GENERATING FUNCTIONS FOR A CLASS OF ARITHMETIC FUNCTIONS

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1. Introduction. In this note the arithmetic functions $L(n)$ and $w(n)$ denote respectively the number and product of the distinct prime divisors of the integer $n > 1$, and $L(1) = 0$, $w(1) = 1$. An arithmetic function f is called multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. It is known ([1], [3], [4]) that every multiplicative function f satisfies the identity

$$(1.1) \qquad f(mn) = \sum_{\substack{a|m \\ b|n}} f(m/a)f(n/b)f'(ab)\,C(a, b),$$

where m and n are arbitrary positive integers, f' is the Dirichlet inverse of f defined by the relation $\Sigma_{d|n} f(d)f'(n/d) = [1/n]$ (here as usual $[x]$ is the greatest integer not exceeding x), and

$$C(m, n) = \begin{cases} (-1)^{L(n)}, & \text{if } w(m) = w(n) \\ 0, & \text{otherwise.} \end{cases}$$

We apply the identity (1.1) to derive some results on the generating function for a class of arithmetic functions closely allied to those previously obtained in [2] by one of the authors.

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2. Main results. An arithmetic function f is said to be unconditionally multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all positive integers m and n. The arithmetic integral of an arithmetic function f is the function h defined by $h(n) = \Sigma_{d|n} f(d)$.

Let us define

$$\epsilon(a, n) = \begin{cases} 1, & \text{if } a \mid n \\ 0, & \text{otherwise.} \end{cases}$$

THEOREM 1. Let $a > 1$ be a fixed integer with $p_1, p_2, \ldots, p_r$ as its distinct prime divisors. Let $g(n)$ be a positive valued unconditionally multiplicative function and $h(n)$ its arithmetic integral. Then

$$(2.1) \qquad \sum_{n=1}^{\infty} h(an)x^n = \sum_{n=1}^{\infty} H(a, n)\, g(n)x^n/(1-x^n)$$

where $H(a, n)$ is a periodic function of n with least period $w(a)$ and in fact

$$(2.2) \quad H(a, n) = h(a) - \sum_{p_i} h(a/p_i)\epsilon(p_i, n) + \sum_{p_i, p_j} h(a/p_ip_j)\epsilon(p_ip_j, n) - \ldots .$$

Proof. Since $g(n)$ is unconditionally multiplicative, it is easily proved that for any prime p, $h'(p^2) = g(p)$ and $h'(p^i) = 0$ for $i > 2$, where h' is the Dirichlet inverse of h. Hence from the identity (1.1) we obtain

$$h(an) = \sum_{\substack{d|a \\ d|n}} h(a/d)h(n/d)g(d)\mu(d)$$

$$= h(a)h(n) - \sum_{p_i} h(a/p_i)h(n/p_i)\epsilon(p_i, n)g(p_i)$$

$$+ \sum_{p_i, p_j} h(a/p_ip_j)h(n/p_ip_j)\epsilon(p_ip_j, n)g(p_ip_j) - \ldots .$$

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Thus

$$\sum_{n=1}^{\infty} h(an)x^n = h(a) \sum_{n=1}^{\infty} h(n)x^n - \sum_{p_i} h(a/p_i)g(p_i) \sum_{n=1}^{\infty} h(n)x^{np_i} + \ldots$$

$$= h(a) \sum_{n=1}^{\infty} g(n)x^n/(1-x^n)$$

$$- \sum_{p_i} h(a/p_i)g(p_i) \sum_{n=1}^{\infty} g(n)x^{np_i}/(1-x^{np_i}) + \ldots .$$

Remembering that g is unconditionally multiplicative, we have $g(p_i)g(n) = g(np_i)$, so

$$\sum_{n=1}^{\infty} h(an)x^n = h(a) \sum_{n=1}^{\infty} g(n)x^n/(1-x^n)$$

$$- \sum_{p_i} h(a/p_i) \sum_{n=1}^{\infty} g(n)\epsilon(p_i, n)x^n/(1-x^n) + \ldots$$

$$= \sum_{n=1}^{\infty} H(a, n)g(n)x^n/(1-x^n),$$

where $H(a, n)$ is given by (2.2).

Since $\epsilon(a, n)$ is a periodic function of n with a as a period, it follows that $H(a, n)$ has $w(a) = p_1 p_2 \ldots p_r$ as a period. That $w(a)$ is in fact the least period of $H(a, n)$ follows from Theorem 2 of [2].

Remark. A part of this theorem (that $H(a, n)$ is periodic with least period $w(a)$) was previously proved by one of us [2] by a different method. However, our theorem here gives an explicit form of $H(a, n)$. The function $g(n)$ need not necessarily be unconditionally multiplicative for $H(a, n)$ to be periodic in n with least period $w(a)$. That $g(n)$ can in fact belong to a wider class of functions is shown by

THEOREM 2. Let $g(n)$ be a positive valued multiplicative function and $h(n)$ its arithmetic integral. Let $H(a, n)$ be defined by

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$$\sum_{n=1}^{\infty} h(an)x^n = \sum_{n=1}^{\infty} H(a, n)g(n)x^n/(1-x^n),$$

where a is an arbitrary integer > 1. Then $H(a, n)$, as a function of n, is periodic with least period $w(a)$ if and only if the function $F(n) \equiv g(nW)/g(W)$, $W = w(n)$, is unconditionally multiplicative.

<u>Proof.</u> Applying Theorem 1 of [2] we have $H(a, n) = h(r)g(sn)/g(n)$, where r is the largest divisor of a which is prime to n and $a = rs$. For a given integer a it is clear that r and s are unaltered by replacing n by $n + w(a)$. Hence if for every a the function $H(a, n)$, as a function of n, has period $w(a)$, it follows by specializing a to be a prime power p^i, $i > 0$, and taking $n = p^j$, $j > 0$, that for all primes p and all $i, j > 0$, $g(p^{i+j})/g(p^j)$ is a function of p^i only and is independent of j. Thus

$$\begin{aligned} F(p^i) &= g(p^{i+1})/g(p) \\ &= [g(p^{i+1})/g(p^i)][g(p^i)/g(p^{i-1})]\ldots[g(p^2)/g(p)] \\ &= F(p)F(p)\ldots F(p) = (F(p))^i . \end{aligned}$$

This result together with the definition of $F(n)$ shows that $F(n)$ is unconditionally multiplicative, thus concluding the proof of the necessity of the condition.

We now proceed to establish the sufficiency. In view of the multiplicative property of $g(n)$ and the assumed properties of $F(n)$ we have, for any given $a > 1$ and with r and s as previously defined,

(2.3) $$H(a, n) = h(r)g(sn)/g(n) = h(r)F(s) .$$

Since r and s are unaltered if n is replaced by $n + w(a)$, it follows that $H(a, n)$ has $w(a)$ as a period.

To prove that $w(a)$ is the least period of $H(a, n)$ we proceed as in [2]. Let R be the least period, so that $H(a, n) = H(a, n+R)$ for all n. Taking $n = a$ and using (2.3), we get $h(1)F(a) = h(t)F(u)$, where t is the largest factor of a such

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that $(t, a+R) = 1$ and $a = tu$. Since $g(n)$ is positive and multiplicative, so is $h(n)$, and $h(1) = 1$ since $g(1) = 1$. Thus $h(t)F(u) = F(a) = F(ut) = F(u)F(t)$, giving $h(t) = F(t)$, so that

$$g(W)h(t) = g(W)F(t) = g(tW), \qquad W = w(t). \tag{2.4}$$

We assert that $t = 1$. For otherwise, if $t = p_1^{c_1} p_2^{c_2} \ldots p_q^{c_q}$ is the prime factor decomposition of $t > 1$, (2.4) gives

$$\prod_{i=1}^{q} g(p_i)[1 + g(p_i) + \ldots + g(p_i^{c_i})] = \prod_{i=1}^{q} g(p_i^{c_i+1}).$$

Since $g(n)$ is positive valued, this is clearly impossible, unless $c_1 = c_2 = \ldots = c_q = 0$. Thus $t = 1$ and hence every prime factor of a is a prime factor of R, proving that $w(a)$ is the least period of $H(a, n)$.

Remark. The class of multiplicative functions $g(n)$ for which Theorem 2 holds may be characterized as follows. Starting with an arbitrary unconditionally multiplicative function $F(n)$, we define $g(p)$ for each prime p in an arbitrary manner, and then define for each $i > 1$, $g(p^i) = g(p)(F(p))^{i-1}$. The particular choice $g(p) = F(p)$ for each prime p makes $g(n)$ unconditionally multiplicative.

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