REPRESENTATION OF BANACH ALGEBRAS WITH AN INVOLUTION

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1. Introduction. In 1943 Gelfand and Neumark **(3)** characterized uniformly closed self-adjoint algebras of bounded operators on a Hilbert space as Banach algebras with an involution (a conjugate linear anti-isomorphism of period two) satisfying several additional conditions. The main purpose of this paper is to point out that if we consider algebras of bounded operators on complex Banach spaces more general than Hilbert space, then we can represent a larger class of algebras by essentially the same methods. The Banach spaces considered are those which have an inner product satisfying the algebraic conditions imposed on a Hilbert space inner product except that positivity is replaced by non-degeneracy and continuity of the inner product. We shall call such Banach spaces BIP-spaces. Now let $\mathfrak{A}$ be a complex Banach algebra with an involution satisfying $||a^*|| = ||a||$. Our main result (Theorem 3.6) is that there is a BIP-space $\mathfrak{X}$ and an involution-preserving isomorphism of $\mathfrak{A}$ onto an algebra of bounded operators on $\mathfrak{X}$ which is norm-preserving on self-adjoint elements.

I should like to take this opportunity to thank C. E. Rickart for many helpful suggestions and to express my deep appreciation for his many acts of kindness.

2. BIP-spaces. In this section we shall define BIP-spaces and present those facts which we shall need to state and prove our representation theorem. It is our intention to present a detailed study of BIP-spaces at a later date. Complex conjugates will be indicated by a superscript c.

Definition 2.1. A BIP-space $\mathfrak{X}$ is a complex Banach space which has an inner product (x, y) defined from $\mathfrak{X} \times \mathfrak{X}$ to the complex numbers such that for arbitrary $x, y, z \in \mathfrak{X}$ and complex λ we have

(1) $(x + \lambda y,\ z) = (x,z) + \lambda(y,z)$,
(2) $(x,y) = (y,x)^c$,
(3) $(x,z) = 0$ for all $z \in \mathfrak{X}$ if and only if $x = 0$,
(4) $|(x,y)| \leqslant ||x||\ ||y||$.

As a result of (4), $x_n \to x$ and $y_n \to y$ imply $(x_n, y_n) \to (x, y)$. This joint continuity of the inner product is equivalent to (4) in the sense that in the presence of joint continuity of the inner product $\mathfrak{X}$ may be given an equivalent norm in which (4) holds.

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For a given inner product all norms under which a linear space L becomes a BIP-space are equivalent. This follows from results of Rickart (**7**, Lemmas 2.1 and 2.4) by noting that for each $x \in L$, $(\cdot, x)$ is a linear functional on L bounded with respect to each such norm and that 0 is the only element at which all of these functionals vanish.

Let S be a set, let $\mathfrak{X}_s$ be a BIP-space for each $s \in S$, and let $\mathfrak{X} = \sum_{s \in S} \mathfrak{X}_s$ be the set of functions x defined on S satisfying $x(s) \in \mathfrak{X}_s$ for each $s \in S$ and $\sum_{s \in S} ||x(s)||$ exists. Define operations on $\mathfrak{X}$ by $(x + y)(s) = x(s) + y(s)$, $(\lambda x)(s) = \lambda x(s)$, $||x|| = \sum_{s \in S} ||x(s)||$ and $(x, y) = \sum_{s \in S} (x(s), y(s))$ for each $x,y \in \mathfrak{X}$ and $s \in S$. Then $\mathfrak{X}$ is a BIP-space which we shall call the direct sum of the $\mathfrak{X}_s$'s.

Let $\mathfrak{X}$ be a BIP-space. By an operator on $\mathfrak{X}$ we shall mean an additive homogeneous function from $\mathfrak{X}$ to $\mathfrak{X}$. If T is an operator which is bounded with respect to the norm on $\mathfrak{X}$ we shall denote the operator bound by $||T||$. We shall denote the algebra of all bounded operators on $\mathfrak{X}$ by $B(\mathfrak{X})$. Let T and S be operators on $\mathfrak{X}$. If there is an operator T^* on $\mathfrak{X}$ such that $(Tx,y) = (x,T^*y)$ for all $x,y \in \mathfrak{X}$, then we shall say T^* is adjoint to T. If T has an adjoint, it is unique and we are justified in using the notation T^* to denote it. If T^* exists, then T is automatically bounded.[1] This follows from the closed graph theorem (**4**, Theorem 2.13.9) by noting that if $x_n \to x$ and $\lim Tx_n$ exists, then $Tx = \lim Tx_n$ since

$$(Tx - \lim Tx_n, y) = (x, T^*y) - \lim (x_n, T^*y) = 0.$$

The following facts are easy to check. If T^* and S^* exist then for complex λ, $(T + \lambda S)^*$ exists and is equal to $T^* + \lambda^c S^*$. If T^* exists then $(T^*)^*$ exists and is equal to T. If T^* and S^* exist, then $(TS)^*$ exists and is equal to S^*T^*. The identity operator, which we shall always denote by I, has an adjoint and $I^* = I$. An algebra of operators on $\mathfrak{X}$ is called self-adjoint if and only if for each T in the algebra, T^* exists and is in the algebra.

The following lemma is a modification of a result of Lax (**5**).

LEMMA 2.2. *Let $\mathfrak{X}$ be a* BIP-*space. If $(x,x) > 0$ for all nonzero $x \in \mathfrak{X}$ and T is an operator on $\mathfrak{X}$ such that T^* exists, then*

$$\sup_{x \neq 0} \frac{(Tx, Tx)}{(x, x)} \leqslant ||T^*T||.$$

Proof. The usual calculation for Hilbert space shows that if $y,z \in \mathfrak{X}$, then $|(y,z)|^2 \leqslant (y,y)(z,z)$. Therefore for any operator S which has an adjoint,

$$(Sx,Sx)^2 = (S^*Sx,x)^2 \leqslant (S^*Sx,S^*Sx)(x,x).$$

Noting that $(S^*S)^* = S^*S$ and applying this repeatedly we see that for any positive integer n,

[1]This result was noted by the referee. It allows us to remove an unnecessary boundedness condition which was originally present in Lemma 2.2.

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$$(Tx, Tx)^{2^n} \leqslant ((T^*T)^{2^{n-1}} x, (T^*T)^{2^{n-1}} x) (x, x)^{2^n - 1}$$
$$= ((T^*T)^{2^n} x, x) (x, x)^{2^n - 1} \leqslant ||T^*T||^{2^n} ||x||^2 (x, x)^{2^n - 1}.$$

Therefore,

$$\frac{(Tx, Tx)}{(x, x)} \leqslant ||T^*T|| \left(\frac{||x||^2}{(x, x)}\right)^{2^{-n}}.$$

The proof is completed by noting that

$$\lim_{n \to \infty} \left(\frac{||x||^2}{(x, x)}\right)^{2^{-n}} = 1.$$

3. *-algebras and the representation theorem.

Definition 3.1. By a **-algebra* $\mathfrak{A}$ we shall mean a complex Banach algebra with identity e, $||e|| = 1$, and a function $a \to a^*$ from $\mathfrak{A}$ into $\mathfrak{A}$ such that for all $a,b \in \mathfrak{A}$ and complex λ we have,

(i) $(a + \lambda b)^* = a^* + \lambda^c b^*$,
(ii) $(ab)^* = b^* a^*$,
(iii) $(a^*)^* = a$,
(iv) $||a^*|| = ||a||$.

A function satisfying (i) to (iii) is called an *involution*. It is apparent that continuity of the involution follows from (iv). Conversely if a complex Banach algebra with e, $||e|| = 1$, has a continuous involution and we replace the norm by the equivalent norm $||a||' = \max(||a||, ||a^*||)$ then we obtain a *-algebra.

The extensiveness of *-algebras may be judged from the fact that any Banach algebra $\mathfrak{B}$ can be embedded in a *-algebra $\mathfrak{B}'$. To see this, let $\mathfrak{B}'$ be $\mathfrak{B} \times \mathfrak{B}$ with

$$(x,y) + (x',y') = (x + x', y + y'),$$
$$\lambda(x,y) = (\lambda x, \lambda^c y), \ (x,y)(x'y') = (xx', y'y),$$
$$||(x,y)|| = \max(||x||, ||y||), \ (x,y)^* = (y,x).$$

An element $h \in \mathfrak{A}$ for which $h^* = h$ is called self-adjoint. It is easy to see that e and 0 are self-adjoint and so are all elements of the form a^*a, aa^*, and $a + a^*$. Let H be the set of all self-adjoint elements of $\mathfrak{A}$. Then H is a real Banach space. For $a \in \mathfrak{A}$ there are unique self-adjoint elements h,k such that $a = h + ik$; in fact, $h = \frac{1}{2}(a + a^*)$ and $k = -\frac{1}{2}i(a - a^*)$. If $a = h + ik$ with $h, k \in H$ we shall call h and k the self-adjoint components of a.

We shall call a linear functional f on $\mathfrak{A}$ (1) a real functional if and only if $f(a^*) = f(a)^c$ for all $a \in \mathfrak{A}$, and (2) a positive functional if and only if $f(a^*a) \geqslant 0$ for all $a \in \mathfrak{A}$. A positive functional is necessarily real and bounded (by $f(e)$) (**6**, pp. 29–32). The extensiveness of real functionals is indicated by the following facts. If f' is a (real) linear functional on H, then there is a unique linear functional f on $\mathfrak{A}$ such that f restricted to H is equal to f'; in fact,

$f(a) = f'(h) + if'(k)$ for $a = h + ik$ with $h, k \in H$. Moreover f is bounded if and only if f' is. A linear functional on $\mathfrak{A}$ is real if and only if it assumes real values on H. If f is a linear functional on A then there are unique real functionals f_1, f_2 such that $f = f_1 + if_2$; in fact,

$$f_1(a) = \tfrac{1}{2}[f(a) + f(a^*)^c], \; f_2(a) = -\tfrac{1}{2}i[f(a) - f(a^*)^c].$$

Moreover f is bounded if and only if both f_1 and f_2 are. A *-algebra need have no (nonzero) positive functionals. To see this consider $\mathfrak{B}'$ as above. Since $(e, -e)^*(e, -e) = -(e, e)$ and (e, e) is the identity for $\mathfrak{B}'$, every positive functional must vanish at (e, e) and therefore is identically zero.

Let f be a real functional on $\mathfrak{A}$ and let $N(f)$ be the set of all $a \in \mathfrak{A}$ such that $f(ba) = 0$ for all $b \in \mathfrak{A}$. Then $N(f)$ is a left ideal. If f is bounded then $N(f)$ is closed and the difference space $\mathfrak{A} - N(f)$ (with elements a_f for $a \in \mathfrak{A}$) is a Banach space under the norm

$$\inf_{b \in N(f)} ||a + b||.$$

We shall call $N(f)$ the reducing ideal of f.

Definition 3.2. Suppose that $a \to T_a$ is a homomorphism of A into an algebra of operators on a BIP-space. Then we shall call the homomorphism a *-homomorphism if and only if for all $a \in \mathfrak{A}$, $T_{a^*} = T_a^*$ and $T_e = I$.

LEMMA 3.3. *Let f be a real functional on $\mathfrak{A}$ with $||f|| \leqslant 1$ and let $N(f)$ be the reducing ideal of f. Then $\mathfrak{A} - N(f)$ is a* BIP-*space and there is a norm-reducing *-homomorphism of $\mathfrak{A}$ onto a self-adjoint algebra of bounded operators on $\mathfrak{A} - N(f)$. The kernel of this homomorphism is contained in $N(f)$.*

Proof. For $a, b \in \mathfrak{A}$, we define (a_f, b_f) to be $f(b^*a)$. If $c, d \in N(f)$, then

$$f((b + d)^*(a + c)) = f(b^*a) + f(a^*d)^c + f((b + d)^*c) = f(b^*a).$$

Therefore (a_f, b_f) is independent of the representatives from a_f and b_f.

Since f is linear,

$$(a_f + \lambda b_f, c_f) = f(c^*a) + \lambda f(c^*b) = (a_f, c_f) + \lambda(b_f, c_f).$$

Since f is a real functional,

$$(a_f, b_f) = f(a^*b)^c = (b_f, a_f)^c.$$

If $(a_f, b_f) = 0$ for all $b \in \mathfrak{A}$, then $f(ca) = 0$ for all $c \in \mathfrak{A}$ and consequently $a_f = 0$. If $c \in a_f$ and $d \in b_f$, then

$$|(a_f, b_f)| = |f(d^*c)| \leqslant ||d|| \, ||c||.$$

Therefore

$$|(a_f, b_f)| \leqslant ||a_f|| \, ||b_f||.$$

Since $\mathfrak{A} - N(f)$ is a Banach space, it follows that it is a BIP-space.

For $b_f = c_f$, we have $(ab)_f = (ac)_f$ since $N(f)$ is a left ideal. Therefore we can define an operator on $\mathfrak{A} - N(f)$ by $T_a(b_f) = (ab)_f$ for each $a \in \mathfrak{A}$.

Since

$$||(ab)_f|| = \inf_{c\in N(f)}||ab + c|| \leqslant \inf_{c\in N(f)}||a(b + c)|| \leqslant ||a||\inf_{c\in N(f)}||b + c|| = ||a||\ ||b_f||,$$

we conclude that $||T_a|| \leqslant ||a||$. Since

$$(T_a(b_f), c_f) = f(c^*ab) = f((a^*c)^*b) = (b_f, T_{a^*}(c_f))$$

we conclude that the mapping $a \to T_a$ is a *-homomorphism.

Finally if a is not in $N(f)$, then $T_a(e_f) = a_f \neq 0$ and therefore $T_a \neq 0$. This means that the kernel of the homomorphism is contained in $N(f)$ which concludes the proof.

When necessary to avoid confusion we shall denote T_a by T_{af}.

LEMMA 3.4. *Let S be a set of real functionals f satisfying $||f|| \leqslant 1$. Let*

$$\mathfrak{X} = \sum_{f\in S} \mathfrak{A} - N(f).$$

*Then there is a norm-reducing *-homomorphism of A onto a self-adjoint algebra of bounded operators on $\mathfrak{X}$ with kernel contained in $\bigcap_{f\in S}N(f)$.*

Proof. For $a \in \mathfrak{A}$, define T_a on $\mathfrak{X}$ by $(T_a\, x)\ (f) = T_{af}(x(f))$ for all $f \in S$, $x \in \mathfrak{X}$. A straightforward check shows that $a \to T_a$ satisfies the stated conditions.

THEOREM 3.5. *If $\mathfrak{A}$ has an essential involution (i.e., for each $a \in \mathfrak{A}$ there is a positive functional f with $f(e) \leqslant 1$ and $f(a^*a) > 0$), then there is a norm-reducing *-isomorphism of A onto a self-adjoint algebra of bounded operators on a Hilbert space.*

Proof. Let S in Lemma 3.4 be the set of all positive functionals f satisfying $f(e) \leqslant 1$. Then the mapping $a \to T_a$ of Lemma 3.4 is a norm-reducing *-isomorphism of A onto a self-adjoint algebra of bounded operators on a BIP-space $\mathfrak{X}$. Moreover, the inner product in $\mathfrak{X}$ is positive definite. By Lemma 2.2,

$$\sup_{x\neq 0}\ (T_a\, x,\ T_a\, x)/(x,x) \leqslant ||T_a{}^*T_a|| \leqslant ||a^*a|| \leqslant ||a||^2.$$

Therefore each T_a can be extended to a bounded operator on the completion of $\mathfrak{X}$ in the Hilbert space norm.

THEOREM 3.6. *Let $\mathfrak{A}$ be a *-algebra. Then there is a* BIP-*space $\mathfrak{X}$ and a norm-reducing *-isomorphism $a \to T_a$ of $\mathfrak{A}$ onto a self-adjoint algebra of bounded operators on $\mathfrak{X}$ which satisfies $||T_h|| = ||h||$ for each self-adjoint $h \in \mathfrak{A}$.*

Proof. Suppose that h is self-adjoint. Let g be a linear functional on $\mathfrak{A}$ satisfying $||g|| = 1$, $g(h) = ||h||$. Let $f(a) = \frac{1}{2}(g(a) + g(a^*)^c)$ for each $a \in \mathfrak{A}$. Then f is a real functional and $||f|| \leqslant 1$ since

$$|f(a)| \leqslant \tfrac{1}{2}(||a|| + ||a^*||) = ||a||.$$

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For arbitrary $a \in N(f)$, we have $||h|| = f(h) = f(h + a) \leqslant ||h + a||$. Therefore $||h_f|| = ||h||$. Let S in Lemma 3.4 be the set of all real functionals with norm at most 1. Define $x \in \mathfrak{X}$ by

$$x(f) = e_f, \quad x(f') = 0, \qquad f' \in S, f' \neq f.$$

Then $||x|| \leqslant 1$ and $||T_h|| \geqslant ||T_h(x)|| = ||T_{h,f}(e_f)|| = ||h_f|| = ||h||$. Therefore $a \to T_a$ is norm-preserving on self-adjoint elements and all that remains is to prove that it is 1:1.

To this end suppose $T_a = 0$. Let $a = h + ik$ with h, k self-adjoint. Then $T_{2h} = T_a + T_a^* = 0$ and $T_{2k} = T_{-ia} + T_{-ia}^* = 0$. Therefore $h = k = 0$ and consequently $a = 0$.

Remark 1. The restriction that $\mathfrak{A}$ has an identity element can be removed. This can be seen by using the standard imbedding of $\mathfrak{A}$ in an algebra $\mathfrak{B}$ with an identity element. $\mathfrak{B}$ consists of all ordered pairs (λ, a) for λ complex and $a \in \mathfrak{A}$. The operations in $\mathfrak{B}$ are defined as if (λ, a) were $\lambda + a$, $(\lambda, a)^* = (\lambda^c, a^*)$, and $||(\lambda, a)|| = |\lambda| + ||a||$.

Remark 2. We can remove the requirement that $\mathfrak{A}$ be complete if we relinquish the requirement that $\mathfrak{X}$ be complete.

Remark 3. A purely algebraic analogue of Theorem 3.6 holds. In this case we let $\mathfrak{X}$ be the finite direct sum of the $\mathfrak{A} - N(f)$'s for all real functionals f.

Remark 4. Finally we indicate a method of obtaining the Gelfand-Neumark result from our considerations. Let $\mathfrak{A}$ be a C^*-algebra (i.e., $\mathfrak{A}$ is a *-algebra, $||a^*a|| = ||a||^2$, and $(e + a^*a)^{-1}$ exists for each $a \in \mathfrak{A}$). It is known that this last condition is superfluous **(8)**. The proof consists of (1) showing that the involution in $\mathfrak{A}$ is essential, (2) appealing to Theorem 3.5, and (3) showing that the isomorphism of Theorem 3.5 is automatically norm-preserving. Except for a minor modification in the method of showing that the T_a's are bounded this is the Gelfand-Neumark proof.

The necessity of demonstrating (3) can be removed if we strengthen (1). Suppose $a \in \mathfrak{A}$. Fukamiya **(2**, Lemma 5**)** has shown that there is a positive functional f satisfying $f(e) = 1$, $f(a^*a) = ||a||^2$. Define $x \in \mathfrak{X}$ by $x(f) = e_f$ and $x(f') = 0$ for $f' \in S$, $f' \neq f$. Then

$$\begin{aligned} \sup_{y \neq 0} ((T_a y, T_a y)/(y,y)) &\geqslant (T_a x, T_a x) \\ = (T_{af} e_f, T_{af} e_f) &= (a_f, a_f) = f(a^*a) = ||a||^2. \end{aligned}$$

Therefore the mapping of Theorem 3.5 is norm-preserving.

4. Representations. A preliminary study of *-algebras indicates that many representations are pathological. We shall limit ourselves to the consideration of a more tractable case. Throughout this section $\mathfrak{A}$ will be a *-algebra satisfying $||a^*a|| \geqslant K\, ||a||^2$ for some $K > 0$. By a representation of $\mathfrak{A}$ we shall mean a continuous *-homomorphism $a \to T_a$ of $\mathfrak{A}$ onto an algebra of bounded operators on a BIP-space. We shall call the representation irreducible if and only if X has no proper subspaces invariant under each T_a. Finally we shall call a set of representations complete if and only if for each $0 \neq b \in \mathfrak{A}$ there is a representation $a \to T_a$ in the set for which $T_b \neq 0$.

LEMMA 4.1. *If L is a proper left ideal in $\mathfrak{A}$, then the closure of $L + L^*$ does not contain e.*

Proof. By using a result of Arens (**1**, Theorem 1) concerning the representation of closed commutative *-subalgebras of $\mathfrak{A}$, it is easy to see that (1) every self-adjoint element has a real spectrum and (2) if $h \in H$ and $||h - e|| < 1$ then there is a $k \in H$ such that k^{-1} exists, $kh = hk$, and $k^2 = h$.

Now suppose e is in the closure of $L + L^*$. Then there are $x_n, y_n \in L$ such that $x_n + y_n^* \to e$. Then $z_n = \frac{1}{2}(x_n + y_n) \in L$ and $z_n + z_n^* \to e$. Pick m such that $||e - (z_m + z_m^*)|| < 1$. Let $z_m = h_1 + ih_2$ with $h_1, h_2 \in H$. Let $h = 2h_1 = z_m + z_m^*$. Pick k as in (2). Since $k^{-1}(2h_2)k^{-1} \in H$ we conclude by (1) that $ie - k^{-1}(2h_2)k^{-1}$ has an inverse. It follows from this that

$$(k(e + ik^{-1}(2h_2)k^{-1})k)^{-1}$$

exists. However,

$$k(e + ik^{-1}(2h_2)k^{-1})k = k^2 + i2h_2 = 2(h_1 + ih_2) = 2z_m.$$

Therefore z_m^{-1} exists which is a contradiction since $z_m \in L$.

LEMMA 4.2. *If L is a proper left ideal in $\mathfrak{A}$, then there is a nonzero bounded real functional f on $\mathfrak{A}$ such that $L \subseteq N(f)$.*

Proof. $(L + L^*) \cap H$ is a (real) subspace of H which is non-dense by Lemma 4.1. Using the Hahn-Banach theorem, pick a nonzero bounded (real) linear functional on H which vanishes on $(L + L^*) \cap H$. Let f be the extension of this functional to $\mathfrak{A}$. For $a \in L$, the self-adjoint components of a are in $(L + L^*) \cap H$. Therefore $f(a) = 0$. Since L is a left ideal, we conclude that $L \subseteq N(f)$.

It should be noted that if $\mathfrak{A}$ were a C^*-algebra then we could require f to be a positive functional (**6**, p. 86).

THEOREM 4.3. *$\mathfrak{A}$ has a complete set of irreducible representations.*

Proof. $\mathfrak{A}$ is semi-simple (**7**, Theorem 5.2). Therefore the intersection of the set $\mathscr{M}$ of maximal left ideals in $\mathfrak{A}$ is 0 (**4**, Corollary 1, p. 485). Suppose $0 \neq b \in \mathfrak{A}$. Pick $M \in \mathscr{M}$ such that $b \notin M$. By Lemma 4.2, there is a bounded real function f on $\mathfrak{A}$ such that $M = N(f)$. The representation $a \to T_a$ associated with f is irreducible since a subspace of $\mathfrak{A} - N(f)$ invariant under each T_a must come from a left ideal of $\mathfrak{A}$ containing M. Since the kernel of this representation is contained in M, $T_b \neq 0$.

REFERENCES

1. R. Arens, *Representation of *-algebras*, Duke Math. J., *14* (1947), 269–282.
2. M. Fukamiya, *On B*-algebra*, Proc. Japan Acad., *27* (1951), 321–327.
3. I. Gelfand and M. Neumark, *On the imbedding of normed rings into the ring of operators in Hilbert space*, Matematičeskii Sbornik, N.S., *12* (*54*) (1943), 197–217.
4. E. Hille, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publications, *31* (New York, 1948).
5. P. D. Lax, *Symmetrizable linear transformations*, Communications on Pure and Applied Mathematics, *7* (1954), 633–647.
6. M. Neumark, *Rings with involution*, Uspehi Matematičeskii Nauk, N.S., *3*, *5* (*27*) (1948), 52–145; translated as Amer. Math. Soc. Translation *25* (1950). All specific references are to the translation.
7. C. E. Rickart, *The uniqueness of norm problem in Banach algebras*, Ann. Math. (2), *51* (1950), 615–628.
8. Review of a paper by Fukamiya in Math. Rev. *14* (1953), 884 and *15* (1954), 1139.

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