The Journal of Symbolic Logic
Volume 14, Number 1, March 1949

FOUNDATIONS OF MATHEMATICS FOR THE WORKING MATHEMATICIAN

N. BOURBAKI

I am very grateful to the Association for Symbolic Logic for inviting me to give this address—an honor which I am conscious of having done very little to deserve. My efforts during the last fifteen years (seconded by those of a number of younger collaborators, whose devoted help has meant more to me than I can adequately express) have been directed wholly towards a unified exposition of all the basic branches of mathematics, resting on as solid foundations as I could hope to provide. I have been working on this as a practical mathematician; in matters pertaining to pure logic, I must confess to being self-taught, and laboring under all the handicaps that this implies; and if, after no little self-questioning, I am speaking here today, I am doing so chiefly in order to enjoy the benefit of your professional advice and criticism, by which I hope to correct my views before I venture into print with them.

Whether mathematical thought is logical in its essence is a partly psychological and partly metaphysical question which I am quite incompetent to discuss. On the other hand, it has, I believe, become a truism, which few would venture to challenge, that logic is inseparable from a coherent exposition of the broad foundations on which mathematical science must rest.

On the true function of logic in this connection, however, there may still be room for some difference of opinion. The history of mathematics would perhaps throw not a little light on this subject; and a detailed study of the pattern according to which the feeling for rigor at times is emphasized and at times recedes, and the foundations of our science as a whole, and of its various branches, are at times scrutinized and then again neglected, would indeed offer a fascinating topic of investigation for a historian more concerned with ideas than with bare facts.

Such a study has not yet been attempted, and would perhaps be premature until some crucial periods in the history of our science be more thoroughly examined. And it is doubtful whether extant documents will ever enable us to draw valid conclusions about those decisive centuries in early Greek science when the need for proofs first reached the level of consciousness and a technique was slowly and laboriously worked out to satisfy that need.

Proofs, however, had to exist before the structure of a proof could be logically analyzed; and this analysis, carried out by Aristotle, and again and more deeply by the modern logicians, must have rested then, as it does now, on a large body of mathematical writings. In other words, logic, so far as we mathematicians are concerned, is no more and no less than the grammar of the language which we use, a language which had to exist before the grammar could be constructed.

Received December 31, 1948. An address delivered, by invitation of the Program Committee, at the eleventh meeting of the Association for Symbolic Logic, at Columbus, Ohio, on December 31, 1948.

https://doi.org/10.2307/2268971 Published online by Cambridge University Press

It serves little purpose to argue that logic exists outside mathematics. Whatever, outside mathematics, is reducible to pure logic is invariably found, on close inspection, to be nothing but a strictly mathematical scheme (mostly combinatorial), so devised as to apply to some concrete situation; one need only think e.g. of the classical syllogism (every man is mortal, Socrates is a man, etc.) to convince oneself of the truth of this statement. Outside mathematics, even in the physical sciences, there is no statement that does not have to be qualified by the knowledge, common to the speaker and to his audience, of some physical or mental context. Logical, or (what I believe to be the same) mathematical reasoning is therefore only possible through a process of abstraction, by the construction of a mathematical model. Every such step involves, in other words, an application of mathematics to something of a different nature. Why do such applications ever succeed? Why is a certain amount of logical reasoning occasionally helpful in practical life? Why have some of the most intricate theories in mathematics become an indispensable tool to the modern physicist, to the engineer, and to the manufacturer of atom-bombs? Fortunately for us, the mathematician does not feel called upon to answer such questions, nor should he be held responsible for such use or misuse of his work.

The primary task of the logician is thus the analysis of the body of existing mathematical texts, particularly of those which by common agreement are regarded as the most correct ones, or, as one formerly used to say, the most "rigorous." In this, he will do well to be guided more by what the mathematician does than by what he thinks, or, as it would be more accurate to say, by what he thinks he thinks; for the mental images which occur to the working mathematician are of psychological rather than logical interest. Also, if logic (as grammar) is to acquire a normative value, it must, with proper caution, allow the mathematician to say what he really wants to say, and not try to make him conform to some elaborate and useless ritual. After the logician has properly discharged such duties, and helped the mathematician to lay suitable foundations for his science, he may then set himself further objectives; and you know, better than I do myself, the brilliant successes that have been so achieved during the last fifty years. Some of these deal with various aspects of the problem of non-contradiction; this is a question which, since the earliest times, has played a prominent part in the relations between logic and mathematics; and it will not be amiss if I discuss it briefly, from the working mathematician's point of view.

Historically speaking, it is of course quite untrue that mathematics is free from contradiction; non-contradiction appears as a goal to be achieved, not as a God-given quality that has been granted us once for all. Since the earliest times, all critical revisions of the principles of mathematics as a whole, or of any branch in it, have almost invariably followed periods of uncertainty, where contradictions did appear and had to be resolved. It is hard to judge whether the need for systematic proofs in early Greek mathematics did or did not arise from the paradoxes connected with the discovery of incommensurable magnitudes, as has been suggested by several historians. But more modern examples, such as the development of the infinitesimal calculus, the theory of series, the theory of sets, all point to the same conclusion. Contradictions do occur;

https://doi.org/10.2307/2268971 Published online by Cambridge University Press

but they cannot be allowed to subsist if the distinction between true and false, proved and unproved is to keep its meaning. There is no sharply drawn line between those contradictions which occur in the daily work of every mathematician, beginner or master of his craft, as the result of more or less easily detected mistakes, and the major paradoxes which provide food for logical thought for decades and sometimes centuries. Absence of contradiction, in mathematics as a whole or in any given branch of it, thus appears as an empirical fact, rather than as a metaphysical principle. The more a given branch has been developed, the less likely it becomes that contradictions may be met with in its further development. At the same time, even in the best established branches of our science, everyone knows that an unskilful, or too skilful, use of the existing terminology and notations can lead to ambiguities and eventually to contradictions. I am not merely referring to those "abus de langage" without which no mathematical text would be readable; a cursory examination of many existing notations will show that few are altogether foolproof, in the sense that the ambiguities inherent in most of them cannot be removed without complicating them to the point of uselessness. As an example to the point, one can mention the use of parentheses and brackets, and the gentleman's agreements in virtue of which these may frequently be omitted.

What will be the working mathematician's attitude when confronted with such dilemmas? It need not, I believe, be other than strictly empirical. We cannot hope to prove that every definition, every symbol, every abbreviation that we introduce is free from potential ambiguities, that it does not bring about the possibility of a contradiction that might not otherwise have been present. Let the rules be so formulated, the definitions so laid out, that every contradiction may most easily be traced back to its cause, and the latter either removed or so surrounded by warning signs as to prevent serious trouble. This, to the mathematician, ought to be sufficient; and it is with this comparatively modest and limited objective in view that I have sought to lay the foundations for my mathematical treatise, in the manner which I shall presently describe.

As every one agrees today, the distinctive character of a mathematical text is that it can be formalized, i.e., translated into a certain kind of sign-language. The first thing I have to do is therefore to lay down the vocabulary and grammar of the sign-language I wish to use, in its pure form at first, and later with all the modifications which usage has taught us to be required. The choice between substantially equivalent sign-languages (i.e., such that unambiguous translation from one to the other is possible) is of course merely a matter of convenience; and in my choice I have been guided chiefly by mathematical rather than logical considerations. The signs I use are as follows: 1° *arguments* (or variables), which are arbitrary signs, usually letters from some alphabet, sometimes modified by subscripts, accents, etc.; 2° the *surrounding line* $\bigcirc$;[1] for the convenience of the printer, and, as experience soon indicates, for greater legibility, this is to be replaced at a very early stage by the usual parentheses, brackets, etc., thereby probably losing once for all the advantages of an entirely unambiguous and

[1] This will usually be of oblong shape, according to the length of the formula inside it.

https://doi.org/10.2307/2268971 Published online by Cambridge University Press

foolproof notation; 3° the *connecting signs*, viz., (a) *not*, *and*, *or*, and (b) the quantifiers $\forall$, $\exists$; 4° the *mathematical signs* $=$ ϵ, $|$; 5° *abbreviations*, to be introduced and defined one by one as they become needed.

Certain combinations of the above signs will be called *relations*, i.e., well-formed formulas;[2] any argument occurring in a relation will be said to be *free* or *bound*; certain relations will be called *true relations*; and I give an operational manual, enabling its users to write relations, and in particular true relations, and to distinguish between free and bound arguments. The manual is as follows:

(a) A relation is formed by writing one argument to the left and one to the right of the sign $=$, or of the sign ϵ, or one to the left, and two to the right of the sign $|$; all arguments are free.

(b) A relation is formed by copying an already written relation, surrounding it by a surrounding line, and writing *not* to the left of it; free and bound variables remain the same as in the former relation.

Using an obvious "short-hand," the first part of rule (b) can be reformulated more briefly as follows: if **R** is a relation, *not* (**R**) is a relation.[3] Here it should be understood that **R** is not an argument (there is no need, at this level, for the "propositional variables" which become indispensable at the higher level of metamathematical reasoning); and to say that it "is" a relation is of course inaccurate, but can, I believe, cause no misunderstanding. The sequence of signs *not* (**R**), with an **R** which may be replaced by a relation as explained above, will be called a *scheme*: it is such that it becomes a relation whenever the letter **R** is replaced by a relation. The same conventions will be used to state more briefly the following rules:

(c) If **R**, **S** are two relations, and no argument is free in one and bound in the other, then (**R**) *and* (**S**), (**R**) *or* (**S**), are relations; the free (bound) arguments in these are the arguments which are free (bound) in one at least of the relations **R**, **S**.

(d) A relation is formed by surrounding a relation **R** by a surrounding line, writing to the left of this any argument other than a bound one in **R**, and, to the left of this, one of the signs $\forall$, $\exists$; this will be expressed more briefly (but inaccurately) as follows: if **R** is a relation, and x an argument which is not bound in **R**, then $\forall x$(**R**), $\exists x$(**R**) are relations. The bound arguments in these are x and the bound arguments in **R**; all others are free.

(e) Whenever an abbreviation is introduced, a rule must be given, stating how it may be used in writing relations, and which arguments are free and which are bound in a relation which is so written.

In the usual manner (by a type of proof which may be described as "experimental induction"), one can show that, in a relation without abbreviations, bound variables are those which occur immediately after an $\forall$ or $\exists$, all others being free.

---

[2]My attention has been drawn to the fact that American logicians use the word "relation" with another meaning. I shall, however, go on using it here in the sense to which I am accustomed, and which is in agreement with French usage.

[3]For typographical reasons, parentheses had to be substituted for surrounding lines wherever these occurred in the present address; thus, "*not* (R)" takes the place of "*not* Ⓡ".

https://doi.org/10.2307/2268971 Published online by Cambridge University Press

From now on, surrounding lines will be replaced by parentheses, these being omitted whenever the meaning is clear without them. By a *scheme* is understood a design wherein certain letters **R**, **S**, etc., occur, and which becomes a relation, according to the above rules, whenever **R**, **S**, etc., are replaced by relations, possibly with some (explicitly stated) restrictions as to the arguments, free and bound, in these relations. Such a scheme must always be a combination of the "fundamental schemes" occurring in the rules (b), (c), (d), and (e). For example,

$$\exists x((\mathit{not}(\forall y(\mathbf{R}))) \ \mathit{or} \ (\mathbf{S}))$$

where y must not be bound in **R**, nor free in **S**; no argument, other than y, can be free in **R** and bound in **S**, or bound in **R** and free in **S**; and x must not be bound either in **R** or in **S**.

Furthermore, an operation is now introduced, the *replacement* of an argument by another in a relation. If **R** is a relation, the replacement of x by y is permissible: (a) if x and y do not both occur in **R**; (b) if x and y are both free in **R**; and it is performed by writing y, in **R**, wherever x was previously written; this does not change **R** if x does not occur in it. It will be convenient, in what follows, to write, e.g., $\mathrm{R}\{x, y, z\}$ for a relation where x, y, z occur either as free arguments or not at all, and $\mathrm{R}\{x, y, y\}$ for the result of the replacement of z by y in this.

Now the rules of inference can be formulated; I have found it convenient to do this in two stages, although the separation between these is somewhat artificial and arbitrary. First I introduce the concept of *synonymous* relations, defined by the following rules (where the restrictions as to arguments in the schemes have been left out):

(s 1–2) Synonymy is "reflexive," "symmetric," and "transitive."

(s 3) If, in any one of the fundamental schemes, a relation **R** is replaced by a synonymous relation **R**$'$, the new relation is synonymous to the earlier one.

(s 4) *not* (*not* **R**) is synonymous to **R**.

(s 5) *not* (**R** *and* **S**) is synonymous to (*not* **R**) *or* (*not* **S**).

(s 6) *not* (($\forall x$(**R**)) is synonymous to $\exists x$(*not* **R**).

(s 7, 9, 10) Commutativity, associativity, distributivity of "*and*" and "*or*".

(s 8) Commutativity of $\forall x$ and $\forall y$, and of $\exists x$ and $\exists y$.

(s 11–12) If x does not occur in **R**, $\forall x$(**R** *and* **S**) is synonymous to **R** *and* ($\forall x$(**S**)); the same holds for $\exists$; the same holds for "*or*".

(s 13) If x is bound in **R**, the replacement of x by an argument y, not occurring in **R**, gives a relation synonymous to **R**.

(s 14) Whenever an abbreviation is introduced, a rule (its "definition") must be given, indicating the way of deriving, from a relation containing it, a synonymous relation written without it.

The abbreviations $\rightarrow$, $\leftrightarrow$ being introduced with their usual meaning, the rules are now formulated for *true* relations:

(v 1) **R** *or not* **R** is true.

(d 1) Every relation which is synonymous to a true relation is true.

(d 2) If **R** and **S** are both true, (**R** *and* **S**) is true.

https://doi.org/10.2307/2268971 Published online by Cambridge University Press

(d 3) If **R** is true, (**R** *or* **S**) is true.

(d 4) If **R**, and **R** $\rightarrow$ **S**, are true, **S** is true.

(d 5) If $\mathbf{R}\{x, y\}$ is a relation, and $\mathbf{R}\{y, y\}$ is true, $\exists x(\mathbf{R}\{x, y\})$ is true.

(d 6) If **R** is true, $\forall x(\mathbf{R})$ is true.

From the above rules, a large number of further rules can easily be derived; this will be omitted here; but, before proceeding to the list of mathematical axioms which I use, I must explain the meaning I give to "axioms," "proofs," and "theories," and a very convenient extension of the above rules to concrete mathematical situations.

If our logical system is to be the grammar of the mathematical language as it is actually used, it must take into account the fact that the truth of a relation is seldom understood in the absolute sense described above, but more usually in a relative sense, which depends upon the assumptions of the moment. Similarly, quantification mostly occurs in a relative sense, the quantified argument being assumed to belong to a given "type." Of course this does not require the introduction of any new logical concepts; but it suggests the use of certain abbreviations which I have found very convenient, and which are as follows.

By a *proof*, I understand a section of a mathematical text, beginning with some relations and schemes of relations, which are called the hypotheses of the proof; the only schemes which occur among the hypotheses of mathematical proofs are two schemes which I shall write later, and they are such that, after the empty spaces in them have been duly filled, the relations which are so obtained contain no free argument; on the other hand, the relations which occur among the hypotheses may contain free arguments.

If P is a proof, a relation **R** will be called P-true: (a) if it is a conjunction of relations occurring among the hypotheses of P, or derived from schemes occurring among these hypotheses by filling up the empty spaces in a permissible manner; (b) if it occurs in a true relation $\mathbf{A} \rightarrow \mathbf{R}$, where **A** is a conjunction of the kind described in (a).

Now, denote by (d′ 1–5) five rules, entirely similar to the rules (d 1–5) for the deduction of true relations, except that P-truth is to be substituted for (absolute) truth; it is easily seen that these rules are valid, as consequences of the earlier ones. As to (d 6), it remains valid in the following restricted form:

(d′ 6) If **R** is P-true, and if the argument x does not occur as a free argument in the hypotheses of P, $\forall x(\mathbf{R})$ is P-true.

Other (derived) rules of deduction also remain valid, some of them with restrictions similar to that occurring in (d′ 6), provided the abbreviations $\rightarrow$, $\leftrightarrow$ are replaced by P-$\rightarrow$, P-$\leftrightarrow$, with obvious meanings. By "abus de langage," the P in these abbreviations, and in "P-true," may be omitted whenever there can be no doubt about what is meant.

Suppose now that an argument u is free in one, and no more than one, of the hypotheses of P; call that hypothesis $\mathbf{H}\{u\}$. Then we introduce two abbreviations, viz. $\forall_{\mathbf{H}}$ and $\exists_{\mathbf{H}}$; these may be used in the same way as the quantifiers $\forall$ and $\exists$; moreover, $\forall_{\mathbf{H}} x(\mathbf{R})$ and $\exists_{\mathbf{H}} x(\mathbf{R})$ are to be synonymous with

https://doi.org/10.2307/2268971 Published online by Cambridge University Press

$\forall x(\mathbf{H}\{x\} \rightarrow \mathbf{R})$ and $\exists x(\mathbf{H}\{x\}$ *and* $\mathbf{R})$ respectively. These signs will be called the "typical" quantifiers, and an argument x which follows such a quantifier will be called a "typical" argument, of the "type" determined by **H**.

It will now be seen that (d′ 6) can be strengthened as follows:

(d″ 6) If **R** is P-true, and if u is a free argument in a hypothesis $\mathbf{H}\{u\}$ of P and in no other, $\forall_{\mathbf{H}} u(\mathbf{R})$ is P-true.

All rules of deduction can be extended in a similar manner; and so can the rules of synonymy, provided that synonymy is replaced by P-equivalence. A proof P then consists of a chain of P-true relations, so arranged that the P-truth of each is apparent from the P-truth of the preceding ones by the application of the above rules. A *theory* is a section of a mathematical text, consisting of a number of proofs which are grouped together for convenience, e.g. because they all have some hypotheses in common; the latter are called the axioms of the theory. If T is a theory, T-truth is to be interpreted, in relation to the axioms of T, exactly as P-truth in relation to the hypotheses of a proof P; and typical quantification will be used similarly. A theory T' is said to be an *extension* of a theory T, and T to be *antecedent* to T', if T' has all the axioms of T, and some more. A theory T is called *contradictory* whenever a relation **R** has been found such that "**R** *and not* **R**" is T-true. A favorite way of proving the T-truth of a relation **R** is to show that the theory T', with the axioms of T, and *not* **R** as additional axiom, is contradictory (*reductio ad absurdum*).

As every one knows, all mathematical theories can be considered as extensions of the general theory of sets, so that, in order to clarify my position as to the foundations of mathematics, it only remains for me to state the axioms which I use for that theory. These are the following:

$E(1)$ $\forall x(x = x)$.

$S(1)$ $\forall \cdots [(x = y) \rightarrow (\mathbf{R}\{x, y, x\} \rightarrow \mathbf{R}\{x, y, y\})]$,

where the sign $\forall \cdots$ means that the quantifier $\forall$ is to be applied to *all* free arguments in the relation to the right of it.

The theory with only the two above axioms (which is antecedent to the theory of sets) is called the theory of equality. In this theory, I speak of a relation **R** as being "functional in an argument x" if x is a free argument in **R**, and if the relation

$$[\exists x(\mathbf{R}\{x\})] \textit{ and } [(\mathbf{R}\{x\} \textit{ and } \mathbf{R}\{y\}) \rightarrow (x = y)]$$

(where y is an argument which is not one of the arguments occurring in $\mathbf{R}\{x\}$) is true. Whenever **R** is functional in the argument x, we allow ourselves to introduce an abbreviation, called a "functional symbol," which may (and will) very much vary in shape and outward appearance, except that it usually will contain the arguments, other than x, which are free in **R**; if e.g. $f_{\mathbf{R}}$ is this symbol, then $x = f_{\mathbf{R}}$ is synonymous with $\mathbf{R}\{x\}$. This is of course nothing else than the ι-symbol, well-known to all logicians; and there is a theorem (for the proof of which I refer you to a paper by B. Rosser) on the possibility of "eliminating" this symbol, provided it is consistently denoted by the ι-notation; but I am not sure of what may be the value of such a proof for the working mathe-

https://doi.org/10.2307/2268971 Published online by Cambridge University Press

matician, who invariably uses far more imperfect but far more compact and practical notations, hardly any one of which is not liable to misinterpretations if certain unwritten rules are not observed in using it.

An extension of the theory of equality is the theory of pairs, where the sign $|$ is used. For this I set down the following axioms:

$E(2)$ $\forall x \forall y \exists z (z|xy)$

$E(3)$ $\forall x \forall y \forall z \forall t [(z|xy \textit{ and } t|xy) \rightarrow (z = t)]$

Hence the relation $z|xy$ is functional in z; I may therefore introduce a functional symbol, for which I choose (x, y), so that $z = (x, y)$ is synonymous with $z|xy$; this being done, the sign $|$ need never be written again. With this notation, the last axiom for the theory of pairs is as follows:

$E(4)$ $\forall x \forall x' \forall y \forall y' [((x, y) = (x', y')) \rightarrow (x = x' \textit{ and } y = y')]$

An extension of the theory of pairs is the theory of the ϵ-relation; in order to write its first axiom, I introduce the abbreviation $\subset$, so that the relation $x \subset y$ is synonymous with $\forall z (z \epsilon x \rightarrow z \epsilon y)$; then the first axiom is:

$E(5)$ $\forall x \forall y [(x \subset y \textit{ and } y \subset x) \rightarrow (x = y)]$.

Now I introduce typical quantifiers $\forall_{set}$, $\exists_{set}$, defined as explained above from the relation $\exists u (u \epsilon x)$; and I set down as axiom the following scheme:

$S(2)$ $\forall \cdots \forall_{set} E \exists X \forall x [(x \epsilon X) \leftrightarrow (x \epsilon E \textit{ and } \mathbf{R})]$

where X is an argument which does not occur in the relation $\mathbf{R}$.

The last axioms are:

$E(6)$ $\forall_{set} X \exists Y \forall Z [(Z \subset X) \leftrightarrow (Z \epsilon Y)]$

$E(7)$ $\forall_{set} X \forall_{set} Y \exists W \forall z [\exists x \exists y (x \epsilon X \textit{ and } y \epsilon Y \textit{ and } z = (x, y)) \leftrightarrow (z \epsilon W)]$

$E(8)$ Zermelo's axiom (in order to formulate it conveniently, it is preferable first to develop the theory with the preceding axioms).

Finally, I write down one axiom, with a free argument E, expressing that E is an infinite set; that is, that there exists a family of subsets of E, containing all sets of one element, and closed with respect to the union, to which E does not belong. On these foundations, I state that I can build up the whole of the mathematics of the present day; and, if there is anything original in my procedure, it lies solely in the fact that, instead of being content with such a statement, I proceed to prove it in the same way as Diogenes proved the existence of motion; and my proof will become more and more complete as my treatise grows.

UNIVERSITY OF NANCAGO

https://doi.org/10.2307/2268971 Published online by Cambridge University Press