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A VERSION OF ROUCHÉ'S THEOREM FOR CONTINUOUS FUNCTIONS

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Abstract

In this paper we give a stronger form of Rouché's theorem for continuous functions.

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1. Introduction

In this paper we extend a version of Rouché's theorem for analytic functions discovered by Irvin Glicksberg (see [1, pages 125 and 126] and [4]) to continuous functions (see Theorem 3.1). Our version of Rouché's theorem is stronger than the versions given in [5, page 48] and [6]. At the end of the paper we give an application of our main result to a harmonic polynomial.

2. A short account of the degree theory in the plane

We start with a short account of the degree theory in the complex plane $\mathbb{C}$. A *curve* is a continuous function $\gamma\colon [a,b] \to \mathbb{C}$, where $[a,b] \subset \mathbb{R}$ is an interval, $-\infty < a < b < +\infty$. The range of a curve $\gamma\colon [a,b] \to \mathbb{C}$ we denote by γ^*, that is, $\gamma^* = \{\gamma(t) : t \in [a,b]\}$. Two curves $\gamma_1\colon [a,b] \to \mathbb{C}$ and $\gamma_2\colon [c,d] \to \mathbb{C}$ are equivalent (we write $\gamma_1 \sim \gamma_2$) if there exists a strictly increasing and continuous function $\tau\colon [a,b] \to [c,d]$ such that $\gamma_1 = \gamma_2 \circ \tau$. Of course, if $\gamma_1 \sim \gamma_2$, then $\gamma_1^* = \gamma_2^*$.

If $\gamma\colon [a,b] \to \mathbb{C}\backslash\{0\}$ is a curve, then there exists a continuous function $\alpha_\gamma\colon [a,b] \to \mathbb{C}$ such that $\gamma(t) = |\gamma(t)| \cdot e^{i\alpha_\gamma(t)}$ for all $t \in [a,b]$. Moreover, if $\widehat{\gamma}\colon [c,d] \to \mathbb{C}$ is a curve and $\gamma \sim \widehat{\gamma}$, then

$$\alpha_\gamma(b) - \alpha_\gamma(a) = \alpha_{\widehat{\gamma}}(d) - \alpha_{\widehat{\gamma}}(c).$$

For a rigorous proof, see [5, pages 27 and 28]. From this it follows that if $\gamma\colon [a,b] \to \mathbb{C}$ is a closed curve (that is, $\gamma(a) = \gamma(b)$) and $z_0 \in \mathbb{C}\backslash\{\gamma^*\}$, then there exists an integer $\mathrm{Ind}(\gamma, z_0)$ such that

$$\mathrm{Ind}(\gamma, z_0) = \frac{\alpha_{T_{z_0}\circ\gamma}(b) - \alpha_{T_{z_0}\circ\gamma}(a)}{2\pi},$$

where T_{z_0} is a translation defined as $T_{z_0}(z) := z - z_0$, $z \in \mathbb{C}$. We call $\operatorname{Ind}(\gamma, z_0)$ the *index of a closed curve γ with respect to z_0* (or the *winding number of γ about z_0*, see [5]). Obviously, if γ_1 is a closed curve, $\gamma_1 \sim \gamma_2$ and $z_0 \in \mathbb{C}\backslash\gamma_1^*$, then $\operatorname{Ind}(\gamma_1, z_0) = \operatorname{Ind}(\gamma_2, z_0)$.

Let $\gamma\colon [a, b] \to \mathbb{C}$ be a curve and let $f\colon D \to \mathbb{C}$ be a continuous function, where $\gamma^* \subset D \subset \mathbb{C}$. Then $f \circ \gamma$ is a curve. Assuming that $f \neq 0$ on γ^*, we define *the degree of f on γ* as the number

$$\deg(f, \gamma) := \frac{\alpha_{f\circ\gamma}(b) - \alpha_{f\circ\gamma}(a)}{2\pi}.$$

If γ is a closed curve, then $f \circ \gamma$ is a closed curve too and

$$\deg(f, \gamma) = \operatorname{Ind}(f \circ \gamma, 0).$$

Let us mention some properties of the degree. Let $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [b, c] \to \mathbb{C}$ be arbitrary curves, $a < b < c$, and let $\gamma_1(b) = \gamma_2(b)$. Consider the curve $\gamma_1 \oplus \gamma_2\colon [a, c] \to \mathbb{C}$ defined as $\gamma_1 \oplus \gamma_2|_{[a,b]} = \gamma_1$ and $\gamma_1 \oplus \gamma_2|_{[b,c]} = \gamma_2$. Then

$$\deg(f, \gamma_1 \oplus \gamma_2) = \deg(f, \gamma_1) + \deg(f, \gamma_2),$$

provided that f is a complex-valued continuous and nonzero function on $(\gamma_1 \oplus \gamma_2)^*$. In particular,

$$\deg(f, \gamma_1) = -\deg(f, \ominus\gamma_1),$$

where $\ominus\gamma_1(t) = \gamma_1(-t + a + b)$, $t \in [a, b]$ (the reverse of γ_1). If in addition $g\colon D \to \mathbb{C}$ is continuous and $g \neq 0$ on γ^*, then

$$\deg(f \cdot g, \gamma) = \deg(f, \gamma) + \deg(g, \gamma), \quad \deg\Big(\frac{f}{g}, \gamma\Big) = \deg(f, \gamma) - \deg(g, \gamma).$$

We recall the definition of zero cycle (see for example [5, page 36]). Let c_i be arbitrary integers and γ_i arbitrary curves for $i = 1, 2, \ldots, n$. Then the formal sum $\gamma = \sum_{i=1}^n (c_i \cdot \gamma_i)$ is called a *chain*. We define the *trace of a chain γ* as $\gamma^* = \bigcup_{i=1}^n \gamma_i^*$. If f is a complex-valued nonzero continuous function on γ^*, then the *degree* of f on the chain γ is defined as follows:

$$\deg(f, \gamma) := \sum_{i=1}^{n} (c_i \cdot \deg(f, \gamma_i)).$$

Let $D \subset \mathbb{C}$ be a domain and let γ and $\widetilde{\gamma}$ be chains, $\gamma^* \subset D$ and $\widetilde{\gamma}^* \subset D$. We say that γ *is homologous to $\widetilde{\gamma}$ relative to D* if $\deg(f, \gamma) = \deg(f, \widetilde{\gamma})$ for every continuous function $f\colon D \to \mathbb{C}\backslash\{0\}$. We say that a chain γ is a *cycle relative to D* if there exists a chain $\mathring{\gamma} = \sum_{i=1}^m (b_i \cdot \mathring{\gamma}_i)$ such that all $\mathring{\gamma}_i$ are closed curves, $\mathring{\gamma}^* \subset D$ and γ is homologous to $\mathring{\gamma}$ relative to D. If γ is a cycle (relative to D) and $z_0 \in \mathbb{C}\backslash\gamma^*$, then *the winding number of γ* about z_0 is defined as

$$\operatorname{Ind}(\gamma, z_0) := \deg(T_{z_0}, \gamma).$$

A cycle γ relative to D is called a *zero cycle in D* if $\operatorname{Ind}(\gamma, z_0) = 0$ for every $z_0 \in (\mathbb{C}\backslash D)$.

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THEOREM 2.1 (The degree principle, see [5, page 37]). *Let $D \subset \mathbb{C}$ be a domain and let $f\colon D \to \mathbb{C}\backslash\{0\}$ be a continuous function. Then*

$$\deg(f, \gamma) = 0$$

for every zero cycle γ in D.

For $z_0 \in \mathbb{C}$ and $\rho > 0$, denote $\mathbb{D}(z_0; \rho) := \{z \in \mathbb{C} : |z - z_0| < \rho\}$ and

$$C_{z_0,\rho}(\theta) := z_0 + \rho e^{i\theta}, \quad \theta \in [0, 2\pi].$$

Let $a \in \mathbb{C}$, $R > 0$ and let f be a complex-valued function continuous and nonzero in the punctured neighbourhood $\mathbb{D}(a; R)\backslash\{a\}$ of a point a. Then the point a is called the *isolated singularity of f* and the number

$$\text{mult}(f, a) := \deg(f, C_{a,r}), \quad \text{where } r \in (0, R),$$

is called the *multiplicity of f* at a (the definition does not depend on r). Let $a \in \mathbb{C}$ be an isolated singularity of f. We say that

$$a \text{ is a } \begin{cases} \textit{removable singularity of } f & \text{if } \exists \lim_{z\to a} f(z) \in \mathbb{C}\backslash\{0\}, \\ \textit{zero of } f & \text{if } \exists \lim_{z\to a} f(z) = 0, \\ \textit{pole of } f & \text{if } \exists \lim_{z\to a} f(z) = \infty, \\ \textit{essential singularity of } f & \text{in all other cases.} \end{cases}$$

THEOREM 2.2 (The topological argument principle, see [5, page 44]). *Suppose that f is a complex-valued continuous and nonzero function in a domain $D \subset \mathbb{C}$ except on a set $E := \{a_i \in D : a_i \neq a_j, i \neq j, i, j \in \mathbb{N}\}$ having no accumulation point in D. If γ is a zero cycle in D and $\gamma^* \subset D\backslash E$, then*

$$\deg(f, \gamma) = \sum_{i=1}^{\infty} (\text{Ind}(\gamma, a_i) \cdot \text{mult}(f, a_i)).$$

Of course, the argument principle has many applications in function theory (for an interesting application, see [2]).

Let D be a bounded Jordan domain in $\mathbb{C}$, that is, there exists a Jordan curve $J\colon [0, 1] \to \mathbb{C}$ such that $J^* = \text{fr}(D)$ ($\text{fr}(D)$ means the topological boundary of D) and $\text{Ind}(J, z_0) = 1$ for $z_0 \in D$. We denote the closure of D by $\text{cl}(D)$. Suppose that $f\colon \text{cl}(D) \to \mathbb{C}$ is a continuous function and the set E of zeros of f is finite, say $E = \{a_i \in \text{cl}(D) : i = 1, 2, \ldots, m\}$. We assume that $E \cap \text{fr}(D) = \varnothing$. Then we define the *number of zeros of f in D* as

$$Z_f(D) := \sum_{i=1}^{m} \text{mult}(f, a_i).$$

The following corollary is an immediate consequence of Theorem 2.2.

COROLLARY 2.3. *Let $D \subset \mathbb{C}$ be a bounded Jordan domain and let $J\colon [0, 1] \to C$ be a Jordan curve such that $J^* = \text{fr}(D)$ and $\text{Ind}(J, z_0) = 1$ for $z_0 \in D$. If $f\colon \text{cl}(D) \to \mathbb{C}$ is a continuous function with finitely many zeros in D, then*

$$Z_f(D) = \deg(f, J).$$

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3. A generalisation of Rouché's theorem

Now we formulate and prove a stronger version of Rouché's theorem for continuous functions compared to its classical version cited in [5, page 48].

THEOREM 3.1. *Let $D \subset \mathbb{C}$ be a bounded Jordan domain and let f and g be complex-valued continuous functions in* $\mathrm{cl}(D)$ *which have finitely many zeros in D. If*

$$|f(z) + g(z)| < |f(z)| + |g(z)| \quad \forall z \in \mathrm{fr}\, D, \tag{3.1}$$

then

$$Z_f(D) = Z_g(D).$$

PROOF. Let $J\colon [0, 1] \to \mathbb{C}$ be a Jordan curve such that $J^* = \mathrm{fr}(D)$ and $\mathrm{Ind}(J, z_0) = 1$ for some $z_0 \in D$. By (3.1), neither f nor g have a zero on J^*. Consider the function $F : \mathrm{fr}(D) \to \mathbb{C}$ defined by $F := (f|_{J^*})/(g|_{J^*})$. Then $F(J^*) \subset \mathbb{C}\backslash[0, +\infty)$, where $[0, +\infty) = \{w \in \mathbb{C} : \mathrm{Im}\, w = 0,\ \mathrm{Re}\, w \geq 0\}$. Indeed, if $F(z_0) = w_0 \geq 0$ for some $z_0 \in J^*$, then, by (3.1), we have $|w_0 + 1| < w_0 + 1$, which is a contradiction. Hence,

$$0 = \deg(F, J) = \deg\Big(\frac{f}{g}, J\Big) = \deg(f, J) - \deg(g, J).$$

Now the assertion of the theorem follows from Corollary 2.3. □

As an application of Theorem 3.1, we consider the following example.

EXAMPLE 3.2. Let us determine the number of zeros of the harmonic polynomial

$$p(z) := z^7 + z - 2 + 4\bar{z}^5, \quad z \in \mathbb{C},$$

in the unit disk $\mathbb{D}(0; 1)$.

First we would like to emphasise that the number of zeros of p is finite ($\leq 7^2 = 49$) because the coefficients at z^7 and $\bar{z}^5$ have different moduli (see [5, pages 50–52]). Set $q(z) := -4\bar{z}^5$, $z \in \mathbb{C}$. We show that the functions p and q satisfy the conditions of Theorem 3.1. Now we check that the sharp triangle inequality (3.1) holds for the functions p and q on $\mathrm{fr}(\mathbb{D}(0; 1))$. Let $z = e^{i\theta}$, $\theta \in \mathbb{R}$. Then

$$|p(z) + q(z)| = |z^7 + z - 2| \leq 4 \leq |p(z)| + |q(z)| = |p(z)| + 4.$$

Note that in this case equality in (3.1) holds if and only if

$$|z^7 + z - 2| = 4 \quad \text{and} \quad |p(z)| = 0.$$

But $|z^7 + z - 2|^2 = 16$ if and only if $-2\cos(7\theta) - 2\cos\theta + \cos(6\theta) = 5$. On the other hand, if $|p(z)| = 0$, then $\cos(7\theta) + 4\cos(5\theta) + \cos\theta - 2 = 0$. Hence, if in (3.1) we have an equality, then $8\cos(5\theta) + \cos(6\theta) = 9$. So, the only possibility is $z = 1$. But for $z = 1$ the inequality (3.1) holds. Hence, by Theorem 3.1,

$$Z_p(\mathbb{D}(0; 1)) = Z_q(\mathbb{D}(0; 1)) = 5.$$

The zeros of p in $\mathbb{D}(0; 1)$, identified by a computer program, are $z_{1,2} \approx -0.7415 \pm 0.53663 \cdot i$, $z_{3,4} \approx -0.2135 \pm 0.8831 \cdot i$ and $z_5 \approx 0.7686$.

REMARK 3.3. In [3, page 414], the authors comment on the classical Rouché's theorem for harmonic functions. However, it would be hard to use that version of Rouché's theorem for the function p considered in Example 3.2.

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