# SUBALGEBRAS, DIRECT PRODUCTS AND ASSOCIATED LATTICES OF MV-ALGEBRAS

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**0. Introduction.** MV-algebras were introduced by C. C. Chang [**3**] in 1958 in order to provide an algebraic proof for the completeness theorem of the Lukasiewicz infinite valued propositional logic. In recent years the scope of applications of MV-algebras has been extended to lattice-ordered abelian groups, AF C*-algebras [**10**] and fuzzy set theory [**1**].

In [**1**] Belluce defined a functor $\gamma$ from MV-algebras to bounded distributive lattices; this functor was used in proving a representation theorem and was also used to show that the prime ideal space of an MV-algebra is homeomorphic to the prime ideal space of some bounded distributive lattice (both spaces endowed with the Stone topology). The problem of what the range of $\gamma$ is arises naturally. This question bears a relation to the question as to whether there is an "MV-space" in the same manner as there are Boolean spaces for Boolean algebras. Some "MV-spaces" are considered by N. G. Martinez [**9**].

A study of this problem was begun by Cignoli, Di Nola and Lettieri [**6**] where it was shown that certain elements in the range of $\gamma$ have a direct decomposition by linear elements in the same range. In [**2**] it is proved that some bounded countable chains are in the range of $\gamma$; moreover a least MV-algebra $A$ for which $\gamma(A)$ is a given bounded countable chain is presented.

In this paper we examine the action of $\gamma$ on direct products and subalgebras of MV-algebras. We operate in an extended category of pairs $(A, \mathscr{I})$ where $A$ is an MV-algebra and $\mathscr{I}$ a non-empty set of prime ideals. We show that this category has product and that $\gamma$ commutes with products. Under certain conditions we show that $\gamma$ preserves monomorphisms. We also give a necessary condition for a bounded distributive lattice to be in the range of $\gamma$, from which it follows that *not* every such lattice is in the range of $\gamma$. And finally, we show that $[0,1]$, as a lattice, is in the range of $\gamma$, as well as every complete bounded chain.

For the basic definition and properties of MV-algebras the reader is referred to [**1**], [**2**], [**3**], [**10**].

We consider an extended category $\mathscr{E}_{\mathrm{MV}}$ of MV-algebras. The objects of $\mathscr{E}_{\mathrm{MV}}$ are pairs $(A, \mathscr{I})$ where $A$ is an MV-algebra and $\mathscr{I}$ a non-empty subset of $\operatorname{Spec} A$, the set of prime ideals of $A$; a morphism $f:(A_1, \mathscr{I}_1) \to (A_2, \mathscr{I}_2)$ of $\mathscr{E}_{\mathrm{MV}}$ is an MV-homomorphism $f: A_1 \to A_2$ such that $f^{-1}(\mathscr{I}_2) \subseteq \mathscr{I}_1$, i.e. if $Q \in \mathscr{I}_2$ then $f^{-1}(Q) \in \mathscr{I}_1$.

From [**1**] we have a functor $\gamma: \mathscr{E}_{\mathrm{MV}} \to \mathscr{D}$ where $\mathscr{D}$ is the category of distributive lattices with $0, 1$. The lattice $\gamma(A, \mathscr{I})$ has as elements equivalence classes $[x]_{\mathscr{I}}$, $x \in A$, where $[x]_{\mathscr{I}} = [y]_{\mathscr{I}}$ if for all $P \in \mathscr{I}$, $x \in P$ iff $y \in P$. Then $[x]_{\mathscr{I}} + [y]_{\mathscr{I}} = [x+y]_{\mathscr{I}}$, $[x]_{\mathscr{I}}[y]_{\mathscr{I}} = [x \wedge y]_{\mathscr{I}}$ are well-defined operations and $\gamma(A, \mathscr{I})$ becomes a distributive lattice with $0 = [0]_{\mathscr{I}}$ and $1 = [1]_{\mathscr{I}}$. If $f:(A_1, \mathscr{I}_1) \to (A_2, \mathscr{I}_2)$ is an $\mathscr{E}_{\mathrm{MV}}$-morphism then $\gamma(f): \gamma(A_1, \mathscr{I}_1) \to \gamma(A_2, \mathscr{I}_2)$ is the lattice homomorphism, $\gamma(f)[x]_{\mathscr{I}_1} = [f(x)]_{\mathscr{I}_2}$. $\gamma(f)$ is an epimorphism if $f$ is. $\gamma(A, \mathscr{I})$ is denoted by $[A]_{\mathscr{I}}$, or, when $\mathscr{I} = \operatorname{Spec} A$, by $[A]$.

The main features about $\gamma(A, \mathscr{I})$ are that some of its structure is reflected in $A$ and its ideal structure parallels that of $A$; in particular $\operatorname{Spec} A$, $\operatorname{Spec}[A]$ are homeomorphic.

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**1.** In this first section we show the range of $\gamma$ is a proper subclass of $\mathscr{D}$.

Let $\mathscr{A}$ be an MV-algebra or a distributive lattice with 0, 1. We shall say that $\mathscr{A}$ has the *prime-extension property* (pep) if whenever $I \subseteq J$ are proper ideals of $\mathscr{A}$ and $I$ is prime then $J$ is prime. We shall show that $\gamma$ preserves pep.

First we recall that in [**11**, Chapter III, §6, Prop. 3] it is shown that the prime deductive systems containing a given prime deductive system form a chain; so we surely can say that:

THEOREM 1.1. *Every* MV-*algebra A has pep.*

THEOREM 1.2. [A] *has pep.*

THEOREM 1.3. *Let $\mathscr{L}_1, \mathscr{L}_2 \in \mathscr{D}$; let $g: \mathscr{L}_1 \to \mathscr{L}_2$ be an epimorphism. Then if $\mathscr{L}_1$ has pep so does $\mathscr{L}_2$.*

*Proof.* Let $L \subseteq S$ be proper ideals of $\mathscr{L}_2$ with $L$ prime. Then $g^{-1}(L) \subseteq g^{-1}(S)$ and both are proper ideals of $\mathscr{L}_1$. But $g^{-1}(L)$ is prime, hence $g^{-1}(S)$ is prime. Let $ab \in S$. $g$ is an epimorphism so there are $x, y \in \mathscr{L}_1$ with $g(x) = a$, $g(y) = b$. Hence $g(xy) \in S$ so $xy \in g^{-1}(S)$. Thus $x \in g^{-1}(S)$ or $y \in g^{-1}(S)$ and it follows that $a \in S$ or $b \in S$, so $S$ is prime. □

Now let $A$ be an MV-algebra, $\mathscr{I} \subseteq \operatorname{Spec} A$, $\mathscr{I} \neq \emptyset$. We clearly have an epimorphism $i: (A, \operatorname{Spec} A) \to (A, \mathscr{I})$ in $\mathscr{E}_{\mathrm{MV}}: i(x) = x$. Thus we have an epimorphism, $[A] \to [A]_{\mathscr{I}}$, $[x] \to [x]_{\mathscr{I}}$. By Theorems 1.2, 1.3 we have the following result.

THEOREM 1.4. *For every $(A, \mathscr{I}) \in \mathscr{E}_{\mathrm{MV}}$, $[A]_{\mathscr{I}}$ has pep.* □

Thus a necessary condition for a bounded distributive lattice to lie in the range of $\gamma$ is for it to have pep. Since there exist distributive lattices with 0, 1 that do not have pep, we have

THEOREM 1.5. *The image of $\gamma: \mathscr{E}_{\mathrm{MV}} \to \mathscr{D}$ is a proper subclass of $\mathscr{D}$.* □

A bounded distributive lattice is called a $P_m$-lattice if each prime ideal is contained in a unique maximal ideal [**8**].

THEOREM 1.6. *Let $\mathscr{L}$ be a bounded distributive lattice with pep. Then $\mathscr{L}$ is a $P_m$-lattice.*

*Proof.* Let $P$ a prime ideal of $\mathscr{L}$, $M_1, M_2$ maximal ideals and assume $P \subseteq M_1$, $P \subseteq M_2$. Suppose $M_1 \neq M_2$: Choose $a \in M_1 - M_2$, $b \in M_2 - M_1$. Then $ab \in M_1 \cap M_2$ is prime. Thus $a \in M_1 \cap M_2$ or $b \in M_1 \cap M_2$, both impossible since $a \notin M_2$ and $b \notin M_i$. Thus $M_1 = M_2$. □

By Corollary 1.3 of [**8**] the maximal ideal space of a pep lattice $\mathscr{L}$ is a Hausdorff space.

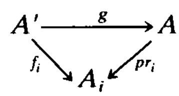
COROLLARY 1.1. *Any lattice in the range of $\gamma$ is a $P_m$-lattice, and so also has a Hausdorff maximal ideal space.* □

By [**1**, Theorems 15 and 20] we now have

COROLLARY 1.2. *The maximal ideal space of an* MV-*algebra A is Hausdorff.* □

**2.** Here we will show that $\mathscr{E}_{\mathrm{MV}}$ is closed order products and that $\gamma$ commutes with the taking of products. Thus we see that the image of $\gamma$ is closed under direct products.

In this section $I$ will be an index set and for each $i \in I$ we have an object $(A_i, \mathscr{I}_i) \in \mathscr{E}_{\mathrm{MV}}$. Let $A = \prod_{i \in I} A_i$. For each $i \in I$ we have projections $\mathrm{pr}_i : A \to A_i$. If $i_0 \in I$ and $Q \in \mathscr{I}_{i_0}$ then $\mathrm{pr}_{i_0}^{-1}(Q) = P$ is a prime ideal of $A$, call *the ideal of $A$ over $Q$*, which we will denote by $\mathrm{Ov}(Q)$. Let $\mathscr{I} = \left\{P : \text{for some } Q \in \bigcup_{i \in I} \mathscr{I}_i,\ P = \mathrm{Ov}(Q)\right\}$. Then $\mathscr{I} \neq \varnothing$ and $\mathscr{I} \subseteq \mathrm{Spec}\, A$. Clearly then the maps $\mathrm{pr}_i : (A, \mathscr{I}) \to (A_i, \mathscr{I}_i)$ are $\mathscr{E}_{\mathrm{MV}}$ morphism. Now let $(A', \mathscr{I}')$ be an $\mathscr{E}_{\mathrm{MV}}$ morphism and such that for each $i \in I$ we have an $\mathscr{E}_{\mathrm{MV}}$ morphism $f_i : (A', \mathscr{I}') \to (A_i, \mathscr{I}_i)$. Since $A$ is the direct product of the $A_i$ and each $f_i$ is an MV-homomorphism of $A'$ to $A_i$ we know there is a unqiue MV-homomorphism $g : A' \to A$ such that for each $i \in I$, the diagram




commutes, i.e. $\mathrm{pr}_i\, g = f_i$.

Now let $P \in \mathscr{I}$. Then for some $i \in I$ and $Q \in \mathscr{I}_i$, we have $P = \mathrm{Ov}(Q)$. Thus $g^{-1}(P) = g^{-1}(\mathrm{Ov}(Q)) = g^{-1}\,\mathrm{pr}_i^{-1}(Q) = (\mathrm{pr}_i\, g)^{-1}(Q) = f_i^{-1}(Q) \in \mathscr{I}'$ since $f_i$ is an $\mathscr{E}_{\mathrm{MV}}$ morphism. Hence $g : (A', \mathscr{I}') \to (A, \mathscr{I})$ is an $\mathscr{E}_{\mathrm{MV}}$-morphism and we see that $(A, \mathscr{I})$ is the product, $\prod_{i \in I} (A_i, \mathscr{I}_i)$. We shall show that $\gamma$ commutes with $\prod$, i.e.:

THEOREM 2.1.

$$\gamma\Big(\prod_{i \in I} (A_i, \mathscr{I}_i)\Big) \cong \prod_{i \in I} \gamma(A_i, \mathscr{I}_i).$$

In the above notation this is $[A]_{\mathscr{I}} \cong \prod_{i \in I} [A_i]_{\mathscr{I}_i}$. First we require.

LEMMA 2.1. *Let $(A, \mathscr{I})$, $(A_i, \mathscr{I}_i) \in \mathscr{E}_{\mathrm{MV}}$ with $\mathscr{I} = \left\{P \mid \text{for some } Q \in \bigcup_{i \in I} \mathscr{I}_i, P = \mathrm{Ov}(Q)\right\}$. Then, if $a, b \in A$, we have $[a]_{\mathscr{I}} = [b]_{\mathscr{I}}$ iff, for each $i \in I$, $[a_i]_{\mathscr{I}_i} = [b_i]_{\mathscr{I}_i}$.*

*Proof.* Suppose $[a]_{\mathscr{I}} = [b]_{\mathscr{I}}$. Let $i \in I$ and let $Q \in \mathscr{I}_i$. Assume $a_i \in Q$. Let $P = \mathrm{Ov}(Q)$. Then $P \in \mathscr{I}$ and $a \in P$. Thus $b \in P$, so $b_i \in Q$. By symmetry we have $[a_i]_{\mathscr{I}_i} = [b_i]_{\mathscr{I}_i}$. Conversely suppose that $[a_i]_{\mathscr{I}_i} = [b_i]_{\mathscr{I}_i}$ for each $i \in I$. Let $P \in \mathscr{I}$ and suppose $a \in P$. For some $i_0 \in I$ and some $Q \in \mathscr{I}_{i_0}$ we have $P = \mathrm{Ov}(Q)$. Thus $a_i \in Q$; hence $b_{i_0} \in q$ and so $b \in P$. By symmetry we conclude $[a]_{\mathscr{I}} = [b]_{\mathscr{I}}$. □

*Proof of Theorem* 2.1. Let $(A, \mathscr{I}) = \prod_{i \in I} (A_i, \mathscr{I}_i)$. Define $h : [A]_{\mathscr{I}} \to \prod_{i \in I} [A_i]_{\mathscr{I}_i}$ by $h([a]_{\mathscr{I}}) = \langle [a_i]_{\mathscr{I}_i} \rangle$ where $\langle [a_i]_{\mathscr{I}_i} \rangle$ is that element of $\prod_{i \in I} [A_i]_{\mathscr{I}_i}$ whose $i$th component is $[a_i]_{\mathscr{I}_i}$. By Lemma 2.1 $h$ is well defined and bijective. It is straight forward to verify that $h$ preserves the lattice operations; hence $h$ is an isomorphism. □

In the sequel, given $(A_i, \mathscr{I}_i)$, $i \in I$, with each $\mathscr{I}_i = \mathrm{spec}\, A_i$, $\mathscr{I}$ will be called the *over-family* of prime ideals of $A$ and will be denoted by Ov(A).

COROLLARY 2.1. *Given* MV-*algebras* $A_i$, $i \in I$, $A = \prod_{i \in I} A_i$, *we have* $[A]_{\mathrm{Ov}(A)} \cong \prod_{i \in I} [A_i]$.

We want now to examine the special case when $I$ is finite. First, two preliminaries.

PROPOSITION 2.1. *Let* $P$ *be a prime ideal in* $A = \prod_{i \in I} A_i$. *Then* $\mathrm{pr}_k(P) \neq A_k$ *for at most one* $k \in I$.

*Proof.* Let $i, k \in I$, $i \neq k$ and suppose $\mathrm{pr}_k(P) \neq A_k$, $\mathrm{pr}_i(P) \neq A_i$. Let $\delta_i \in A$ be such that the $i$th component of $\delta_i$ is 1 and the $j$th component, $j \neq i$, is 0. Similarly for $\delta_k$. Clearly $\delta_i \wedge \delta_k = 0$; thus $\delta_i \wedge \delta_k \in P$ so either $\delta_i \in P$ or $\delta_k \in P$. But then $1 \in \mathrm{pr}_i(P)$ or $1 \in \mathrm{pr}_k(P)$ which is impossible. □

PROPOSITION 2.2. *If* $I$ *is finite and* $P \subseteq A = \prod_{i \in I} A_i$ *is a prime ideal of* $A$, *then there is exactly one* $h \in I$ *with* $\mathrm{pr}_h(P) \neq A_h$.

*Proof.* We known there is at most one such $h$. Suppose that $\mathrm{pr}_i(P) = A_i$ for each $i \in I$. Choose $q_i \in P$ such that $\mathrm{pr}_i(q_i) = 1$. Then $q = \sum_{i \in I} q_i \in P$, and $q = 1$, absurd. □

THEOREM 2.2. *If* $I$ *is finite*, $A = \prod_{i \in I} A_i$, *then* $\mathrm{Ov}(A) = \mathrm{Spec}\, A$.

*Proof.* Let $P \in \mathrm{Spec}\, A$. By Proposition 2.2 there is a unique $h \in I$ with $\mathrm{pr}_h(P) \neq A_h$. Let $Q = \mathrm{pr}_h(P)$ and $Q' = \mathrm{pr}_h^{-1}(Q)$. Then $Q'$ is a proper ideal of $A$, $P \subseteq Q'$. Since $Q \in \mathrm{Spec}\, A_h$, $Q' \in \mathrm{Ov}(A)$. Let $a \in Q'$. For $i \neq h$, $\mathrm{pr}_i(P) = A_i$ so we can find a $u_i \in P$ with $\mathrm{pr}_i(u_i) = \mathrm{pr}_i(a)$. For $h$ we know that $\mathrm{pr}_h(a) \in Q$ so we can find a $u_h \in P$ with $\mathrm{pr}_h(u_h) = \mathrm{pr}_h(a)$. Let $u = \sum_{i \in I} \delta_i u_i$. Then $u \in P$. Now if $j \in I$ we have $\mathrm{pr}_j(u) = \sum_{i \in I} \mathrm{pr}_j(\delta_i)\, \mathrm{pr}_j(u_i) = \mathrm{pr}_j(u_i) = \mathrm{pr}_j(a)$. Thus $u = a$ so $a \in P$; that is $Q' = \mathrm{Ov}(Q) = P$ so $P \in \mathrm{Ov}(A)$. □

COROLLARY 2.2. *If* $I$ *is finite and* $\mathcal{I}_i = \mathrm{Spec}\, A_i$ *for each* $i \in I$, *then* $[A] \cong \prod_{i \in I} [A_i]$.

*Proof.* By Theorem 2.2 and Corollary 2.1. □

COROLLARY 2.3. *If* $A$ *is a finite* MV-*algebra*, *then* $\mathrm{Spec}\, A$ *has a base of clopen sets.*

*Proof.* By [**5**, Corollary 2.7], $A = \prod_{i \in I} A_i$, where $A_i$ is a linearly ordered MV-algebra for every $i \in I$. Then, by Corollary 2.2, $[A] \simeq \prod_{i \in I} [A_i] = \prod_{i \in I} \{0, 1\}$, i.e. $[A]$ is a boolean algebra. Thus $\mathrm{Spec}\, A$, which is homeomorphic to $\mathrm{Spec}[A]$, has a base of clopen sets. □

**3.** We would like now to examine the behavior of $\gamma$ with respect to subobjects, i.e. monomorphisms. It is easy to see in general that $\gamma$ does not preserve monomorphisms. For example if $\mathcal{I}$ is any proper subset of $\mathrm{Spec}\, A$ we have an $\mathscr{E}_{\mathrm{MV}}$ monomorphism $i: (A, \mathrm{Spec}\, A) \to (A, \mathcal{I})$ where $i$ is the identity map. But, in general, $[x] \to [x]_{\mathcal{I}}$ will not be one-one. For some subobjects however, $\gamma$ will preserve monicity. In particular we will show if $A$ is a subalgebra of $B$ then $[A]$ is isomorphic to a sublattice of $[B]$.

To begin, let $(A_1, \mathcal{I}_1)$, $(A_2, \mathcal{I}_2)$ be in $\mathscr{E}_{\mathrm{MV}}$. We call $(A_1, \mathcal{I}_1)$ a *full subobject* of $(A_2, \mathcal{I}_2)$ if there is a monomorphism $f: (A_1, \mathcal{I}_1) \to (A_2, \mathcal{I}_2)$ such that $f^{-1}(\mathcal{I}_2) = \mathcal{I}_1$, i.e. if $P \in \mathcal{I}_1$ there is a $Q \in \mathcal{I}_2$ with $f^{-1}(Q) = P$.

THEOREM 3.1. *If $(A_1, \mathcal{I}_1)$ is a full subobject of $(A_2, \mathcal{I}_2)$ then there is an injective homomorphism of $[A_1]_{\mathcal{I}_1}$ into $[A_2]_{\mathcal{I}_2}$.*

*Proof.* There is a monomorphism $f:(A_1, \mathcal{I}_1) \to (A_2, \mathcal{I}_2)$. Thus $f: A_1 \to A_2$ is an MV-monomorphism and $f^{-1}(\mathcal{I}_2) = \mathcal{I}_1$. We have a homomorphism $\gamma(f):[A_1]_{\mathcal{I}_1} \to [A_2]_{\mathcal{I}_2}$ where $\gamma(f)([x]_{\mathcal{I}_1}) = [f(x)]_{\mathcal{I}_2}$. Suppose $[f(x)]_{\mathcal{I}_2} = [f(y)]_{\mathcal{I}_2}$. Let $P \in \mathcal{I}_1$. Assume $x \in P$. Now $P = f^{-1}(Q)$ for some $Q \in \mathcal{I}_2$. So $f(x) \in Q$. Thus $f(y) \in Q$ and so $y \in f^{-1}(Q) = P$. By symmetry we have $[x]_{\mathcal{I}_1} = [f(x)]_{\mathcal{I}_2}$ and so $\gamma(f)$ is one-one. □

Now let $A \subseteq B$, $A$ a subalgebra of $B$. We have the inclusion map $i: A \to B$, $i(x) = x$. If $Q \in \operatorname{Spec} B$ then $i^{-1}(Q) = A \cap Q \in \operatorname{Spec} A$. Thus $i^{-1}(\operatorname{Spec} B) \subseteq \operatorname{Spec} A$ so $i:(A, \operatorname{Spec} A) \to (B, \operatorname{Spec} B)$ is a subobject of $(B, \operatorname{Spec} B)$.

THEOREM 3.2. *If $A, B$ are MV-algebras, $A$ a subalgebra of $B$, then*

$$\operatorname{Spec} A = \{A \cap Q \mid Q \in \operatorname{Spec} B\}.$$

*Proof.* Clearly $\{A \cap Q \mid Q \in \operatorname{Spec} B\} \subseteq \operatorname{Spec} A$. Let $P \in \operatorname{Spec} A$. Let $H$ be the ideal in $B$ generated by $P$. Let $G$ be the lattice-filter in $B$ generated by $A - P$. If $x \in H \cap G$ then there is a $p \in P$ with $x \le p$ and a $z \in A - P$ with $z \le x$. This implies $z \le p$, so $z \in P$ which is impossible. So $H \cap G = \emptyset$. By [7, Theorem 2.5], there is a prime ideal $Q \in \operatorname{Spec} B$ *with* $H \subseteq Q$, $Q \cap G = \emptyset$. $A = P \cup (A - P)$, so $A \cap Q = P \cap Q = P$ since $P \subseteq Q$ and $Q \cap (A - P) = \emptyset$. □

COROLLARY 3.1. *If $A$ is a subalgebra of $B$ then $(A, \operatorname{Spec} A)$ is a full subobject of $(B, \operatorname{Spec} B)$.*

*Proof.* Let $i: A \to B$ be the inclusion map. If $P \in \operatorname{Spec} A$ then by the above theorem, $P = i^{-1}(Q)$ for some $Q \in \operatorname{Spec} B$. □

COROLLARY 3.2. *If $A$ is a subalgebra of $B$ there is an injective homomorphism of $[A]$ into $[B]$.*

*Proof.* Clear from the above corollary and Theorem 3.1. □

**4.** Given that not every lattice in $\mathcal{D}$ is in the range of $\gamma$ it becomes pertinent to know which lattices are. We know that some countable chains lie in the range of $\gamma$. In this section we show the same is true for the chain $[0, 1]$, in fact for any complete bonded chain.

To this end let $\mathbb{N}$ be the set of positive integers, $\mathcal{F}$ a maximal filter in $2^{\mathbb{N}}$ that contains all cofinite subsets of $\mathbb{N}$. Let $A$ be the ultrapower $[0, 1]^{\mathbb{N}}/\mathcal{F}$. Then $A$ is a linearly ordered MV-algebra. For each $r \in [0, 1]$ let $\tau_r$ be the element of $A$ determined by the sequence $\langle r, r^2, r^3, \ldots \rangle \in [0, 1]^{\mathbb{N}}$. We then have

PROPOSITION 4.1. *Let $r, s \in [0, 1)$, $0 < r < s \le 1$. Let $P_r$, $P_s$ be the ideals of $A$ generated by $\tau_r$, $\tau_s$ respectively. Then $P_r \subseteq P_s$ and $\tau_s \notin P_r$.*

*Proof.* Since $0 < r < s$ we have $1 < s/r$. Let $h$ be any positive integer. Then there is a least integer $n_0 \in \mathbb{N}$ such that $h < (s/r)^n$ for all $n \ge n_0$. Hence $\{n \mid hr^n < s^n\} = \{n \mid n \ge n_0\} \in \mathcal{F}$. Thus $h\tau_r < \tau_s$ and so $\tau_s \notin P_r$. Clearly $\tau_r < \tau_s$ so $\tau_r \in P_s$; thus $P_r \subseteq P_s$. □

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Let $s \in [0,1)$. Set $P'_s = \bigcap_{s<r} P_r$. From the above we have $P_s \subseteq P'_s$. Since $A$ is linearly ordered all of its ideals are prime, if proper. Thus each $P'_s$, $s \in [0,1)$, is a prime ideal. Let $\mathcal{J} = \{P'_s \mid s \in [0,1)\}$. Then $\mathcal{J} \neq \emptyset$, and $\mathcal{J} \subseteq \operatorname{Spec} A$.

PROPOSITION 4.2. *For each $x \in A$ there is an $s \in [0,1]$ with $[x]_{\mathcal{J}} = [\tau_s]_{\mathcal{J}}$.*

*Proof.* If $x \notin P_r$ for any $r \in [0,1)$ then $x \notin P'_s$ for any $s \in [0,1)$. Thus $[x]_{\mathcal{J}} = 1 = [\tau_1]$. Otherwise let $s = \inf\{r \mid x \in P_r\}$. Consider $\tau_s$. Let $\tau_s \in P'_t \in \mathcal{J}$. If $t < s$ choose $r, t < r < s$. By Proposition 4.1 $\tau_s \notin P_r$, hence $\tau_s \notin P'_t$. Thus $s \leq t$. If $s = t$ then for $t < r \leq 1$ we have $x \in P_r$ and so $x \in \bigcap_{t<r} P_r = P'_t$. If $s < t$ choose $r, s < r < t$ with $x \in P_r$. Since $P_r \subseteq P_t \subset P'_t$ we have $x \in P'_t$. Conversely suppose $x \in P'_t \in \mathcal{J}$. Then for all $r \in [0,1)$, $t < r$, we have $x \in P_r$. Hence for all $r \in [0,1)$, $t < r$, we have $s \leq r$, so $\tau_s \in P_r$. Thus $\tau_s \in \bigcap_{t<r} P_r = P'_t$. Hence $[x]_{\mathcal{J}} = [\tau_s]_{\mathcal{J}}$. □

From the above proposition we see that $[A]_{\mathcal{J}} = \{[\tau_s]_{\mathcal{J}} \mid s \in [0,1]\}$. Since it is evident that $[\tau_s] \leftrightarrow s$ is an order preserving bijection we obtain

THEOREM 4.3. *There is a linearly ordered* MV*-algebra $A$ and non-empty subset $\mathcal{J} \subseteq \operatorname{Spec} A$ such that $[A]_{\mathcal{J}} \cong [0,1]$.*

To extend the above result to any complete bounded chain we set some premises.

Let $\mathcal{L}$ be a first order language for the theory of MV-algebras. Extend $\mathcal{L}$ to $\mathcal{L}^+$ by adding constant symbols, $c_r$, one for each $r \in \mathcal{C}$ where $\mathcal{C}$ is a given complete bounded chain. Let $\Delta_1$ be the first order axioms for linearly ordered MV-algebras. Let $\Delta_2 = \{nc_r < c_s \mid n = 1, 2 \ldots ; r, s \in \mathcal{C}, r < s\}$. Let $\Delta = \Delta_1 \cup \Delta_2$. We now have

PROPOSITION 4.3. *Every finite subset $\Delta'$ of $\Delta$ has a model.*

*Proof.* Let $c_{r_1}, c_{r_2}, \ldots, c_{r_n}$ be the constant symbols occurring in the formulas of $\Delta'$ (we can suppose $r_n < r_{n-1} \ldots < r_1$).

Let $E_n$ be a subalgebra of a proper ultrapower $[0,1]^*$ of $[0,1]$ generated by $\varepsilon, \varepsilon^2, \ldots \varepsilon^n$ where $\varepsilon$ is a non-zero infinitesimal. Then interpreting $c_{r_k}$ by $\varepsilon^k E_n$ becomes a model for $\Delta'$. □

Thus, by the compactness theorem, we have the following

COROLLARY 3.3. *$\Delta$ has a model.* □

THEOREM 4.4. *Let $\mathcal{C}$ be a complete bounded chain. Then there exist an* MV*-algebra $\mathcal{A}$ and a family $\mathcal{J}$ of prime ideals of $\mathcal{A}$ such that $[\mathcal{A}]_{\mathcal{J}}$ is isomorphic to $\mathcal{C}$.*

*Proof.* Let $\mathcal{A} = \langle A, +, ., -, 0, 1\{a_r : r \in \mathcal{C}\}\rangle$ be a model of $\Delta$, by Corollary 3.3. Moreover if $r < s$, then, for any positive $n, na_r < a_s$. Hence if we set $P_r = \langle a_r \rangle$, the principal ideal generated by $a_r$, we get prime ideals $P_r \subset P_s$ if and only if $r < s$, $r, s \in \mathcal{C}$. For each $x \in A$ let us define the set $B(x) = \{r \in \mathcal{C} \mid x \in P_r\}$, and set $m(x) = \inf B(x)$. Consider the map $f$ defined by

$$f : [x] \in [A] \rightarrow f([x]) = m(x).$$

Then:
(i) $f$ is well defined: indeed if $x \equiv y$ (Spec $A$) then $B(x) = B(y)$, which implies $m(x) = m(y)$.
(ii) $f$ is a homomorphism: indeed it is increasing because if $[x] < [y]$ then $B(y) \subseteq B(x)$ so $m(x) \le m(y)$.
(iii) $f$ is onto: let $r \in \mathscr{C}$, then we have that $r = \min B(a_r) = m(a_r)$. Thus, by [**2**, Theorem 3.1], there is a set $\mathscr{I} \subseteq \operatorname{Spec} A$ such that $[A]_{\mathscr{I}} \cong \mathscr{C}$. □

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