A MIXED PROBLEM FOR NORMAL HYPERBOLIC LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

G. F. D. DUFF

In the theory of hyperbolic differential equations a mixed boundary value problem involves two types of auxiliary conditions which may be described as initial and boundary conditions respectively. The problem of Cauchy, in which only initial conditions are present, has been studied in great detail, starting with the early work of Riemann and Volterra, and the well-known monograph of Hadamard **(4)**. A modern treatment of great generality has been given by Leray **(7)**. In contrast mixed problems have received comparatively little attention, and the nature of the boundary conditions to be imposed on equations of order higher than the second is known only for equations in two independent variables **(8)**. For second order normal hyperbolic equations both linear and non-linear, the problem has been studied, using the method of analytic approximation, by Schauder and Krzyzanski **(5)** who assigned as boundary condition that the unknown function should take given values on a timelike boundary surface. The monograph of Ladyshenskaya **(6)** treats certain cases of the problem where the normal derivative is given, for instance when the coefficients are independent of the time variable.

In this paper a different boundary condition is considered; this condition involves the derivative of the dependent variable in a given direction, which is defined on the boundary but is not tangential to the boundary. There are restrictions in the large on this direction, made necessary by the properties of certain families of characteristic surfaces. However, the condition includes as a special case the problem of the normal derivative, which arises in the theory of supersonic flow.

As in **(5)** the analytic case is treated first, by means of dominant power series. The nature of the boundary conditions is taken into account by a certain order of choice among the dominating series. For the non-analytic case a suitable modification of the estimates of **(5)** is arranged, while the construction of the solution is as before.

1. The mixed problem. We study the linear normal hyperbolic partial differential equation

$$L(u) = a^{ik} \frac{\partial^2 u}{\partial x^i \, \partial x^k} + b^i \frac{\partial u}{\partial x^k} + cu = f, \tag{1.1}$$

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with one dependent variable u and N independent variables $x^i (i = 1, \ldots, N)$. Summation over repeated indices is understood in (1.1). The coefficients a^{ik}, b^i, c, and f are functions of the x^i, differentiable k times throughout the domain of x^i space to be considered. The normal hyperbolic character of (1.1) is expressed by the signature of the quadratic form

$$a^{ik}\,\xi_1\,\xi_k, \tag{1.2}$$

which signature is $(N - 1,1)$ with one negative term.

With the Riemann metric

$$ds^2 = a_{ik}\,dx^i dx^k \tag{1.3}$$

based on the associate covariant tensor a_{ik}, we have a classification of directions v^i as spacelike, null, or timelike, according as

$$v^2 = a_{ik}v^i v^k = a^{ik}v_i v_k$$

is positive, zero or negative. Also surfaces $S: \phi(x^i) = 0$ shall be spacelike, null or timelike according as

$$a_{ik}\frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^k} \tag{1.4}$$

is negative, zero, or positive. The normal n^i is defined by

$$n^i = a^{ik}\frac{\partial\phi}{\partial x^k}. \tag{1.5}$$

Let $S: \phi(x^i) = 0$ be an initial spacelike surface and let $T: \psi(x^i) = 0$ be a timelike surface intersecting S in a rim C of $N - 2$ dimensions. We shall suppose that S is bounded by C and that T is bounded "toward the past" by C, in a suitable orientation of "time." Let G be the characteristic surface, passing through the rim C, which lies in the region enclosed by S and T, that is, which bounds the domain of dependence D_S on S according to the theory of the Cauchy problem. We note that G is composed of characteristic strips tangent to the rim C.

On S we assign values of u and $\partial u/\partial n$; these are the usual Cauchy data, and they determine u in D_S. On T we assign a boundary condition "of the second kind" as follows. Let v be a vector field defined on T and subject to restrictions stated below. Then, if the directional derivative of u in the direction of v is denoted by $\partial u/\partial v$, we set

$$\frac{\partial u}{\partial v} \equiv v^i\frac{\partial u}{\partial x^i} = f(x^i), \tag{1.6}$$

where $f(x^i)$ is a datum function given on T.

On the rim C this datum function $f(x^i)$ and its derivatives transverse to C in T shall be subject to certain conditions of compatibility with respect to the given differential equation and Cauchy data. These conditions were imposed to ensure that the derivatives of u up to a certain order shall be

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continuous across G. The value for u on C shall be that assigned by the Cauchy data, and the first compatibility condition is that f shall be equal to the value of $\partial u/\partial v$ on C derived from the Cauchy data. The second compatibility condition brings in the differential equation since it postulates that $\partial f/\partial t$ equal $\partial^2 u/\partial v \partial t$, the latter being calculated on C from the Cauchy data and the differential equation. Here t denotes a suitable timelike variable. Likewise the kth condition of compatibility determines the $(k - 1)$st derivative of f as equal to the corresponding derivative of u, calculated from the Cauchy data and the differential equation, by successive differentiation with respect to t and substitution of values already found.

If the first k compatibility conditions hold, it is evident that u and its first k derivatives are continuous across G at the rim C. Now the method of construction of the solutions leads to their being continuous across G, at every point of G. To show that the transverse derivatives of u up to order k are also continuous across G, we note that through each point of G there passes a bicharacteristic ray issuing from the rim C. If a solution u is continuous and has continuous tangential derivatives on G, then its first transverse derivative u_n is continuous along the entire bicharacteristic if it is so at one point. The conditions necessary for this conclusion will be satisfied in our construction, and we infer that u_n is continuous across G. In succession the higher transverse derivatives, up to and including order k, are proved continuous across G.

The initial and boundary conditions determine u in a larger region. Construct the retrograde characteristic cone C^P with vertex P. If C^P, T, and S together bound a region, then P lies in this domain of dependence on S and T. Since the Cauchy initial value problem can be regarded as solved, we subtract its solution from the dependent variable and so find a reduced boundary value problem which may be stated as follows. We must find a solution of (1.1), with $f = 0$, which vanishes on G_0, satisfies (1.6) on T, and is defined in the region V intermediate to S and T. The compatibility condition is now that $f(x^i) = 0$ on C, while the corresponding condition of order k is that the derivative of $f(x^i)$ of order $k - 1$ in a direction within T but transverse to C should vanish. We note that the rim C, being a subspace of S, is spacelike, and that all directions tangent to G are either spacelike or null. Indeed G, being an integral surface of a first order partial differential equation, is composed of the bicharacteristic curves passing through C which determine characteristic strips tangent to C.

Now let C_t be a family of spacelike $(N-2)$-dimensional surfaces filling T, and such that $C_0 = C$. We may construct characteristic surfaces G_t containing C_t and these will fill up the region V. The condition which we impose on the vector field v is that *v should not be tangent to the local G_t at any point of T.*

That some restriction of the vector field relative to characteristic surfaces is necessary can be seen from the equation of the vibrating string;

$$u_{xx} = u_{tt}.$$

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If we require $u(x, 0) = u_t(x, 0) = 0$ for $t = 0$, $x > 0$, and

$$\alpha(t)u_x + \beta(t)u_t = f(t), \qquad x = 0, \; t > 0,$$

then

$$u(x, t) = \int_0^{t-x} \frac{f(\tau)}{\beta(\tau) - \alpha(\tau)} d\tau, \qquad t > X,$$

and the denominator of the integrand vanishes if $\alpha(\tau) = \beta(\tau)$; that is, if the vector field takes the direction of the forward characteristic entering the region through T. This condition holds for all such equations in two variables; it is easily seen that the directional derivative along any forward characteristic is determined by the Cauchy problem with data taken at the instant t where the characteristic curve meets T.

If $N > 2$, the situation is more involved. For the analogous condition, namely that v should not touch G_t, we may ask: what conditions on v enable us to construct a family of spacelike varieties C_t on T such that v will never be tangent to the G_t constructed on C_t as base? Such a field will be called admissible.

To answer the question we recall that the G_t, being integral surfaces of a first order partial differential equation, are constructed as envelopes of portions of the characteristic conoids with vertex on C_t. The portion concerned is that part $C^P{}_i$ of the forward half-cone lying in the interior of our region.

Let us consider the tangent space at a typical point P of T: $\psi \equiv X = X^{N-1} = 0$. In the surface T the tangent plane to C_t will take the form (with P as origin)

$$t - \sum c_\alpha x_\alpha = 0, \qquad \alpha = 1, \ldots, N-2,$$

where $\sum c_\alpha^2 < 1$ since C_t is spacelike. The tangent plane to the cone $C^P{}_i$ in the full space, which also meets T tangent to C_t, will have the equation

$$t - \sum c_\alpha x_\alpha - \sqrt{(1 - \sum c_\alpha^2)}\, x = 0.$$

We must determine those regions of space such that there exist values of the c_α which render the function

$$f \equiv t - \sum c_\alpha x_\alpha - \sqrt{(1 - \sum c_\alpha^2)}\, x$$

consistently of a given sign.

There will later appear the restriction that the vector v should not touch T, and we shall, for convenience, make this assumption here. The region of space to be considered may now be taken as the side $x > 0$ of T. Two cases arise, according as

$$f_0[v] = v_t - v_x,$$

the initial value of f with the components of v substituted for the coordinates, is positive or negative. These correspond to v lying initially "later" than G or "earlier," and will be referred to as the positive and negative cases respectively.

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Taking first the case of negative values, we shall minimize f with respect to the c_α. Since

$$\frac{\partial f}{\partial c_\alpha} = -x_\alpha + \frac{c_\alpha}{\sqrt{(1 - \sum c_\alpha^2)}} x$$

and

$$\frac{\partial^2 f}{\partial c_\alpha \partial c_\beta} = \frac{\delta_{\alpha\beta} x}{\sqrt{(1 - \sum c_\alpha^2)}} + \frac{c_\alpha c_\beta x}{(1 - \sum c_\alpha^2)^{3/2}},$$

we find, first, that an extremum is present for

$$c_\alpha = \frac{\pm x_\alpha}{\sqrt{(x^2 + \sum x_\alpha^2)}},$$

and secondly, that this is a minimum value. The actual minimum is therefore

$$f_{\min} = t - \sqrt{(x^2 + \sum x_\alpha^2)}$$

which will be negative if

$$t < \sqrt{(x^2 + \sum x_\alpha^2)}.$$

That is, points on the forward cone $C^P{}_i$, or within it, are excluded.

The positive case is a little different, since no true maximum of f exists. As we have $x > 0$ the third term of f is negative, and it follows that if

$$f_1 \equiv t - \sum c_\alpha x_\alpha$$

takes positive values for some c_α, then so does f for sufficiently small positive x. Now the sum in f will take its greatest value when $\sum c_\alpha^2$ is allowed its greatest value. Thus we may take $\sum c_\alpha^2 = 1$ and so find the extrema of

$$f_2 = t - \sum c_\alpha x_\alpha + \lambda(1 - \sum c_\alpha^2).$$

Hence

$$\frac{\partial f_2}{\partial c_\alpha} = -x_\alpha - 2\lambda c_\alpha = 0,$$

and so, with $\sum c_\alpha^2 = \sum x_\alpha^2/4\lambda^2 = 1$, we find

$$c_\alpha = \frac{-x_\alpha}{\sqrt{(\sum x_\alpha^2)}}$$

and the maximum of f_1 is

$$f_{1\max} = t - \sum c_\alpha x_\alpha = t + \sqrt{(\sum x_\alpha^2)}.$$

Thus we get positive values for f_1, and since $\sum c_\alpha^2 = 1$, also for f, provided

$$t > -\sqrt{(\sum x_\alpha^2)}.$$

The bounding surface so defined is cylindrical in the x direction, and touches the cone $C^P{}_i$ along its intersection with the surface T.

If the condition of not touching holds at a point P, then by continuity it holds in a neighbourhood of P. Over a compact portion of T we can find uniform moduli of continuity, provided that the limiting cases mentioned above do not arise. A neighbourhood of uniform size can thus be defined, and the construction extended to the whole of the compact region by repeated application of the existence theorem.

This result may be summarized as follows:

LEMMA I. *A vector field v not tangent to G_0 or to T is admissible if*

(a) *being initially positive, it satisfies*

$$v_t > -\sqrt{(\sum v_\alpha^2)} \qquad \text{on } T;$$

(b) *being initially negative, it satisfies*

$$v_t < \sqrt{(v_x^2 + \sum v_\alpha^2)} \qquad \text{on } T.$$

We remark that the normal vector field, with one non-vanishing component $v_x > 0$, falls under case (b).

2. The analytic case. Let all coefficients in the differential equation, and the surfaces S and T, be analytic. Then characteristic surfaces such as the G_t are also analytic provided that the rims C_t are analytic. This can be arranged and will be assumed.

Before reducing the differential equation to a standard form (**4**, p. 76) we shall simplify the boundary condition

$$\frac{\partial u}{\partial v} = f. \tag{2.1}$$

Here f vanishes to order $k + 1$ on C_0 according to the compatibility conditions. We note that the vector field u is not parallel to G_0 on T and thus we can construct a C^k function u_1 which vanishes on G_0 and also satisfies (2.1) on T. Subtracting this function from u, we obtain for the new dependent variable a differential equation of the form

$$L(u) = f_1 \tag{2.2}$$

while the new boundary conditions are (cf. §6),

$$u = 0, \qquad \text{on } G_0, \tag{2.3}$$

with

$$\frac{\partial u}{\partial v} = 0, \qquad \text{on } T. \tag{2.4}$$

We now change the independent variables so as to give T the equation $x = x^{N-1} = 0$ while the analytic family of characteristic surfaces G_t have equation

$$G_t\colon t = x^N = \text{const.}$$

This forces the coefficient a^{NN} to vanish identically in the new system. Since the rim C_t is spacelike, and so never tangent to a bicharacteristic direction, we can choose the remaining variables $x^1, \ldots, x^{N-2}$ so that the bicharacteristics on G_t are

$$x^\rho = \text{const.}, \qquad \rho = 1, \ldots, N-2. \tag{2.5}$$

This results in the vanishing of the coefficients $a^{N\rho}$. Following Hadamard **(4)**, we divide by $a^{N,N-1}$ which cannot now vanish since $L(u)$ is not parabolic; and we replace u by

$$u \exp\left[\int b^N dx\right]$$

which causes the term in $L(u)$ containing $\partial u/\partial t$ to disappear. Then the differential equation becomes

$$\frac{\partial^2 u}{\partial x \partial t} = L_1(u) + f_2, \tag{2.6}$$

where the operator $L_1(u)$ contains no differentiations with respect to t. With this form of the equation Hadamard and others studied the indeterminacy of Cauchy's problem for characteristic surfaces.

The boundary conditions to go with (2.6) are now

$$u = 0 \text{ for } t = 0 \tag{2.7}$$

and

$$\frac{\partial u}{\partial v} = v^i \frac{\partial u}{\partial x^i} = b^N u, \qquad x = 0. \tag{2.8}$$

In order to express this latter condition more conveniently, we note that by hypothesis the component v^N does not vanish—this is our condition on the vector v. Dividing by v^N and transposing some terms, we have

$$\frac{\partial u}{\partial t} = \sum_{k=1}^{N-1} \beta^k \frac{\partial u}{\partial x^k} + hu, \qquad x = 0. \tag{2.9}$$

We now expand u in a series of powers of t, and determine the coefficients in succession. Let

$$u = \sum_{n=1}^{\infty} u_n t^n, \quad f_2 = \sum_{n=0}^{\infty} f_n t^n, \tag{2.10}$$

and also let

$$L_1(u) = \sum_{n=0}^{\infty} t^n L_{n1}(u).$$

Then the u_n satisfy

$$n \frac{\partial u_n}{\partial x} = L_{01}(u_{n-1}) + f_{n-1} + \ldots, \tag{2.11}$$

where the terms omitted contain the $u_k (k = 0, 1, \ldots, n-2)$. We have taken $n_0 \equiv 0$ to satisfy (2.7). Substituting these expansions into (2.8), we get the conditions

$$(2.12)\qquad nu_n = \sum_k \beta^k{}_0 \frac{\partial u_{n-1}}{\partial x^k} + h_0 u_{n-1} + \ldots, \qquad x = 0.$$

Here $\beta^k{}_0$ and h_0 are initial terms in the expansions of the β^k and h in powers of t, while the terms omitted in (2.12) again contain u_k $(k = 0, 1, \ldots, n - 2)$. Thus the u_n are uniquely determined by integration of (2.11) for successive values of n, in the form

$$(2.13)\qquad nu_n(x) = nu_n(0) + \int_0^x [L_{01}(u_{n-1}) + f_{n-1} + \ldots]\, dx',$$

and the functions so found are analytic in x as well as in the remaining variables.

The techniques of dominating series will now be applied to show that the series solution thus found is convergent in a certain domain. We note that the operations in (2.13) are such as to preserve any dominant relation; thus if we dominate the coefficients in (2.6) and (2.9) the new solution will dominate that already found. Now the two auxiliary conditions will be dominated in the following way. We shall seek a solution with positive coefficients of the dominating differential equation. This solution will automatically dominate the condition (2.7). We will also show that if the left side of (2.9) is computed (in the dominant case) it will dominate the right side, and therefore will dominate the actual condition (2.9). This requires a certain order of choice among the various dominating constants which will appear. The proof will also show that the series has a radius of convergence independent of the function f_2 in (2.6), and hence independent of the data prescribed for the original problem.

We choose as origin a point of C_0 and set

$$y = x^1 + \ldots + x^{N-2},$$

and let ρ, σ be sufficiently small positive numbers. Then the dominant boundary condition can be written

$$(2.14)\qquad \frac{\partial u}{\partial t} = \left(1 - \frac{t}{\sigma}\right)\left(1 - \frac{y}{\rho}\right)^{-1} \left[\sum_i G_i \frac{\partial u}{\partial x^i} + Hu\right],$$

where $G_i (i = 1, \ldots, N - 1)$ and H are positive constants. Letting

$$(2.15)\qquad \tau = -\sigma \log\left(1 - \frac{t}{\sigma}\right) = t + \frac{t^2}{2\sigma} + \ldots,$$

we can write this

$$(2.16)\qquad \frac{\partial u}{\partial \tau} = \left(1 - \frac{y}{\rho}\right)^{-1} \left[\sum_i G_i \frac{\partial u}{\partial x^i} + Hu\right], \qquad x = 0.$$

Denote $\sum_i G_i$ by G.

In proving the convergence theorem we will actually assume that the left side of (2.16) dominates the right side. Since the series in (2.15) has positive coefficients, this will imply that the left side of (2.14) dominates the right side, and hence that the boundary condition (2.9) will be dominated as required.

The dominating differential equation takes the form

$$(2.17)\qquad \frac{\partial^2 u}{\partial x \partial t} = \left(1 - \frac{t}{\sigma}\right)^{-1}\left(1 - \frac{x+y}{\rho}\right)^{-1} \left[\sum_{i,k} A_{ik}\frac{\partial^2 u}{\partial x^i \partial x^k} + \sum_i B_i \frac{\partial u}{\partial x^i} + Cu + F\right],$$

where the A_{ik}, B_i, C and F are constants. Here only F depends on f_2 in (2.6) and we have therefore to find a radius of convergence independent of F. Let us assume that U is a function of τ, x, and y only. Then (2.17) becomes, with use of (2.15),

$$(2.18)\qquad \frac{\partial^2 u}{\partial x \partial \tau} = \left(1 - \frac{x+y}{\rho}\right)^{-1}\left[A_{11}\frac{\partial^2 u}{\partial x^2} + A_{12}\frac{\partial^2 u}{\partial x \partial y} + A_{22}\frac{\partial^2 u}{\partial y^2} + B_1\frac{\partial u}{\partial x} + B_2\frac{\partial u}{\partial y} + Cu + F\right],$$

where the A's are chosen anew if necessary. Since we can always increase them, we shall require that

$$(2.19)\qquad A = A_{11} + A_{12} + A_{22} > G.$$

If we further assume that U is a function of the combination

$$w = \tau + \alpha(x + y)$$

alone, where $\alpha > 0$, then we can find an ordinary differential equation for $U(w)$ which dominates (2.18) and therefore still more dominates (2.6). To do this we replace $x + y$ in the denominator of (2.18) by $x + y + \tau/\alpha = w/\alpha$. Collecting terms in the ordinary differential equation to which (2.18) now leads, we find

$$(2.21)\qquad \left(1 - \frac{w}{\alpha\rho} - \alpha A\right) U'' = BU' + \frac{C}{\alpha}U + \frac{F}{\alpha}.$$

Here primes denote differentiation with respect to w. We now choose α so small that

$$(2.22)\qquad 1 - \alpha A \geqslant \tfrac{1}{2}.$$

According to (2.18) we would set B in (2.21) equal to $B_1 + B_2$. Since we can increase B freely without destroying the dominance over (2.6), we shall stipulate that

$$(2.23)\qquad B > H\frac{1 - \alpha A}{1 - \alpha G} + \frac{1}{\alpha\rho}.$$

Defining $r = \alpha\rho(1 - \alpha A)$ we can write (2.21) in the form

$$(2.24)\qquad U'' = \left(1 - \frac{w}{r}\right)^{-1}\left[\frac{B}{1 - \alpha A}U' + \frac{C}{\alpha(1 - \alpha A)}U + \frac{F}{\alpha(1 - \alpha A)}\right].$$

If in this equation $U(0)$ and $U'(0)$ are positive, then all coefficients of the series solution are positive.

Indeed, if we set

$$U = \sum_{n=0}^{\infty} a_n w^n, \tag{2.25}$$

the recursion formula for the a_n is seen to be

$$(n+2)(n+1)\, a_{n+2} = (n+1)\left[\frac{n}{r} + \frac{B}{1-\alpha A}\right] a_{n+1} + \frac{C}{\alpha(1-\alpha A)}\, a_n + \frac{F\delta_n^0}{\alpha(1-\alpha A)}\, .$$

where the last term on the right is present only if $n = 0$. Assuming now that a_0 and a_1 are non-negative, we find

$$(n+2)a_{n+2} \geqslant \left[\frac{n}{r} + \frac{B}{1-\alpha A}\right] a_{n+1}. \tag{2.26}$$

This relation is used below.

Now consider the boundary condition (2.16). The formulae (2.9) and (2.12) show that the initial values for the u_n will be dominated if the left side of (2.16) dominates the right side. Still more will this hold if y in (2.16) is replaced by $x + y + \tau/\alpha = w/\alpha$. With this modification we get for $U(w)$ the condition

$$U'(w) \gg \left(1 - \frac{w}{\alpha\rho}\right)^{-1} [\alpha\, GU' + HU] \tag{2.27}$$

which will hold if

$$\left(1 - \frac{w}{\alpha\rho} - \alpha G\right) U'(w) \gg HU(w). \tag{2.28}$$

To verify that (2.28) implies (2.27) we recall that U is a series with positive coefficients; thus if we add to each side the series $\alpha GU'$ and then multiply on right and left by the series for $(1 - w/\alpha\rho)^{-1}$ we will not destroy the dominating relation.

To demonstrate (2.28) we calculate the coefficient of w^n on the left; it is

$$(1-\alpha G)(n+1)\, a_{n+1} - \frac{1}{\alpha\rho}\, n a_n$$

which by (2.26), with $n+1$ changed into n, is not less than

$$(1-\alpha G)\left[\frac{n-1}{r} + \frac{B}{1-\alpha A}\right] a_n - \frac{1}{\alpha\rho}\, n a_n$$

From (2.23) we find that this in turn exceeds

$$\left[(1-\alpha G)\left(\frac{n-1}{\alpha\rho(1-\alpha A)} + \frac{H}{1-\alpha A}\cdot\frac{1-\alpha A}{1-\alpha G} + \frac{1}{\alpha\rho(1-\alpha A)}\right) - \frac{n}{\alpha\rho}\right] a_n$$
$$\geqslant \left[H + \frac{n}{\alpha\rho}\left(\frac{1-\alpha G}{1-\alpha A} - 1\right) + \frac{\alpha G}{\alpha\rho(1-\alpha A)}\right] a_n.$$

Since $G < A$ the middle term in the bracket is positive and thus the coefficient exceeds H which is the coefficient of w^n on the right of (2.28). This proves that (2.28) and (2.27) hold in general and hence that (2.16) and so (2.9) are dominated in the required way when $x = 0$.

For dominating power series we therefore choose a solution $U(w)$ having positive values for $U(0)$ and $U'(0)$. The radius of convergence of this series is equal to $r = \alpha\rho(1 - \alpha A)$, from the theory of linear differential equations, and this is independent of F.

Repeating this work at other points of C_0 we can show that the unique analytic solution thus found exists in a neighbourhood $t < \delta_1$, of C_0. Here δ_1 is independent of the datum functions of the Cauchy problem as well as the mixed boundary condition. If we select any compact portion of T such that the above hypotheses are uniformly satisfied when any one of the characteristic surfaces G_t is chosen in place of G_0 then we can find a δ_1 which will serve for them all.

Combining the local solution just constructed with the solution of Cauchy's problem for the analytic case, we see that the resulting composite solution is analytic except possibly on G_0. If the datum function $f(x^i)$ originally given satisfies the compatibility conditions of §1 up to the order k inclusive, then, by well-known properties of the discontinuities across characteristic surfaces of derivatives of u, it follows that u and its derivatives up to order k inclusive are continuous across G_0. We state this result as a lemma:

LEMMA II. *Let the compatibility conditions up to order k inclusive be satisfied in the analytic case; then there exists a unique solution analytic for* $0 < t < \delta_1$ *except on* G_0, *where the derivatives of order up to k inclusive are continuous.*

The domain of definition of this local solution will be extended in §4. We note that for the purposes of this local analytic solution it is sufficient to have (2.9) and thus v may be tangent to T.

3. Estimates of solutions. To extend the result to non-analytic equations and data, we give estimates of the square integrals of the solution and its derivatives up to a certain order. These are found by a modification of the method used by Krzyzanski and Schauder **(5)**, which in turn is based on the work of Friedrichs and Lewy **(3)**. For brevity we shall indicate only the alterations necessary for our purposes. In this section we take the geometric background to be Euclidean. It is also convenient to suppose that T is cylindrical in the sense that S spans T and the rim $C_0 = S \cap T$ is closed.

Since the Cauchy problem is regarded as solved, we can take as initial spacelike surface any spacelike surface which spans T, meeting T in the rim C_0. We shall construct a family of surfaces S_t spanning T, with $S_0 \cap T = C_0$, and such that the given vector v is never tangent to the S_t. The direction field v is again assumed admissible; thus there exists a family G_t of characteristic surfaces, with $G_0 \cap T = C_0$, such that v is not tangent to the G_t. Let us extend

v to a field defined throughout a region of space containing the G_t; if this region is sufficiently small we can, even in the analytic case, determine v analytically so that it is never tangent to G_t in the region. Denoting the minimum angle of v to G_t by θ_0, we construct spacelike surfaces S_t as follows: S_t shall contain $C_t = T \cap G_t$; and S_t shall be inclined to G_t at an angle between $\frac{1}{3}\theta_0$ and $\frac{2}{3}\theta_0$ at every point. These surfaces S_t may be chosen to be analytic in the analytic case. Now we see that v is never tangent to the S_t.

We now set up a coordinate system on the family of surfaces S_t with equation t = const. We choose the coordinate network $x^1, \ldots, x^{N-1}$ on S_t in such a way that the parametric lines of t cross T at every point from inside to outside with increasing t. This can be achieved by a change of scale in a suitable "radial" coordinate in S_t, and does not alter the spacelike character of S_t. Since v is never tangent to S_t, we could take as Nth coordinate, in place of t, a suitable parameter ξ^N along the integral curves of v. The transformation of coordinates so defined is clearly non-singular; and will be used below in certain surface integrals taken over T. By measuring arc along the v curves starting on T we ensure that T has in these coordinates the equation

$$T: \xi^N = 0.$$

However, this requires that v should not be tangent to T, which we therefore assume for the rest of this section. This condition has been anticipated in the form of the statement of Lemma I.

Let V_t be the region bounded by S, T, and S_t; and let n_1, n_2, n_3 denote the outward Euclidean normals on the surface of S, T, and S_t respectively. If $\cos(nx^i)$ denotes the cosine of n with the parametric line of x^i, then we have

$$\cos(n_1 t) < 0, \quad \cos(n_2 t) > 0, \quad \cos(n_3 t) > 0. \tag{3.1}$$

We now multiply the differential equation (1.1) by $\partial u/\partial t$ and integrate over V_t. After some partial integrations we find

$$\int_{S+T+S_t} \left[2 \sum_{i,k=1}^{N} a^{ik} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial t} \cos(nx^k) - \sum_{i,k=1}^{N} a^{ik} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^k} \cos(nt) \right] dS = \int_{V_t} \left(\Phi + f \frac{\partial u}{\partial t} \right) dV, \tag{3.2}$$

where Φ is a quadratic expression in u and its first derivatives, involving also the coefficients in (1.1) and their first derivatives.

A separate choice of variables is now made in each of the three surface integrals on the left in (3.2). The coordinates $x^1, \ldots, x^{N-1}$ are not changed but the last coordinate is taken to be

$$\eta_s^N = g_s(x^1, \ldots, x^{N-1}, t), \qquad s = 1, 2, 3. \tag{3.3}$$

where the g_s are the functions giving the equations of S, T and S_{t_0} when equated to zero. Thus

$$g_1 = t, \quad g_2 = \xi, \quad g_3 = t - t_0.$$

After some calculation, which we omit, it is possible to verify as in (5) that the integrand on the left in (3.2) becomes

$$\left[a(g_s)\left(\frac{\partial u}{\partial \eta_s^N}\right)^2 - \sum_{i,k=1}^{N-1} a^{ik} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^k}\right] \cos(n_s t)$$

where

$$a(g_s) = \sum_{i,k=1}^{N} a^{ik} \frac{\partial g_s}{\partial x^i} \frac{\partial g_s}{\partial x^k}$$

is the characteristic quadratic form. Since S and S_{t_0} are spacelike and T timelike, we see that

$$a(g_1) < 0, \quad a(g_2) > 0, \quad a(g_3) < 0,$$

in view of the convention of sign in (1.2). Noting that the quadratic form

$$\sum_{i,k=1}^{N-1} a^{ik} \xi_i \xi_k$$

is positive definite, and $\cos(n_3 t)$ is positive, we see that for $s = 3$, that is for the integral over S_{t_0}, the integrand is negative definite. We shall now drop the subscript 0 on the t in S_t.

Since $\cos(n_2 t)$ is positive, the term

$$-\sum_{i,k=1}^{N-1} a^{ik} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^k} \cos(n_2 t)$$

is also negative definite. We transpose to the right side of (3.2) the other term in the integral over T and also the whole of the integral over S, and find, after changing the sign throughout,

$$\begin{aligned} &\int_{S_t}\left[-a(g_3)\left(\frac{\partial u}{\partial \eta_3^N}\right)^2 + \sum_{i,k=1}^{N-1} a^{ik} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^k}\right] \cos(n_3 t)\, dS \\ (3.4) \qquad &+ \int_T \sum_{i,k=1}^{N-1} a^{ik} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^k} \cos(n_2 t)\, dS \\ &= -\int_{V_t}\left(\Phi + f\frac{\partial u}{\partial t}\right) dV + \int_S + \int_T a(g_2)\left(\frac{\partial u}{\partial \eta_2^N}\right)^2 dS. \end{aligned}$$

The left hand side of this equation is now positive definite in all of the derivatives appearing, and in particular the integral over T on the left is non-negative. Thus we can drop this term provided $\leqslant$ is substituted for the equality sign, and the inequality so obtained is in the right direction for our purposes.

Indeed, there is a positive constant c such that the left side of (3.4) exceeds

$$c\sum_{i=1}^{N} \int_{S_t}\left(\frac{\partial u}{\partial x^i}\right)^2 dS.$$

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Now we find an upper bound for the right hand side by conventional methods, and so obtain an estimate

$$(3.5)\quad \sum_{i,k=1}^{N}\int_{S_t}\left(\frac{\partial u}{\partial x^i}\right)^2 dS < K\Bigg[\int_{V_t}\sum_{i=1}^{N}\left(\frac{\partial u}{\partial x^i}\right)^2 dV + \int_{V_t} f^2 dV + \int_S\left[\sum_{i=1}^{N}\left(\frac{\partial u}{\partial x^i}\right)^2 + u^2\right]dS + \int_T\left(\frac{\partial u}{\partial \xi^N}\right)^2 dS\Bigg]$$

for some constant K independent of u. The expression Φ may contain u itself and so we have estimated integrals of the type

$$\int_V u^2 dV$$

by writing

$$u = u_0 + \int_0^t u_t dt'$$

and

$$\int_{V_t} u^2\, dV \leqslant 2\int_{V_t} u_0^2\, dV + 2\int_{V_t}\left(\int_0^t u_t\, dt'\right)^2 dV$$
$$\leqslant 2t\int_S u_0^2\, dS + 2t^2\int_{V_t} u_t^2\, dV.$$

These terms are incorporated on the right in (3.5). Integrating from $t = 0$ to $t = t_0$ in (3.5), we get

$$(3.6)\quad \int_{V_{t_0}}\sum_{i=1}^{N}\left(\frac{\partial u}{\partial x^i}\right)^2 dV \leqslant K_1\Bigg[\int_{V_{t_0}} f^2 dV + \int_{S_0}\left[\sum_{i=1}^{N}\left(\frac{\partial u}{\partial x^i}\right)^2 + u^2\right]dS + \int_T\left(\frac{\partial u}{\partial \xi^N}\right)^2 dS\Bigg],$$

provided that t_0 is sufficiently small. Replacing the first term on the right in (3.5) by means of (3.6), we finally get a similar estimate for the surface integral on the left in (3.5).

Similar estimates for the higher derivatives of u are needed; as they can be found by modifying the calculations of **(5**, §3**)** in the manner indicated above, the details will be omitted. We quote the result as follows: Let $D_h u$ denote a typical partial derivative of order h, and let ρ be a positive integer. Let the coefficients a^{ik}, b^k, c, and f of (1.0) have bounded derivatives up to and including the order $N + 1$; and let these derivatives of order up to $N + \rho - 1$ be square integrable over the domain. Then there holds the estimate

$$(3.7)\quad \sum_{h=0}^{N+\rho}\int_V (D_h u)^2 dV < K_\rho\Bigg[\sum_{h=0}^{N+\rho}\int_V (D_h f)^2 dS + \sum_{h=0}^{N+\rho-1}{}'\int_T\left(D_h\frac{\partial u}{\partial \xi^N}\right)^2 dS + \sum_{h=0}^{N+\rho}\int_S (D_h u)^2 dS\Bigg],$$

and a similar estimate holds for

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$$(3.8) \qquad \sum_{h=0}^{N+\rho} \int_{S_t} (D_h u)^2 dS_t.$$

In these formulae the summation $\sum$ over h is to be taken over all derivatives of the order indicated. However, in the integral over T the summation $\sum'$ over h shall include only derivatives tangential to T and so only one differentiation with respect to $\xi^N = \eta^N{}_2$ will appear. Again, by integrating (3.8) over t, we find the sharp form

$$(3.9) \qquad \sum_{h=0}^{N+\rho} \int_{V_t} (D_h u)^2 dV < t\bar{K}[\qquad]$$

of the estimate for the volume integrals. This is to be used in connection with quasi-linear equations, which we shall mention in §5.

4. Extension of the domain. The following lemmas of Schauder and Krzyzanski **(5**, §6**)** will be used here and in the next section. Let R_k denote the class of functions $v(x^i)$ having absolutely continuous derivatives of order $\leqslant k-1$ in the sense of Tonelli; and quadratically integrable derivatives of order $\leqslant k$; all on a given closed domain such as the region V_t. Such a function is absolutely continuous in the above sense when it is absolutely continuous on almost every parametric line of the chosen coordinates.

With the norm

$$(4.1) \qquad ||v||_k = \sum_{h=0}^{k} \int_{V_t} (D_h v)^2 dV,$$

the class R_k becomes a linear space R_k. We have

LEMMA III. *Polynomials $p(x^i)$ are dense in R_k. Provided that $v \in C^r$ on a compact subset V_1 of V, an approximating sequence $p_j(x^i)$ can be found such that the derivatives of order $\leqslant r$ of the p_j converge uniformly to the derivatives of v.*

LEMMA IV. *Let $k \geqslant N+1$, and let $v_n \in R_k$ be a sequence with uniformly bounded norms $||v_n||_k < K$. Then there exists a subsequence v_{n_j} uniformly convergent to a limit $v \in R_k$.*

The uniform convergence is established by Lemma I of **(9)** while the fact that v belongs to R_k follows **(1)** from the theory of strong convergence in L^2.

We now extend the domain of definition of the solution of Lemma II. Let $0 < \delta < \delta_1$ and let us divide the larger domain V into p slices V_j of width δ:

$$V_j: (j-1)\,\delta \leqslant t \leqslant j\delta, \qquad j = 1, 2, \ldots, p.$$

Let S_j be the surface $t = j\delta$ and $C_j = S_j \cap T$. In each of these domains we shall construct solutions which will subsequently be pieced together. The solution in the large so found will be class R_k, provided that the compatibility conditions of order $\leqslant k$ hold initially, and that $k \geqslant N+2$.

Let u_1 be the local solution defined in V_1. We cannot apply Lemma II to V_2 by taking values of u_1 and $\partial u_1/\partial t$ on S_2 as Cauchy data since these may

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not be analytic in the large. However, as is remarked in **(5**, §7**)**, u_1 and $\partial u_1/\partial t$ have derivatives of all orders on S_1 except on $G_0 \cap S_1$; and satisfy the compatibility conditions of all orders on C_1. We therefore approximate them by polynomial sequences ϕ_{1s} and ψ_{1s} such that (a) the derivatives of orders $\leqslant k$ and $k - 1$ respectively converge uniformly on S_1 while (b) on C_1 the derivatives of orders $\leqslant k + N(p - 2)$ converge. These approximations are possible, by Lemma III.

To each pair ϕ_{1s}, ψ_{1s} there corresponds a solution u_{2s} of the Cauchy problem with data on S_1. We define these solutions throughout V_2 by selecting analytic boundary values χ_s which satisfy the compatibility conditions relative to u_{2s}, on C_1, up to the order $k + N(p - 2)$. Thus the derivatives of χ_s of order $\leqslant k + N(p - 2)$ converge to the corresponding derivatives of f on C_1. Now Lemma II shows that u_{2s} is defined and of class $R_{k+N(p-2)}$ in V_2.

To extend these solutions to V_3 and beyond, we approximate u_{2s} and $\partial u_{2s}/\partial t$ on S_2 by sequences ϕ_{3sr} and ψ_{3sr} of polynomials. By Lemma III these approximations can be made uniform for derivatives of order $\leqslant k + N(p - 2)$. Again we define solutions u_{3sr} in V_3, with Cauchy data ϕ_{3sr} and ψ_{3sr}, and boundary data χ_{3sr}, where χ_{3sr} is a polynomial satisfying the appropriate compatibility conditions of order $\leqslant k + N(p - 2)$. By Lemma II, the solutions u_{3sr} exist in V_3 and are of class $R_{k+N(p-2)}$ there. Also, by (3.7),

$$||u_{3sr}||_{k+N(p-2)} \leqslant K, \tag{4.2}$$

where K is independent of r and s. By Lemma IV, there exists for each s a subsequence u_{3sr_ρ} convergent to a limit u_{3s} of class $R_{k+N(p-3)}$ in V_3; thus u_{3s} also satisfies the differential equation and the estimate (4.2). We now approximate to values of u_{3s} and $\partial u_{3s}/\partial t$ on S_3 in order to define Cauchy data for a sequence of solutions u_{4sr} of class $R_{k+N(p-3)}$ in V_4. The approximations to the boundary condition are again of class $C_{k+N(p-3)}$ and there exists a sequence of solutions u_{4s} of class $C_{k+N(p-4)}$ in V_4 which satisfy a uniform estimate of the type (4.2).

Proceeding in this way we define a sequence of solutions u_{js} of class $R_{k+N(p-j)}$ in V_j, all satisfying an estimate of the type (4.2). We now piece together the solutions u_{js}, for fixed s, to give solutions u_s defined in $V_2 + V_3 + \ldots + V_p$, which are of class R_k. By Lemma IV there exists a subsequence uniformly convergent to a limit u of class R_k in $V_2 + V_3 + \ldots + V_p$. Assuming that $k \geqslant N + 2$, this function u has continuous first and second derivatives, and satisfies the differential equation and the boundary condition. Also, by the manner of its construction, this solution merges with the original solution in V_1 to yield a solution U of class R_k in $V = V_1 + V_2 + \ldots + V_p$. This completes the proof that the solution can be extended to a domain of arbitrary extent.

5. The non-analytic case. In the differential equation (1.1) let all coefficients and f be of class R_{k-1} (where $k \geqslant N + 2$) in V_1 and let the data

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of the mixed problem be of class C^{k+N}. Let W (in Schauder's notation) be a domain which contains S and is contained in the region of dependence on S alone. We suppose that the differential equation is of class C^{k+N} in W. Finally, the conditions of compatibility up to order k inclusive shall be satisfied. We state the result in this case as follows.

THEOREM. *There exists a solution u of the given mixed problem, which is of class R_k in V, of class R_{k+N} and C^k in W, and which satisfies an estimate of the form* (3.7).

The proof involves approximation to the coefficients of the differential equation by polynomials. From Lemmas III and IV we see that the approximating coefficients a_s^{ik}, b_s^i, c_s and f_s $(s = 1, 2, \ldots)$, together with derivatives up to order $k - 1$ inclusive, can be chosen to converge to their respective limits (a) in V, in mean and (b) in W, uniformly. As in the preceding extension of the domain of §4, the data can be approximated by polynomials which retain the k compatibility conditions. According to §4, there exists a solution u_s of the approximate problem, with

$$||u_s||_k < K.$$

From Lemma III we infer the existence of a uniformly convergent subsequence tending to a limit $u \in R_k$. Since $k \geqslant N + 2$, u is C^2 and satisfies the differential equation and the boundary conditions. In fact u is of class R_{k+N} in W as follows from the theory of the Cauchy problem **(9)**. That the solution is unique follows from the estimate (3.7).

A boundary condition of the third kind (in potential theory) can be reduced to that treated here. If

$$\frac{\partial u}{\partial v} + hu = f,$$

where h and f are functions of position on T, then the reduction in **(2)** will apply.

With boundary value problems for hyperbolic equations there is an evident analogy with potential theory, and Hadamard **(4**, p. 248 ff**)** discusses these problems in that light. However, the case of a plane boundary treated by him is essentially easier than the general case since in effect it can be solved by the method of images. The result found in this paper has a greater generality than one would expect by this analogy, since the case of the oblique derivative, which in potential theory requires special methods, is included.

In conclusion we note that Schauder has also treated the quasi-linear and non-linear mixed problems with the values of u assigned on T, **(5, 10)**. His methods extend without difficulty to the boundary condition studied here. Indeed, in the quasi-linear case, the linear solution is used to define a functional transformation, and then with the help of the sharp estimate (3.9) it is shown that a fixed point of the transformation, and hence a solution, exists for sufficiently small domains. The non-linear problem is reduced by differentiation to a quasi-linear integro-differential system which can be solved under the same conditions as the quasi-linear hyperbolic equation. Schauder's proof of the integrability conditions for this system, which establish the existence of the solution for the non-linear equation, requires no modification in the present case.

6. Removal of the compatibility conditions. The preceding result has been stated under conditions similar to those in **(5)** with the boundary condition of the first kind. In both of these theorems the compatibility conditions of order up to k are somewhat inappropriate in view of the theory of discontinuities of derivatives of solutions of hyperbolic equations. To remove this limitation, we shall need to strengthen the differentiability conditions.

Consider, therefore, the first boundary condition when $q + 1$ $(0 \leqslant q < k)$ compatibility conditions hold. We shall actually treat the case $q = 0$ since the continuity across G_0 of the derivatives up to order q is easily established later in the appropriate cases. Thus, taking the Cauchy data to vanish and considering the homogeneous differential equation, we assume only that $f(x^i)$ vanishes to the first order on C_0. Let the differential equation and data be of class C^{2k}, and let us reduce this problem to that treated in **(5)** by setting

$$u = u_1 + v.$$

We shall arrange that $L(v)$ be C^k everywhere and that u shall satisfy a boundary condition on T which is compatible of order k.

More precisely, we set $v \equiv 0$ in the Cauchy domain between S and G, and require that $L(v)$ should vanish to the order k as G is approached from above. The function v itself shall be continuous, shall vanish on G and shall satisfy on T the boundary condition

$$v = \sum_{n=1}^{k} f_n t^n,$$

where the f_n are the coefficients of f in a Taylor series expansion in powers of t. For u_1 we now have

$$u_1 = \bar{f} \equiv f - \sum_{n=1}^{k} f_n t^n \qquad \text{on } T.$$

Since we have assumed $L(u) = 0$ in the reduced form of the boundary value problem, the first compatibility conditions for u_1 will be the vanishing of the appropriate derivatives of $\bar{f}$. Thus the problem for u_1 is of the above type, since we have in effect taken $u = 0$ in the region W of the theorem. This shows that the problem is in this case reduced to finding v.

For this purpose we note that all functions and coefficients can be expanded in a Taylor series of powers of t (where $t = 0$ is the equation of G) up to terms of order t^k and with a remainder of this order in t. The coefficients of terms containing t^r are derivatives of order r, and so are C^{2k-r}. Hence all such coefficients are C^k. Following Hadamard (**4**, pp. 78–79), we construct the first k terms of the series in powers of t, of the problem

$$L(w) = 0,$$

with $w = 0$ on G and $w = f$ on T.

By the manner of its construction, the function

$$v = \sum_{n=1}^{k} w_n t^n$$

satisfies

$$v = \sum_{n=1}^{k} f_n t^n \qquad \text{on } F,$$

and

$$L(v) = t^{k+1} r$$

where r is a C^k remainder term, in the region between G and T. Thus $L(v)$ has continuous derivatives up to order k across G and the reduction is established.

If the first $q + 1$ compatibility conditions hold, then it is easy to show that u is C^q across G, $q < k$, by considering the discontinuities of successive derivatives across G and noting that since they vanish on C they must vanish along all bicharacteristics issuing from C. We may now state the existence theorem of **(5)** with this modification.

Let the differential equation and boundary datum function f be C^{2k}, $k \geqslant N + 2$, and let the first $q + 1$ compatibility conditions hold. Then there exists a unique solution of $L(u) = 0$ in V with given Cauchy data and with $u = f$ on T. The solution is of class R_k in V except that if $q < k - N$ it is of class C^q across G.

The domain of this solution is, however, restricted to the domain wherein v has been defined and so does not include any multiple points of the characteristic surface G.

A similar reduction for the boundary condition of the second kind, considered in this paper, is possible. Here, however, it is not necessary that any compatibility condition should hold. We calculate v as the first k terms of the analytic series expansion in §2, and proceed as above. The result may be stated as follows when q compatibility conditions hold.

Let the differential equation be C^{2k} and the boundary datum function be C^{2k-1}, $k \geqslant N + 2$. Let the first q compatibility conditions hold on C. Then there exists a unique solution of $L(u) = 0$ in V with given Cauchy data and with

$$\frac{\partial u}{\partial v} = f$$

on T. The solution is of class R_k in V, except that if $q < k - N$ it is of class C^q across G.

Again the domain is limited by multiple points or self-intersections of the characteristic surface G.

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University of Toronto