/) \le \frac{c}{(1-\beta)^{r/2}}\,\omega^\varphi_{k,r}(f^{(r)}, 1/n) \end{aligned}$$

and, in particular,

$$(3.10)\qquad |\mathcal{L}_{k-1}(\beta) - f^{(r)}(\beta)| \le \frac{c}{(1-\beta)^{r/2}}\omega^\varphi_{k,r}(f^{(r)}, 1/n).$$

Now, since for any $g \in \mathbb{C}^r[a,b]$ and $x \in [a,b]$,

$$g(x) = \sum_{i=0}^{r-1} \frac{g^{(i)}(a)}{i!}(x-a)^i + \frac{1}{(r-1)!}\int_a^x (x-u)^{r-1} g^{(r)}(u)\,du,$$

using (3.10), we conclude that, for any $x_1 \le x < 1$, the following holds:

$$\begin{aligned}
|f(x) - p_{k+r}(x)| &= \frac{1}{(r-1)!}\left|\int_{x_1}^x (x-u)^{r-1}\left(f^{(r)}(u) - p_{k+r}^{(r)}(u)\right)\,du\right| \\
&\le c\int_{x_1}^x (x-u)^{r-1}\left|f^{(r)}(u) - \mathcal{L}_{k-1}(u)\right|\,du \\
&\le c\omega_{k,r}^{\varphi}(f^{(r)}, 1/n)\int_{x_1}^x \frac{(x-u)^{r-1}}{(1-u)^{r/2}}\,du \\
&\le c\omega_{k,r}^{\varphi}(f^{(r)}, 1/n)\int_{x_1}^1 (1-u)^{r/2-1}\,du \\
&\le c(1-x_1)^{r/2}\omega_{k,r}^{\varphi}(f^{(r)}, 1/n) \\
&\le cn^{-r}\omega_{k,r}^{\varphi}(f^{(r)}, 1/n).
\end{aligned}$$

Note that since $f - p_{k+r}$ is continuous, the above inequality is also valid for $x = 1$. The proof of the lemma is now complete. ∎

Lemma 3.2 *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha + \beta > 1$, and let $1 \le \nu \le \lfloor \frac{n}{2} \rfloor$. Then,*

$$\sum_{i=\nu}^{\lfloor \frac{n}{2} \rfloor} |I_i|^{-\alpha}\varphi^{-\beta}(x_i) \sim n^{2\alpha+\beta}\sum_{i=\nu}^{\lfloor \frac{n}{2} \rfloor} i^{-\alpha-\beta} \le Cn^{2\alpha+\beta}\nu^{1-\alpha-\beta}, \tag{3.11}$$

where C and the constants in the equivalence relation depend only on α and β, and are independent of ν and n. Furthermore,

$$\sum_{i=1}^{n-1} |I_i|^{-\alpha}\varphi^{-\beta}(x_i) \le Cn^{2\alpha+\beta}. \tag{3.12}$$

Proof Taking into account that, for any $1 \le i \le \lfloor \frac{n}{2} \rfloor$,

$$|I_i| \sim \frac{\varphi(x_i)}{n} \quad \text{and} \quad \varphi(x_i) = \sqrt{1 - x_i^2} = \sin(i\pi/n) \sim \frac{i}{n},$$

we have

$$\sum_{i=\nu}^{\lfloor \frac{n}{2} \rfloor} |I_i|^{-\alpha}\varphi^{-\beta}(x_i) \sim \sum_{i=\nu}^{\lfloor \frac{n}{2} \rfloor} n^{\alpha}\varphi^{-\alpha-\beta}(x_i) \sim n^{2\alpha+\beta}\sum_{i=\nu}^{\lfloor \frac{n}{2} \rfloor} i^{-\alpha-\beta} \le C\frac{n^{2\alpha+\beta}}{\nu^{\alpha+\beta-1}}.$$

Inequality (3.12) immediately follows from (3.11) with $\nu = 1$ taking into account that $\varphi(x_i) = \varphi(x_{n-i})$ and $|I_{n-i}| = |I_{i+1}| \sim |I_i|$. ∎

Lemma 3.3 *Let $k \geq 2$, $r \in \mathbb{N}$, $1 \leq m \leq k-1$, $n \geq 3m$, and let $j \in \mathbb{N}$ be such that $[x_{j+m}, x_j] \subset [-1/2, 1/2]$ (which is equivalent to $n/3 \leq j \leq 2n/3 - m$). Then, for any $f \in \mathbb{C}(-1,1)$,*

$$\big|[x_j, x_{j+1}, \ldots, x_{j+m}\,; f]\big| \leq c\left(n^k \omega_{k,r}^{\varphi}(f, 1/n) + \|f\|_{[-1/2,1/2]}\right), \tag{3.13}$$

where c depends only on k and r.

Note that Lemma 3.3 is valid if $m = 0$ and $m = k$ as well. However, since (3.13) becomes quite weak in these cases, they are excluded from the statement.

Proof First, recall the Marchaud inequality for a function $f \in \mathbb{C}[a,b]$ and $1 \leq m < k$:

$$\omega_m(f, t, [a,b]) \leq ct^m \left\{ \int_t^{b-a} \frac{\omega_k(f, s, [a,b])\, ds}{s^{m+1}} + (b-a)^{-m} \|f\|_{[a,b]} \right\}.$$

Now, taking $[a,b] = [-1/2, 1/2]$ in the above estimate, using (3.8), and taking into account that $|x_j - x_{j+m}| \sim 1/n$ for all j such that $x_j \in [-1/2, 1/2]$, we have

$$\begin{aligned}
\big|[x_j, x_{j+1}, \ldots, x_{j+m}\,; f]\big| &\leq c(x_j - x_{j+m})^{-m} \omega_m(f, x_j - x_{j+m}, [x_{j+m}, x_j]) \\
&\leq cn^m \omega_m(f, 1/n, [-1/2, 1/2]) \\
&\leq \int_{1/n}^1 \frac{\omega_k(f, s, [-1/2, 1/2])}{s^{m+1}}\, ds + c\|f\|_{[-1/2,1/2]} \\
&= c\int_{1/n}^1 \frac{\omega_k(f, s, [-1/2, 1/2])}{s^k} \cdot s^{k-1-m}\, ds \\
&\qquad + c\|f\|_{[-1/2,1/2]} \\
&\leq cn^k \omega_k(f, 1/n, [-1/2, 1/2]) \int_{1/n}^1 s^{k-1-m}\, ds \\
&\qquad + c\|f\|_{[-1/2,1/2]} \\
&\leq cn^k \omega_k(f, 1/n, [-1/2, 1/2]) + c\|f\|_{[-1/2,1/2]} \\
&\leq cn^k \omega_{k,r}^{\varphi}(f, 1/n) + c\|f\|_{[-1/2,1/2]},
\end{aligned}$$

where the last inequality follows from (1.5). ∎

The following lemma generalizes Lemma 3.3 to the cases when x_j can be close to the endpoints of $[-1,1]$. Note that the condition $r > k - 2m$ in its statement is essential. In fact, this is the main reason why (2.2) is no longer valid with $\omega_{k,2}^{\varphi}$, $k \geq 4$, instead of $\omega_{3,2}^{\varphi}$.

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Lemma 3.4 *Let $k \geq 2$, $n \geq 3k$, $1 \leq m \leq k-1$, and let $r \in \mathbb{N}_0$, $r > k-2m$. Then, for $f \in \mathbb{C}(-1,1)$ and for every $1 \leq j \leq n-m-1$,*

$$\begin{aligned}(3.14)\qquad &|[x_j, x_{j+1}, \dots, x_{j+m}\,; f]| \\ &\qquad \leq cn^k\Big(\frac{n}{\min\{j, n-j\}}\Big)^{r+2m-k}\omega^{\varphi}_{k,r}(f, 1/n) + c\|f\|_{[-1/2,1/2]}.\end{aligned}$$

Proof It is sufficient to prove this lemma for $1 \leq j \leq \lfloor\frac{n}{2}\rfloor$, the other case being symmetric. Also, recall that, for all $1 \leq i \leq \lfloor\frac{n}{2}\rfloor$, $\varphi(x_i) \sim i/n$ and $|I_i| \sim i/n^2$.

Now, for all $1 \leq j \leq n-k-1$, the inequalities (3.8) and (3.3) imply

$$\begin{aligned}|[x_j, x_{j+1}, \dots, x_{j+k}\,; f]| &\leq c|I_j|^{-k}\omega_k(f, x_j - x_{j+k}, [x_{j+k}, x_j]) \\ &\leq c|I_j|^{-k}w^{-r}(x_{j+k}, x_j)\omega^{\varphi}_{k,r}(f, /[x_{j+k}, x_j]/) \\ &\leq c|I_j|^{-k}\varphi^{-r}(x_j)\omega^{\varphi}_{k,r}(f, 1/n).\end{aligned}$$

Therefore, if $1 \leq j \leq \min\big\{n-k-1, \lfloor\frac{n}{2}\rfloor\big\}$, then

$$|[x_j, x_{j+1}, \dots, x_{j+k}\,; f]| \leq cn^k\Big(\frac{n}{j}\Big)^{k+r}\omega^{\varphi}_{k,r}(f, 1/n),$$

which is stronger than (3.14) for $m = k$. We now use induction in m (the case $m = k$ being its base). Suppose that k and r are fixed, that m is such that $1 \leq m \leq k-1$ and $m > (k-r)/2$, and that (3.14) is valid with m replaced by $m+1$. We will now show that it has to be valid for m as well, which will complete the proof of the lemma.

Let $1 \leq \mu \leq \lfloor\frac{n}{2}\rfloor$ be an index such that $[x_{\mu+m}, x_\mu] \subset [-1/2, 1/2]$. Lemma 3.3 implies that

$$|[x_\mu, \dots, x_{\mu+m}\,; f]| \leq c(n^k\omega^{\varphi}_{k,r}(f, 1/n) + \|f\|_{[-1/2,1/2]}).$$

Also,

$$\begin{aligned}-[x_j, x_{j+1}, \dots, x_{j+m}\,; f] &= \big([x_\mu, \dots, x_{\mu+m}\,; f] - [x_j, \dots, x_{j+m}\,; f]\big) \\ &\qquad - [x_\mu, \dots, x_{\mu+m}\,; f] \\ &= \sum_{i=j}^{\mu-1}(x_{i+m+1} - x_i)[x_i, x_{i+1}, \dots, x_{i+m+1}\,; f] \\ &\qquad - [x_\mu, \dots, x_{\mu+m}\,; f].\end{aligned}$$

Therefore,

$$\begin{aligned}\big|[x_j, x_{j+1}, \dots, x_{j+m}\,; f]\big| &\leq c\sum_{i=j}^{\lfloor\frac{n}{2}\rfloor}|I_i|\,|[x_i, x_{i+1}, \dots, x_{i+m+1}\,; f]| \\ &\qquad + cn^k\omega^{\varphi}_{k,r}(f, 1/n) + c\|f\|_{[-1/2,1/2]},\end{aligned}$$

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and using our induction hypothesis we have

$$
\begin{aligned}
&\big|[x_j, x_{j+1}, \dots, x_{j+m}\,; f]\big| \\
&\le c \sum_{i=j}^{\lfloor \frac{n}{2} \rfloor} |I_i| \left(n^k \Big(\frac{n}{\min\{i, n-i\}} \Big)^{r+2(m+1)-k} \omega_{k,r}^{\varphi}(f, 1/n) + \|f\|_{[-1/2,1/2]} \right) \\
&\qquad + cn^k \omega_{k,r}^{\varphi}(f, 1/n) + c\|f\|_{[-1/2,1/2]} \\
&\le cn^k \omega_{k,r}^{\varphi}(f, 1/n) \sum_{i=j}^{\lfloor \frac{n}{2} \rfloor} |I_i| \Big(\frac{n}{i} \Big)^{r+2(m+1)-k} + cn^k \omega_{k,r}^{\varphi}(f, 1/n) + c\|f\|_{[-1/2,1/2]} \\
&\le cn^k \omega_{k,r}^{\varphi}(f, 1/n) \sum_{i=j}^{\lfloor \frac{n}{2} \rfloor} \frac{n^{r+2m-k}}{i^{r+2m-k+1}} + cn^k \omega_{k,r}^{\varphi}(f, 1/n) + c\|f\|_{[-1/2,1/2]} \\
&\le cn^k \omega_{k,r}^{\varphi}(f, 1/n) \Big(\frac{n}{j} \Big)^{r+2m-k} + c\|f\|_{[-1/2,1/2]},
\end{aligned}
$$

and the proof is now complete. ■

We need the following special case of Lemma 3.4 for $j = 1$, $k = 3$, $r = 2$, and $m = 1$ or 2.

Corollary 3.5 *Let $n \ge 9$, $m = 1$ or $m = 2$, and $f \in \mathbb{C}(-1, 1)$. Then,*

$$
|[x_1, x_2, \dots, x_{m+1}\,; f]| \le cn^{2m+2}\, \omega_{3,2}^{\varphi}(f, 1/n) + c\|f\|_{[-1/2,1/2]}. \tag{3.15}
$$

4 Proofs of Positive Results

Let $\Sigma_{k,n}$ be the collection of all continuous piecewise polynomials of degree $k - 1$ on the Chebyshev partition $\{x_j\}_{j=0}^n = \{\cos(j\pi/n)\}_{j=0}^n$.

The following lemma is a corollary of (more general) [13, Theorem 3].

Lemma 4.1 *For every $k \in \mathbb{N}$ there are constants $c = c(k)$ and $c_* = c_*(k)$, such that if $n \in \mathbb{N}$ and $S \in \Sigma_{k,n} \cap \Delta^2$, then there is a polynomial $P_n \in \Delta^2$ of degree $\le c_* n$, satisfying*

$$
\|S - P_n\| \le c\omega_k^{\varphi}(S, 1/n). \tag{4.1}
$$

Hence, in order to obtain direct estimates for polynomial approximation, we only need to construct suitable piecewise polynomials $S \in \Sigma_{k,n} \cap \Delta^2$.

In order to construct such piecewise polynomials, we use the following result which was proved in [14, Corollary 2.4].

Lemma 4.2 *Let $k \in \mathbb{N}$, and let $f \in \mathbb{C}^2[a, a+h]$, $h > 0$, be convex. Then there exists a convex polynomial P of degree $\le k+1$ satisfying $P(a) = f(a)$, $P(a+h) = f(a+h)$, $P'(a) \ge f'(a)$, and $P'(a+h) \le f'(a+h)$, and such that*

$$\|f - P\|_{[a,a+h]} \le c(k)h^2\omega_k(f'', h, [a, a+h]).$$

Lemma 4.3 *If $f \in \mathbb{C}^2_\varphi \cap \Delta^2$, then for $n \ge 9$, there is a continuous piecewise quartic polynomial $s_n \in \Sigma_{5,n} \cap \Delta^2$, such that*

$$\|f - s_n\|_{[x_{n-1},x_1]} \le \frac{c}{n^2}\,\omega^\varphi_{3,2}(f'', 1/n) \tag{4.2}$$

and, for $j = 1$ and $j = n$,

$$\|f - s_n\|_{I_j} \le cn^{-2}\,\omega^\varphi_{3,2}(f'', 1/n) + cn^{-6}\|f''\|_{[-1/2,1/2]}. \tag{4.3}$$

Proof First, we apply Lemma 4.2 on each interval I_j, $j = 2, \dots, n-1$ (*i.e.*, set $a := x_j$ and $h := |I_j| = x_{j-1} - x_j$), with $k = 3$, and define $s_n\,|_{I_j} := P$.

On the intervals I_1 and I_n, we define s_n as follows

$$s_n^{(i)}(x_1+) = f^{(i)}(x_1),\ i = 0, 1 \quad \text{and} \quad s_n''(x) = f''(x_1),\ x \in I_1,$$

and

$$s_n^{(i)}(x_{n-1}-) = f^{(i)}(x_{n-1}),\ i = 0, 1 \quad \text{and} \quad s_n''(x) = f''(x_{n-1}),\ x \in I_n.$$

Then, s_n is a continuous piecewise quartic polynomial on $[-1, 1]$ which is convex (since s_n' is non-decreasing for all $x \in [-1, 1]$) and such that, for every $j = 2, \dots, n-1$,

$$\|f - s_n\|_{I_j} \le c|I_j|^2\omega_3(f'', |I_j|, I_j).$$

Now, we use the estimate (3.4) with $l = r = 2$ and $k = 3$ to conclude

$$\|f - s_n\|_{I_j} \le \frac{c}{n^2}\,\omega^\varphi_{3,2}(f'', 1/n) \tag{4.4}$$

for $j = 2, \dots, n-1$. Hence, (4.2) is proved, and it remains to estimate $\|f - s_n\|_{I_1}$ and $\|f - s_n\|_{I_n}$. We only estimate the former since the latter can be dealt with analogously. First, Lemma 3.1 with $k = 3$ and $r = 2$ implies that

$$\|f - l_1\|_{I_1} \le \frac{c}{n^2}\,\omega^\varphi_{3,2}(f'', 1/n), \tag{4.5}$$

where the polynomial l_1 of degree ≤ 4 is such that $l_1^{(i)}(x_1) = f^{(i)}(x_1)$, $i = 0, 1$, and

$$\begin{aligned} l_1''(x) := L_2(f''; x_1, x_2, x_3)(x) &= f''(x_1) + (x - x_1)[x_1, x_2; f''] \\ &+ (x - x_1)(x - x_2)[x_1, x_2, x_3; f'']. \end{aligned}$$

Now, using Corollary 3.5, we have for every $x \in I_1$

$$\begin{aligned}|l_1''(x) - s_n''(x)| &= |l_1''(x) - f''(x_1)|\\ &= \big|(x-x_1)[x_1,x_2\,;f''] + (x-x_1)(x-x_2)[x_1,x_2,x_3\,;f'']\big|\\ &\le n^{-2}\big|[x_1,x_2\,;f'']\big| + n^{-4}\big|[x_1,x_2,x_3\,;f'']\big|\\ &\le cn^{-2}\big(n^4\omega_{3,2}^{\varphi}(f'',1/n) + \|f''\|_{[-1/2,1/2]}\big)\\ &\qquad + cn^{-4}\big(n^6\omega_{3,2}^{\varphi}(f'',1/n) + \|f''\|_{[-1/2,1/2]}\big)\\ &\le cn^2\omega_{3,2}^{\varphi}(f'',1/n) + cn^{-2}\|f''\|_{[-1/2,1/2]},\end{aligned}$$

and, therefore,

$$\begin{aligned}|l_1(x) - s_n(x)| &= \left|\int_{x_1}^{x}(x-u)\,\big(l_1''(u) - s_n''(u)\big)\,du\right|\\ &\le cn^{-4}\|l_1'' - s_n''\|_{I_1}\\ &\le cn^{-2}\omega_{3,2}^{\varphi}(f'',1/n) + cn^{-6}\|f''\|_{[-1/2,1/2]}.\end{aligned}$$

Combining this with (4.5) and using the triangle inequality we get (4.3). ■

Proof of Theorem 2.1 By virtue of Lemma 4.1, the inequality

$$\|f - s_n\| \le cn^{-2}\omega_{3,2}^{\varphi}(f'',1/n) + cn^{-6}\|f''\|_{[-1/2,1/2]},$$

which follows from Lemma 4.3, and the estimate

$$\omega_5^{\varphi}(s_n,1/n) \le c\|f-s_n\| + c\omega_5^{\varphi}(f,1/n) \le c\|f-s_n\| + cn^{-2}\omega_{3,2}^{\varphi}(f'',1/n)$$

(see (1.10)), we conclude that there exists a polynomial $P_n \in \Delta^2$ of degree $\le cn$ such that

$$\begin{aligned}\|f - P_n\| &\le \|f - s_n\| + \|s_n - P_n\|\\ &\le \|f - s_n\| + c\omega_5^{\varphi}(s_n,1/n)\\ &\le cn^{-2}\omega_{3,2}^{\varphi}(f'',1/n) + cn^{-6}\|f''\|_{[-1/2,1/2]}.\end{aligned}$$

This completes the proof of the estimate (2.1).

In order to prove (2.2) note that (1.6) implies that

$$n^3\omega_{3,2}^{\varphi}(f'',1/n) \ge C(f), \quad \text{for all } n \in \mathbb{N}.$$

Hence, for $n \ge \|f''\|_{[-1/2,1/2]}/C(f) =: N(f)$,

$$\frac{1}{n}\|f''\|_{[-1/2,1/2]} \le C(f) \le n^3\omega_{3,2}^{\varphi}(f'',1/n).$$

Therefore,

$$\|f - P_n\| \le cn^{-2}\omega_{3,2}^{\varphi}(f'',1/n), \quad n \ge N(f),$$

which completes the proof of the theorem. ■

5 Proofs of Negative Results

In order to prove Theorem 2.3, let $b \in (0,1)$ and, for $x \in [-1,1]$, set

$$g_b(x) := \ln \frac{b}{1+x+b} \quad \text{and} \quad G_b(x) := \int_{-1}^{x} (x-u) g_b(u)\, du.$$

Clearly, G_b is in $\mathbb{C}^\infty[-1,1]$ (note also that G_b is concave on $[-1,1]$).

First, we prove

Lemma 5.1 *The following estimates hold:*

$$\omega_{2,4}^{\varphi}(G_b^{(4)}, t) \le c \tag{5.1}$$

and

$$b \ln \frac{1}{b} |g_b(x)| \le (1+x) \ln \frac{3e^2}{1+x}. \tag{5.2}$$

Proof First, taking into account that $G_b^{(4)} = g_b''(x) = (1+x+b)^{-2}$ we have

$$\omega_{2,4}^{\varphi}(G_b^{(4)}, t) = \omega_{2,4}^{\varphi}(g_b'', t) \le c \|\varphi^4 g_b''\| \le c\,.$$

To prove (5.2) we have to check the inequality

$$b \ln \frac{1}{b} \ln \frac{1+x+b}{b} \le (1+x) \ln \frac{3e^2}{1+x}.$$

Indeed, this inequality holds for $x = -1$. At the same time, for the derivatives of both sides we have

$$\begin{aligned} \frac{d}{dx}\Big(b \ln \frac{1}{b} \ln \frac{1+x+b}{b} \Big) &= \frac{b \ln 1/b}{1+x+b} \le \ln \frac{3e}{1+x+b} \\ &\le \ln \frac{3e}{1+x} = \frac{d}{dx}\Big((1+x) \ln \frac{3e^2}{1+x} \Big), \end{aligned}$$

which completes the proof. ∎

Denote by $\mathbb{P}_n^*$ the set of polynomials p_n of degree $\le n-1$, such that

$$p_n''(-1) \ge 0.$$

Clearly, every polynomial p_n from $\mathbb{P}_n \cap \Delta^2$ is also in $\mathbb{P}_n^*$.

In the proof of Lemma 5.3 below we need the following Dzyadyk type inequality (see [16, Lemma 14.1 and (14.9)]) which is a generalization of the classical Dzyadyk inequality [4].

Lemma 5.2 (Dzyadyk type inequality) *Let $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, and y be any point in $[-1,1]$. Then, for any $\nu \in \mathbb{N}_0$, and any polynomial P_n of degree $\le n-1$,*

$$|P_n^{(\nu)}(y)| \le C(m,\nu)\rho_n^{m-\nu}(y)\left\| \frac{P_n(\,\cdot\,)}{\big[|\cdot - y| + \rho_n(y)\big]^m}\right\|, \tag{5.3}$$

where $\rho_n(x) := \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}$.

Since the references [4, 16] may not be readily accessible, for the sake of completeness we give a short proof of Lemma 5.2 here.

Proof Everywhere in this proof, C denotes constants that may depend only on m and ν. Now, we recall that $x_j = \cos(j\pi/n)$, $0 \le j \le n$, $I_j = [x_j, x_{j-1}]$, and let $\tilde{x}_j := \cos\left(j - \frac{1}{2}\right)\frac{\pi}{n}$, $1 \le j \le n$, be the zeros of the Chebyshev polynomial $T_n(x) := \cos(n \arccos x)$ of degree n.

For any $1 \le j \le n$ and $x \in I_j$,

$$\frac{4}{3}|x - \tilde{x}_j| \le |T_n(x)||I_j| \le 4|x - \tilde{x}_j|, \tag{5.4}$$

$$|I_j| \le 4 \min\left\{|x_{j-1} - \tilde{x}_j|, ||x_j - \tilde{x}_j|\right\}, \tag{5.5}$$

and

$$\rho_n(x) \le |I_j| \le \frac{\pi^2}{2}\rho_n(x). \tag{5.6}$$

We note that all constants in (5.4)–(5.6) are exact. Estimates (5.5)–(5.6) are simple trigonometric inequalities and so we omit their proofs. Inequalities (5.4) can also be verified using standard calculus techniques, and are certainly known (perhaps not with the exact constant). For reader's convenience, we give a short proof in the appendix.

The right inequality in (5.4), (5.5), and the observation $\|T_n\| = 1$ imply that, for any $x \in [-1,1]$ and $1 \le j \le n$,

$$|T_n(x)|(|I_j| + |x - \tilde{x}_j|) \le 5|x - \tilde{x}_j|\,. \tag{5.7}$$

Now, let $y \in [-1,1]$ be fixed and denote by μ the index such that $y \in I_\mu$, $1 \le \mu \le n$. Without loss of generality, we assume that $y \ne \tilde{x}_\mu$ and

$$\left\| \frac{P_n(\,\cdot\,)}{\big[|\cdot - y| + \rho_n(y)\big]^m}\right\| = \rho_n^{-m}(y).$$

Hence, we need to show that, for every $\nu \in \mathbb{N}_0$,

$$|P_n^{(\nu)}(y)| \le C\rho_n^{-\nu}(y). \tag{5.8}$$

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Now,

$$q_n(x) := \frac{y-\tilde{x}_\mu}{x-\tilde{x}_\mu}\frac{T_n(x)}{T_n(y)}$$

is a polynomial of degree $n-1$ such that $q_n(y)=1$, and inequalities (5.4) and (5.7) imply

$$|q_n(x)| = \left|\frac{y-\tilde{x}_\mu}{T_n(y)}\right| \cdot \left|\frac{T_n(x)}{x-\tilde{x}_\mu}\right| \le \frac{3|I_\mu|}{4} \cdot \frac{5}{|x-\tilde{x}_\mu|+|I_\mu|} = \frac{15}{4} \cdot \frac{|I_\mu|}{|x-\tilde{x}_\mu|+|I_\mu|}$$

for all $x \in [-1,1]$. Using (5.6) we get

$$\begin{aligned}\frac{|x-y|+\rho_n(y)}{\rho_n(y)} &\le \frac{|x-\tilde{x}_\mu|}{\rho_n(y)} + \frac{|\tilde{x}_\mu - y|}{\rho_n(y)} + 1 \le \frac{\pi^2}{2}\frac{|x-\tilde{x}_\mu|}{|I_\mu|} + \frac{\pi^2}{2} + 1 \\ &\le 6 \cdot \frac{|x-\tilde{x}_\mu|+|I_\mu|}{|I_\mu|}\end{aligned}$$

and, therefore,

$$|q_n(x)| \le \frac{23\rho_n(y)}{|x-y|+\rho_n(y)}, \quad x \in [-1,1].$$

Hence, $Q_n(x) := (q_n(x))^m$ is a polynomial of degree $m(n-1)$, and the following inequalities are satisfied:

$$\|P_n Q_n\| \le 23^m \rho_n^m(y) \left\| \frac{P_n(\,\cdot\,)}{\big[|\cdot - y| + \rho_n(y)\big]^m} \right\| = C$$

and

$$\|Q_n\| \le C.$$

We now use the well-known Markov-Bernstein inequality

$$\|\rho_n^\nu p_n^{(\nu)}\| \le C\|p_n\|$$

which is satisfied for every polynomial p_n of degree $\le n-1$, to conclude that

$$|[P_n(y)Q_n(y)]^{(\nu)}| \le C\rho_{n+m(n-1)}^{-\nu}(y) \le C\rho_n^{-\nu}(y)$$

and

$$|Q_n^{(\nu)}(y)| \le C\rho_{m(n-1)+1}^{-\nu}(y) \le C\rho_n^{-\nu}(y)$$

for $\nu \in \mathbb{N}_0$. We will now use strong induction in ν to prove (5.8). If $\nu = 0$, (5.8) is obvious. Suppose now that (5.8) is proved for all $0 \le l \le \mu - 1$. Using the Leibniz identity for derivatives

$$[f(x)g(x)]^{(\nu)} = \sum_{l=0}^{\nu} \binom{\nu}{l} f^{(l)}(x) g^{(\nu-l)}(x),$$

the fact that $Q_n(y) = 1$, and the induction hypothesis we have

$$|P_n^{(\nu)}(y)| = |P_n^{(\nu)}(y)Q_n(y)| \le \left|[P_n(y)Q_n(y)]^{(\nu)}\right| + \sum_{l=0}^{\nu-1} \binom{\nu}{l} |P_n^{(l)}(y)|\,|Q_n^{(\nu-l)}(y)|$$

$$\le C\rho_n^{-\nu}(y) + C\sum_{l=0}^{\nu-1}\binom{\nu}{l}\rho_n^{-l}(y)\rho_n^{-\nu+l}(y) = C\rho_n^{-\nu}(y),$$

and so (5.8) is verified for $l = \nu$ as well. This completes the proof of the lemma. ■

Lemma 5.3 *For each $b \in (0, n^{-2})$, and every polynomial $p_n \in \mathbb{P}_n^*$, we have*

$$\|G_b - p_n\| \ge \frac{c}{n^4}\ln\frac{1}{n^2 b} - \frac{1}{n^4}.$$

Proof Put

$$g_b^*(x) := -\ln\left(n^2(1+x+b)\right), \quad l(x) := g_b(x) - g_b^*(x) = \ln n^2 b,$$

so that $l(x)$ is a constant. Let

$$G_b^*(x) := \int_{-1}^x (x-u)g_b^*(u)\,du \quad \text{and} \quad L(x) := \int_{-1}^x (x-u)l(u)\,du = \frac{1}{2}(x+1)^2\ln n^2 b.$$

Then we have

$$G_b^*(x) + L(x) = G_b(x).$$

Also, for every $p_n \in \mathbb{P}_n^*$,

$$p_n''(-1) - L''(-1) \ge -l(-1) = \ln 1/n^2 b. \tag{5.9}$$

Straightforward computations yield

$$\int_{-1}^x |g_b^*(u)|\,du \le c/n^2 + cn^2(1+x)^2, \quad -1 \le x \le 1,$$

whence

$$|G_b^*(x)| \le (x+1)\int_{-1}^x |g_b^*(u)|\,du \le c(1+x)/n^2 + cn^2(1+x)^3 \le \frac{c}{n^4}(1+n^2(1+x))^3.$$

Hence,

$$|p_n(x) - L(x)| \le \|p_n - G_b\| + |G_b^*(x)| \le cn^6\Big(\|p_n - G_b\| + \frac{1}{n^4}\Big)(1/n^2 + (1+x))^3.$$

We now apply (5.3) with $y = -1$, $k = 2$, and $m = 3$:

$$|p_n''(-1) - L''(-1)| \le cn^{-2}\left\|\frac{p_n(x) - L(x)}{(x+1+n^{-2})^3}\right\| \le cn^4\Big(\|p_n - G_b\| + \frac{1}{n^4}\Big).$$

This combined with (5.9), in turn completes the proof of the lemma. ■

We are now ready to prove Theorem 2.3 by constructing a counterexample.

Proof of Theorem 2.3 Let $b_n \in (0, 1/e)$, $n \geq 2$, be such that

$$b_n \ln \frac{1}{b_n} = \frac{1}{n^2},$$

and set

$$f_n(x) := c\frac{1}{n^2} G_{b_n}(x),$$

where $c < 1$ (which is independent of n) is taken so small that (5.12) and (5.13) below are fulfilled. We summarize the properties of f_n as follows from Lemma 5.1. Namely, for every $n \geq 2$,

$$f_n \in \mathbb{C}^\infty[-1, 1],$$

$$|f_n''(x)| \leq (1+x) \ln \frac{3e^2}{1+x}, \tag{5.10}$$

$$f_n(-1) = f_n'(-1) = f_n''(-1) = 0, \tag{5.11}$$

$$\|f_n^{(j)}\| < 1, \quad j = 0, 1, 2, \quad \text{and} \quad \|\varphi^{2j-4} f_n^{(j)}\| < 1, \quad j = 3, 4, \tag{5.12}$$

and

$$\omega_{2,4}^{\varphi}(f_n^{(4)}, 1/n) \leq n^{-2}. \tag{5.13}$$

The definition of b_n yields

$$\ln \ln n \leq \ln \ln n^2 \leq \ln \ln 1/b_n = \ln \frac{1}{n^2 b_n}.$$

Hence, by virtue of Lemma 5.3, there exists $n_1 \geq 2$ such that, for all $n \geq n_1$ and $p_n \in \mathbb{P}_n^*$,

$$\|f_n - p_n\| \geq c_* \frac{\ln \ln n}{n^6}, \tag{5.14}$$

for some suitable constant $c_* > 0$, where we used the fact that $q_n := cn^2 p_n \in \mathbb{P}_n^*$.

Now, we put $D_0 := 1$ and

$$D_\sigma := \frac{D_{\sigma-1}}{n_\sigma^6} = \frac{1}{n_1^6} \cdots \frac{1}{n_\sigma^6},$$

where n_σ is defined by induction as follows. Suppose that $n_1, \dots, n_{\sigma-1}$, $\sigma > 1$, have been selected. We write

$$F_{\sigma-1}(x) := \sum_{j=1}^{\sigma-1} D_{j-1} f_{n_j}(x)$$

and select $n_\sigma > n_{\sigma-1}$ to be so large that the following inequalities are satisfied:

$$\max\{\sigma, \|F_{\sigma-1}^{(6)}\|\} < D_{\sigma-1} \ln\ln\ln n_\sigma, \tag{5.15}$$

$$\|F_{\sigma-1}^{(9)}\| < D_{\sigma-1} n_\sigma. \tag{5.16}$$

Note that (5.16) implies

$$E_{n_\sigma-2}(F''_{\sigma-1}) \le \frac{c}{n_\sigma^7}\|F_{\sigma-1}^{(9)}\| < c\frac{D_{\sigma-1}n_\sigma}{n_\sigma^7} = cD_\sigma. \tag{5.17}$$

Now, let

$$\Phi_\sigma(x) := \sum_{j=\sigma}^{\infty} D_{j-1} f_{n_j}(x),$$

where the uniform convergence of the series as well as its four times term-by-term differentiation for $x \in (-1, 1)$, is justified by (5.12). In fact, since

$$\begin{aligned}\sum_{j=\sigma}^{\infty} D_{j-1} = D_{\sigma-1}\sum_{j=\sigma}^{\infty}\frac{1}{n_\sigma^6\cdots n_{j-1}^6} &< D_{\sigma-1}\sum_{j=\sigma}^{\infty}\Big(\frac{1}{n_\sigma^6}\Big)^{j-\sigma} \\ &= \frac{D_{\sigma-1}n_\sigma^6}{n_\sigma^6-1} < 2D_{\sigma-1},\end{aligned} \tag{5.18}$$

the inequalities (5.12) imply that

$$\|\Phi_\sigma\| < 2D_{\sigma-1} \quad \text{and} \quad \|\varphi^4\Phi_\sigma^{(4)}\| < 2D_{\sigma-1}. \tag{5.19}$$

Setting

$$\mathfrak{f}_1(x) := \Phi_1(x),$$

and using properties of the $\omega_{2,4}^\varphi$ modulus, we have by (5.13), (5.15) and (5.19):

$$\begin{aligned}\omega_{2,4}^\varphi&\Big(\mathfrak{f}_1^{(4)}, \frac{1}{n_\sigma}\Big) \\ &\le \omega_{2,4}^\varphi\Big(F_{\sigma-1}^{(4)}, \frac{1}{n_\sigma}\Big) + \omega_{2,4}^\varphi\Big(D_{\sigma-1}f_{n_\sigma}^{(4)}, \frac{1}{n_\sigma}\Big) + \omega_{2,4}^\varphi\Big(\Phi_{\sigma+1}^{(4)}, \frac{1}{n_\sigma}\Big) \\ &\le \frac{1}{n_\sigma^2}\|F_{\sigma-1}^{(6)}\| + \frac{D_{\sigma-1}}{n_\sigma^2} + cD_\sigma \\ &\le \frac{D_{\sigma-1}}{n_\sigma^2}(\ln\ln\ln n_\sigma + c) \le 2\frac{D_{\sigma-1}}{n_\sigma^2}\ln\ln\ln n_\sigma\end{aligned} \tag{5.20}$$

for sufficiently large σ.

On the other hand, by virtue of (5.11) and (5.17) there exists a polynomial q_{n_σ} of degree $n_\sigma - 3$, such that

$$q_{n_\sigma}(-1) = F''_{\sigma-1}(-1) = 0,$$

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and

$$\|F''_{\sigma-1} - q_{n_\sigma}\| \le 2cD_\sigma.$$

Hence, for

$$Q_{n_\sigma}(x) := \int_{-1}^{x} (x-u) q_{n_\sigma}(u)\, du,$$

we have

$$Q''_{n_\sigma}(-1) = q_{n_\sigma}(-1) = 0,$$

and

$$\|F_{\sigma-1} - Q_{n_\sigma}\| \le 4cD_\sigma, \tag{5.21}$$

where we have used the relations

$$F_{\sigma-1}(-1) = F'_{\sigma-1}(-1) = 0.$$

Now, if $p_{n_\sigma} \in P^*_{n_\sigma}$, then

$$\mathfrak{f}_1 - p_{n_\sigma} = (F_{\sigma-1} - Q_{n_\sigma}) + D_{\sigma-1}(f_{n_\sigma} - R_{n_\sigma}) + \Phi_{\sigma+1},$$

where we observe that $R_{n_\sigma} := \frac{1}{D_{\sigma-1}}(p_{n_\sigma} - Q_{n_\sigma}) \in P^*_{n_\sigma}$. Therefore, by virtue of (5.14), (5.19) and (5.21),

$$\begin{aligned} \|\mathfrak{f}_1 - p_{n_\sigma}\| &\ge D_{\sigma-1}\|f_{n_\sigma} - R_{n_\sigma}\| - \|F_{\sigma-1} - Q_{n_\sigma}\| - \|\Phi_{\sigma+1}\| \\ &\ge D_{\sigma-1}\frac{c_*}{n_\sigma^6}\ln\ln n_\sigma - (4c+2)D_\sigma = D_\sigma(c_*\ln\ln n_\sigma - (4c+2)) \\ &\ge \frac{1}{2}D_\sigma c_* \ln\ln n_\sigma, \end{aligned} \tag{5.22}$$

where the last inequality holds for all sufficiently large σ.

The inequalities (5.10) and (5.18) imply that

$$|\mathfrak{f}_1''(x)| \le 2(1+x)\ln\frac{3e^2}{1+x},$$

so that if we put

$$\mathfrak{f}_2''(x) := 2(1+x)\ln\frac{3e^2}{1+x}$$

and

$$\mathfrak{f}_2(x) := \int_{-1}^{x}(x-u)\mathfrak{f}_2''(u)\, du,$$

and if we denote

$$\mathfrak{f}(x) := \mathfrak{f}_1(x) + \mathfrak{f}_2(x),$$

then $\mathfrak{f}$ is a convex function on $[-1, 1]$.

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Note also that $\mathfrak{f}_2$ is twice continuously differentiable in $[-1, 1]$, and it is in $\mathbb{B}^6$. We will show that there exists a polynomial $\mathcal{Q}_n \in \mathbb{P}_n$ such that $\mathcal{Q}_n''(-1) = 0$ and

$$\|\mathfrak{f}_2 - \mathcal{Q}_n\| \le \frac{c}{n^6}. \tag{5.23}$$

It follows from [9, Theorem 1] that there exists a polynomial R_n such that

$$\|\mathfrak{f}_2 - R_n\| \le c\omega_6^{\varphi}(\mathfrak{f}_2, 1/n)$$

and

$$\|\mathfrak{f}_2'' - R_n''\| \le c\omega_4^{\varphi}(\mathfrak{f}_2'', 1/n).$$

Therefore, since (see (1.8))

$$\omega_6^{\varphi}(\mathfrak{f}_2, 1/n) \le cn^{-6}\|\varphi^6\mathfrak{f}_2^{(6)}\| \le cn^{-6}$$

and (see (1.9))

$$\omega_4^{\varphi}(\mathfrak{f}_2'', 1/n) \le cn^{-2}\omega_{2,2}^{\varphi}(\mathfrak{f}_2^{(4)}, 1/n) \le cn^{-2}\|\varphi^2\mathfrak{f}_2^{(4)}\| \le cn^{-2},$$

we conclude that R_n satisfies the inequalities $\|\mathfrak{f}_2 - R_n\| \le cn^{-6}$ and $\|\mathfrak{f}_2'' - R_n''\| \le cn^{-2}$. In particular, since $\mathfrak{f}_2''(-1) = 0$, the estimate $|R_n''(-1)| \le cn^{-2}$ holds.

Now, it follows from [6, Lemma 8] (or see [8, Lemma 14c]) that there exists a polynomial $M_n \in \mathbb{P}_n$ such that $\|M_n\| \le cn^{-4}$ and $M_n''(-1) \ge 2^{-10}$.

We now define $\mathcal{Q}_n$ as follows:

$$\mathcal{Q}_n(x) := R_n(x) - \frac{R_n''(-1)}{M_n''(-1)} M_n(x).$$

Clearly, $\mathcal{Q}_n''(-1) = 0$, and

$$\|\mathfrak{f}_2 - \mathcal{Q}_n\| \le \|\mathfrak{f}_2 - R_n\| + \left|\frac{R_n''(-1)}{M_n''(-1)}\right| \|M_n\| \le \|\mathfrak{f}_2 - R_n\| + cn^{-2}\|M_n\| \le cn^{-6},$$

which proves (5.23).

Now, (1.8) implies that

$$\omega_{2,4}^{\varphi}\Big(\mathfrak{f}_2^{(4)}, \frac{1}{n}\Big) \le cn^{-2}\|\varphi^6\mathfrak{f}_2^{(6)}\| \le cn^{-2},$$

and hence, using (5.20) and (5.15), we have

$$\begin{aligned}
\omega_{2,4}^{\varphi}\Big(\mathfrak{f}^{(4)}, \frac{1}{n_\sigma}\Big) &\le \omega_{2,4}^{\varphi}\Big(\mathfrak{f}_1^{(4)}, \frac{1}{n_\sigma}\Big) + \omega_{2,4}^{\varphi}\Big(\mathfrak{f}_2^{(4)}, \frac{1}{n_\sigma}\Big) \\
&\le \frac{2}{n_\sigma^2}\big(D_{\sigma-1} \ln\ln\ln n_\sigma + c\big) \\
&\le \frac{3D_{\sigma-1}}{n_\sigma^2} \ln\ln\ln n_\sigma,
\end{aligned}$$

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for sufficiently large σ.

On the other hand, (5.22) and (5.23) imply that, if $P_{n_\sigma} \in \mathbb{P}^*_{n_\sigma}$, then

$$\|\mathfrak{f} - P_{n_\sigma}\| \geq \|\mathfrak{f}_1 - (P_{n_\sigma} - \mathcal{Q}_{n_\sigma})\| - \|\mathfrak{f}_2 - \ QQ_{n_\sigma}\|$$
$$\geq \frac{c_*}{2} \frac{D_{\sigma-1}}{n_\sigma^6} \ln\ln n_\sigma - \frac{C}{n_\sigma^6} \geq \frac{c_*}{4} \frac{D_{\sigma-1}}{n_\sigma^6} \ln\ln n_\sigma$$

for sufficiently large σ, where we used the fact that $P_{n_\sigma} - \mathcal{Q}_{n_\sigma} \in \mathbb{P}^*_{n_\sigma}$.

Thus, taking into account that $E_n^{(2)}(f) \geq \inf_{p_n \in \mathbb{P}^*_n} \|f - p_n\|$ we have

$$\frac{n_\sigma^4 E_{n_\sigma}^{(2)}(\mathfrak{f})}{\omega_{2,4}^{\varphi}(\mathfrak{f}^{(4)}, 1/n_\sigma)} \geq \frac{c_*}{12} \cdot \frac{\ln\ln n_\sigma}{\ln\ln\ln n_\sigma} \to \infty \quad \text{as} \quad \sigma \to \infty.$$

This concludes the proof of Theorem 2.3. ∎

A Appendix

Lemma A.1 *Let $n \in \mathbb{N}$ and $1 \leq j \leq n$. Denote*

$$\mathrm{t}_j(x) := \frac{T_n(x)}{x - \tilde{x}_j}|I_j|, \ x \neq \tilde{x}_j, \quad \textit{and} \quad \mathrm{t}_j(\tilde{x}_j) := T_n'(\tilde{x}_j)|I_j|,$$

where $T_n(x) := \cos n \arccos x$ is the Chebyshev polynomial of degree n, and $\tilde{x}_j := \cos(j - 1/2)\frac{\pi}{n}$ is the zero of T_n lying in I_j. Then, for every $x \in I_j$, we have

$$\frac{4}{3} < |\mathrm{t}_j(x)| < 4. \tag{A.1}$$

Moreover, the constants in (A.1) *are exact and cannot be improved.*

Proof First, we observe that $|\mathrm{t}_{n-j+1}(-x)| = |\mathrm{t}_j(x)|$ and $\{-x \mid x \in I_{n-j+1}\} = I_j$. Hence, without loss of generality, one can assume that $j \leq \lfloor (n+1)/2 \rfloor$. Note that t_j is a polynomial of degree $n-1$ having exactly $n-1$ real zeros $\tilde{x}_i$, $1 \leq i \leq n$ and $i \neq j$. Therefore, by Rolle's theorem, t_j' has exactly $n-2$ distinct zeros. In particular, t_j' has a unique zero in $[\tilde{x}_{j+1}, \tilde{x}_{j-1}] \supset I_j$ if $j \geq 2$, and so $|\mathrm{t}_j(x)| \geq \min\{|\mathrm{t}_j(x_j)|, |\mathrm{t}_j(x_{j-1})|\}$ for $x \in I_j$ and $j \geq 1$. Hence, for $x \in I_j$,

$$|\mathrm{t}_j(x)| \geq |\mathrm{t}_j(x_j)| = \frac{|I_j|}{\tilde{x}_j - x_j} \geq \frac{|I_1|}{\tilde{x}_1 - x_1} = \frac{4\cos^2(\pi/4n)}{4\cos^2(\pi/4n) - 1} > \frac{4}{3},$$

which is the lower estimate in (A.1).

To prove the upper estimate, we use

(i) $\sin(\alpha\theta) \leq \alpha \sin\theta$, for all $\alpha \geq 1$ and $0 \leq \theta \leq \pi/\alpha$, and
(ii) $|\sin\theta| \leq |\theta|$, for all $\theta \in \mathbb{R}$.

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Denoting $\tau := \arccos x$ (and hence $x \in I_j$ implies that $(j-1)\pi/n \le \tau \le j\pi/n$) and $\tau_j := (j-1/2)\frac{\pi}{n}$ we have, for $x \in I_j$,

$$\begin{aligned}
\left|\mathrm{t}_j(x)\right| &= |I_j| \left| \frac{\sin n(\tau-\tau_j)/2}{\sin(\tau-\tau_j)/2} \frac{\sin n(\tau+\tau_j)/2}{\sin(\tau+\tau_j)/2} \right| \le \frac{n|I_j|}{\sin(\tau+\tau_j)/2} \\
&\le \frac{n|I_j|}{\sin((j-1)\frac{\pi}{n}+\tau_j)/2} \\
&= \frac{2n \sin\frac{\pi}{2n} \sin(j-1/2)\frac{\pi}{n}}{\sin(j-3/4)\frac{\pi}{n}} \le \pi \frac{j-1/2}{j-3/4}.
\end{aligned}$$

Therefore, if $j \ne 1$, then $\|\mathrm{t}_j\|_{I_j} \le 6\pi/5 < 4$.

Finally, if $j = 1$, since t_1 is positive and strictly increasing on $[\tilde{x}_2, 1]$, we have

$$0 < \mathrm{t}_1(x) \le \mathrm{t}_1(1) = \frac{|I_1|}{1-\tilde{x}_1} = 4\cos^2\left(\frac{\pi}{4n}\right) < 4, \quad x \in I_1 .$$

The inequalities (A.1) are now verified. The constants in (A.1) are exact since $\mathrm{t}_1(1) = 4\cos^2\left(\frac{\pi}{4n}\right) \to 4$ and $\mathrm{t}_1(x_1) = 4\cos^2\left(\frac{\pi}{4n}\right)/\left(4\cos^2\left(\frac{\pi}{4n}\right) - 1\right) \to 4/3$ as $n \to \infty$. ∎

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