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INEQUALITIES INVOLVING THE INVERSES OF POSITIVE DEFINITE MATRICES[1]

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Notation. Let G be a permutation group of degree m. Let χ be an irreducible complex character of G. If $A = (a_{ij})$ is an m-square matrix, the generalised matrix function of A based on G and χ is defined by

$$d(A) = \sum_{g \in G} \chi(g) \prod_{t=1}^{m} a_{tg(t)}$$

For example if $G = S_m$, the full symmetric group, and χ is the alternating character, then d = determinant. If $G = S_m$ and χ is identically 1, then d = permanent.

Let n be a positive integer. Denote by Γ the set of all functions from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$. If $X = (x_{ij})$ is an n-square matrix and $\beta, \gamma \in \Gamma$, then $X[\beta|\gamma]$ is the m-square matrix whose i, j entry is $x_{\beta(i), \gamma(j)}$. Fix $\alpha \in \Gamma$. Let f be the function of the n-square nonsingular matrices defined by $f(X) = d(X^{-1}[\alpha|\alpha])$. Finally, let H_n denote the (convex) set of positive definite Hermitian n-square matrices.

Theorem. *Let λ and μ be nonnegative numbers which sum to 1. If $A, B \in H_n$, then*

$$f(\lambda A + \mu B) \leq f(A)^\lambda f(B)^\mu \tag{1}$$

This result was obtained in (**10**) when d = determinant. If a and b are nonnegative numbers, then

$$a^\lambda b^\mu \leq \lambda a + \mu b \tag{2}$$

(1). It follows that f is convex on H_n. If we specialise to the case $n = m$, α = identity, then (1) becomes

$$d((\lambda A + \mu B)^{-1}) \leq d(A^{-1})^\lambda \, d(B^{-1})^\mu. \tag{3}$$

Further specialisation to d = det yields

$$\det(\lambda A + \mu B) \geq (\det A)^\lambda (\det B)^\mu,$$

an inequality attributed to H. Bergström (**10**; **1**; **2**; **8**).

If X is m-square and p is an integer, $1 \leq p \leq m$, let X_p be the leading p-square principal submatrix of X.

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Corollary. *Let $A \in H_m$. Let p be a positive integer, $p \leq m$. Then*

$$d[(A_p)^{-1}] \leq d(A^{-1})_p. \tag{4}$$

Specialising to $d = \det$, we obtain

$$1 \leq \det(A^{-1})_p \det A_p,$$

an inequality of N. G. de Bruijn (**3**, Theorem 10.6), (**10**, Inequality (5.3)).

Proof. Let V be an n-dimensional complex inner product space. Let $\otimes^m V$ be the mth tensor power of V and denote by $v_1 \otimes \cdots \otimes v_m$ the decomposable (or pure) tensor product of the indicated vectors. The inner product in V induces an inner product in $\otimes^m V$ which has the following effect on decomposable tensors:

$$(v_1 \otimes \cdots \otimes v_m, w_1 \otimes \cdots \otimes w_m) = \prod_{t=1}^{m} (v_t, w_t) \tag{5}$$

For each $g \in S_m$, let $P(g)$ denote the action of g on $\otimes^m V$, i.e., $P(g^{-1})v_1 \otimes \cdots \otimes v_m = v_{g(1)} \otimes \cdots \otimes v_{g(m)}$ for all decomposable $v_1 \otimes \cdots \otimes v_m$. Then with respect to the inner product (5), $P(g)^* = P(g^{-1})$ (**4**). It follows that

$$T(G, \chi) = \frac{\chi(\mathrm{id})}{o(G)} \sum_{g \in G} \chi(g) P(g)$$

is Hermitian. By the generalised orthogonality relations (**11**, p. 16) and the fact that $P(g_1 g_2) = P(g_1)P(g_2)$, $T(G, \chi)$ is idempotent.

If $E = \{e_1, \ldots, e_n\}$ is an orthonormal basis of V, then $\{e_\gamma^\otimes = e_{\gamma(1)} \otimes \cdots \otimes e_{\gamma(m)} : \gamma \in \Gamma\}$ is a basis of $\otimes^m V$. It follows that $\{e_\gamma^* = T(G, \chi)\, e_\gamma^\otimes : \gamma \in \Gamma\}$ spans $V_\chi(G)$, the range of $T(G, \chi)$.

For each $\gamma \in \Gamma$, define $G_\gamma = \{g \in G : \gamma g = \gamma\}$. Compute

$$\begin{aligned}
\|e_\gamma^*\|^2 &= (T(G, \chi)e_\gamma^\otimes, T(G, \chi)e_\gamma^\otimes) \\
&= (e_\gamma^\otimes, T(G, \chi)e_\gamma^\otimes) \\
&= \frac{\chi(\mathrm{id})}{o(G)} \sum_{g \in G} \chi(g) \prod_{t=1}^{m} (e_{\gamma(t)}, e_{\gamma g(t)}) \\
&= \frac{\chi(\mathrm{id})}{o(G)} \sum_{g \in G_\gamma} \chi(g).
\end{aligned}$$

Let

$$\Omega = \left\{\gamma \in \Gamma : \sum_{g \in G_\gamma} \chi(g) \neq 0\right\}.$$

Then $e_\gamma^* \neq 0$ if and only if $\gamma \in \Omega$.

If S is a linear operator on V, let $K(S)$ denote the induced linear operator on $\otimes^m V$, i.e.,

$$K(S)(v_1 \otimes \cdots \otimes v_m) = (Sv_1) \otimes \cdots \otimes (Sv_m),$$

for all $v_1, \ldots, v_m \in V$. Suppose now that $A = (a_{ij})$ is an n by n matrix. Let S be the linear operator on V whose matrix representation with respect to E is A^T, i.e.,

$(Se_i, e_j) = a_{ij}$. Since $K(S)$ commutes with $T(G, \chi)$, we have

$$\begin{aligned}(K(S)e^*_\beta, e^*_\gamma) &= \frac{\chi(\mathrm{id})}{o(G)} \sum_{g \in G} \chi(g)(Se_{\beta(1)} \otimes \cdots \otimes Se_{\beta(m)}, e_{\gamma g(1)} \otimes \cdots \otimes e_{\gamma g(m)}) \\ &= \frac{\chi(\mathrm{id})}{o(G)} \sum_{g \in G} \chi(g) \prod_{t=1}^{m} (Se_{\beta(t)}, e_{\gamma g(t)}) \\ &= \frac{\chi(\mathrm{id})}{o(G)} d(A[\beta|\gamma]). \end{aligned} \tag{6}$$

It follows from (6) that $d(A[\beta|\gamma])$ is zero if either γ or β fails to lie in Ω. (In case $G = S_m$ and χ is the alternating character, Ω is the set of one-to-one functions.)

Lemma. (Generalised Cauchy-Binet Theorem). *Let A and B be n-square matrices. Let G be a subgroup of S_m and suppose χ is an irreducible character of G. If $\beta, \gamma \in \Gamma$, then*

$$d((AB)[\beta|\gamma]) = \frac{\chi(\mathrm{id})}{o(G)} \sum_{\omega \in \Omega} d(A[\beta|\omega])\, d(B[\omega|\gamma]),$$

and both sides are zero if either β or γ fails to lie in Ω.

Proof. Let S and T be the linear operators on V whose matrix representations with respect to E are, respectively, A^T and B^T. Then

$$\begin{aligned}\frac{\chi(\mathrm{id})}{o(G)} d((AB)[\beta|\gamma]) &= (K(TS)e^*_\beta, e^*_\gamma) \\ &= (K(S)e^*_\beta, K(T^*)e^*_\gamma) \\ &= \sum_{\omega \in \Gamma} (K(S)e^*_\beta, e^\otimes_\omega)(e^\otimes_\omega, K(T^*)e^*_\gamma) \end{aligned} \tag{7}$$

by Parseval's Identity. Since $T(G, \chi)$ is Hermitian and idempotent, $e^\otimes_\omega$ in (7) can be replaced with e^*_ω. But, $e^*_\omega = 0$ unless $\omega \in \Omega$. We proceed:

$$= \sum_{\omega \in \Omega} \frac{\chi(\mathrm{id})^2}{o(G)^2} d(A[\beta|\omega])\, \overline{d(B^*[\gamma|\omega])}$$

Since $B^*[\gamma|\omega] = B[\omega|\gamma]^*$ and $d(X^*) = \overline{d(X)}$, the proof is complete.

To complete the proof of the Theorem, we follow the technique employed by Muir: First observe that there exists a nonsingular matrix P such that $P^*AP = I$ and $P^*BP = C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$ with $c_i > 0$, $1 \leq i \leq n$. It follows that $(\lambda A + \mu B)^{-1} = P(\lambda I + \mu C)^{-1} P^*$. Let $H = \mathrm{diag}(h_1, h_2, \ldots, h_n)$, where $h_i = (\lambda + \mu c_i)^{-1}$.
Then

$$\begin{aligned} d((\lambda A + \mu B)^{-1}[\alpha|\alpha]) &= d((PHP^*)[\alpha|\alpha]) \\ &= \frac{\chi(\mathrm{id})^2}{o(G)^2} \sum_{\beta, \gamma \in \Omega} d(P[\alpha|\beta])\, d(H[\beta|\gamma])\, d(P^*[\gamma|\alpha]). \end{aligned}$$

Observe that, since H is diagonal,

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$$d(H[\beta|\gamma]) = \begin{cases} \sum_{\sigma\in G_\beta} \chi(\tau^{-1}\sigma)\prod_{t=1}^{m} h_{\beta(t)}, & \text{if } \gamma = \beta\tau \text{ for some } \tau \in G \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$d((\lambda A+\mu B)^{-1}[\alpha|\alpha]) = \frac{\chi(\text{id})^2}{o(G)^2}\sum_{\beta\in\Omega}\frac{1}{o(G_\beta)}\sum_{\tau\in G} d(P[\alpha|\beta])\,\overline{d(P[\alpha|\beta\tau])}\sum_{\sigma\in G_\beta}\chi(\tau^{-1}\sigma)\prod_{t=1}^{m} h_\beta(t)$$

$$= \sum_{\beta\in\Omega} u_{\alpha\beta}\prod_{t=1}^{m} h_{\beta(t)}, \quad \text{where} \tag{8}$$

$$u_{\alpha\beta} = \frac{\chi(\text{id})^2}{o(G)^2 o(G_\beta)}\sum_{\tau\in G} d(P[\alpha|\beta])\,\overline{d(P[\alpha|\beta\tau])}\sum_{\sigma\in G_\beta}\chi(\tau^{-1}\sigma)$$

$$= \frac{\chi(\text{id})^2}{o(G)^2 o(G_\beta)}\sum_{\sigma\in G_\beta}\sum_{\pi\in G}\chi(\pi)\, d(P[\alpha|\beta])\,\overline{d(P[\alpha|\beta\sigma\pi^{-1}])}$$

$$= \frac{\chi(\text{id})^2}{o(G)^2}\sum_{\pi\in G}\chi(\pi)\, d(P[\alpha|\beta])\,\overline{d(P[\alpha|\beta\pi^{-1}])}$$

because $\beta\sigma = \beta$ for all $\sigma \in G_\beta$. Now,

$$d(P[\alpha|\beta\pi^{-1}]) = \sum_{\tau\in G}\chi(\tau)\prod_{k=1}^{m} P_{\alpha(k)\beta\pi^{-1}\tau(k)}$$

$$= \sum_{\tau\in G}\chi(\pi\tau)\prod_{k=1}^{m} P_{\alpha(k)\beta\tau(k)}.$$

Hence,

$$u_{\alpha\beta} = \frac{\chi(\text{id})^2}{o(G)^2} d(P[\alpha|\beta])\sum_{\tau\in G}\left(\overline{\sum_{\pi\in G}\overline{\chi(\pi)}\chi(\pi\tau)}\right)\prod_{k=1}^{m}\bar{P}_{\alpha(k)\beta\tau(k)}$$

$$= \frac{\chi(\text{id})}{o(G)} d(P[\alpha|\beta])\sum_{\tau\in G}\overline{\chi(\tau)}\prod_{k=1}^{m}\bar{P}_{\alpha(k)\beta\tau(k)}$$

$$= \frac{\chi(\text{id})}{o(G)}|d(P[\alpha|\beta])|^2.$$

Continuing from (8) and substituting for H, we obtain

$$d((\lambda A+\mu B)^{-1}[\alpha|\alpha]) = \sum_{\beta\in\Omega}\left(u_{\alpha\beta}/\prod_{k=1}^{m}(\lambda+\mu c_{\beta(k)})\right) \tag{9}$$

$$\leq \sum_\beta u_{\alpha\beta}\left(\prod_k c_{\beta(k)}\right)^{-\mu} \quad \text{by} \quad (2)$$

$$= \sum_\beta u^{\lambda}_{\alpha\beta}\left(u_{\alpha\beta}/\prod_k c_{\beta(k)}\right)^{\mu}$$

$$\leq \left\{\sum_\beta u_{\alpha\beta}\right\}^{\lambda}\left\{\sum_\beta \left(u_{\alpha\beta}/\prod_k c_{\beta(k)}\right\}^{\mu} \tag{10}$$

by Hölder's Inequality. Successively choosing $\lambda = 1$, $\mu = 0$ and $\lambda = 0$, $\mu = 1$ in (9), we obtain

$$d(A^{-1}[\alpha|\alpha]) = \sum_\beta u_{\alpha\beta}, \quad \text{and} \tag{11}$$

$$d(B^{-1}[\alpha|\alpha]) = \sum_{\beta}\left(u_{\alpha\beta}/\prod_{k} c_{\beta(k)}\right). \tag{12}$$

A combination of (10)–(12) yields the result.

The proof of the corollary exactly parallels the development in (**10**, §5).

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