

REGULAR GRAPHS AND STABILITY

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Abstract

We show that a graph G is semi-stable at vertex v if and only if the set of vertices of G adjacent to v is fixed by the automorphism group of G_v , the subgraph of G obtained by deleting v and its incident edges. This result leads to a neat proof that regular graphs are semi-stable at each vertex. We then investigate stable regular graphs, concentrating mainly on stable vertex-transitive graphs. We conjecture that if G is a non-trivial vertex-transitive graph, then G is stable if and only if $\Gamma(G)$ contains a transposition, offering some evidence for its truth.

1. Introduction

Throughout this paper, all graphs considered are finite, undirected and are without loops and multiple edges. If G is such a graph, we denote by $V(G)$ its vertex set; unless otherwise specified, we shall assume $V(G) = \{v_i\}_{i=1}^p$. All basic graph-theoretical concepts used in this paper are defined in Behzad and Chartrand (1971), and we use the permutation group terminology of Wielandt (1964).

If u and v are adjacent vertices of G , we write $u \sim v$; otherwise we write $u \not\sim v$. If $u \in V(G)$, we define the *open neighbourhood* of u , $N_G(u)$, to be the set $\{v \in V(G) : u \sim v\}$. The *closed neighbourhood* of u , $\overline{N_G(u)}$, is $N_G(u) \cup \{u\}$. We denote by $\deg_G(u)$, the *degree* of u (that is, the cardinality of $N_G(u)$). For convenience, in these definitions we omit the subscript G if no confusion can arise. If $W \subseteq V(G)$, then G_W denotes the subgraph of G induced by $V(G) \setminus W$. For convenience, if W consists of a single vertex v , we write G_v rather than $G_{\{v\}}$. Denoting the automorphism group of G by $\Gamma(G)$, we define $\Gamma(G)_W$ to be the subgroup of $\Gamma(G)$ all of whose elements fix each element of $W \subseteq V(G)$; here, contrary to the usual convention, we consider $\Gamma(G)_W$ to act only on $V(G) \setminus W$. If $W = \{v\}$, we write $\Gamma(G)_v$ rather than $\Gamma(G)_{\{v\}}$.

We say that G is *semi-stable* at $v \in V(G)$ if $\Gamma(G_v) = \Gamma(G)_v$, (see Holton (1973a)). Thus G is semi-stable at v if and only if every automorphism of G_v ex-

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tends to an automorphism of G . We say that G is *stable* if there exists a sequence S of all the vertices of G which, by relabelling of $V(G)$ if necessary, we can assume to be $\{v_1, v_2, \dots, v_p\}$, such that $\Gamma(G_{S_j}) = \Gamma(G)_{S_j}$ for $j = 1, 2, \dots, p$, where $S_j = \{v_1, v_2, \dots, v_j\}$. We say that S is a *stabilising sequence* for G . The concept of stability is not to be confused with the similarly named, related, but different concept of Sheehan (1972).

Here we show that G is semi-stable at v if and only if $N_G(v)$ is fixed by $\Gamma(G_v)$, and use this result to generalise the theorem of Holton (1973a) that all regular graphs are semi-stable at each of their vertices. We then show that if G is a non-trivial stable graph, then $\Gamma(G)$ contains a transposition, this result having been used before, Holton (1973b), but not proved. This necessary condition for stability, which was shown in Holton (1973b) to be also sufficient if G is a tree, is shown by means of examples not to be sufficient if G is a regular graph. Consequently, we attempt to prove that it is sufficient if G is a vertex-transitive graph. This necessitates the characterisation of vertex-transitive graphs whose automorphism groups contain a transposition, which we accomplish. Unfortunately, we cannot resolve whether or not the possession by $\Gamma(G)$ of a transposition is sufficient for the vertex-transitive graph G to be stable, but we obtain several subsidiary results which suggest that this might be true, and indicate the directions in which research on the problem might be pursued.

2. A characterisation of semi-stability

In this section we present a characterisation of semi-stability. We use the following lemma of Holton (1973a).

LEMMA 1. *If $v \in V(G)$, then $\Gamma(G)_v \leq \Gamma(G_v)$.*

THEOREM 2. (*The characterisation*). *G is semi-stable at v if and only if $N_G(v)$ is fixed by $\Gamma(G)$.*

PROOF. If v is an isolated vertex of G , then trivially $\Gamma(G_v) \leq \Gamma(G)_v$, and, as $N_G(v) = \emptyset$, $N_G(v)$ is fixed by $\Gamma(G_v)$. Thus suppose henceforth that $N_G(v) \neq \emptyset$.

(i) Assume G is semi-stable at v . Let $g \in \Gamma(G)$, with $v^g = v$, and let $g' \in \Gamma(G)_v$ be the restriction of g to $V(G) \setminus \{v\}$. Now, if $u \sim v$ in G , it follows that $u^{g'} \sim v$ in G . Hence $N_G(v)$ is fixed by g' . We deduce that $N_G(v)$ is fixed by $\Gamma(G)_v$. However, as G is semi-stable at v , $\Gamma(G)_v = \Gamma(G_v)$, so that $N_G(v)$ is fixed by $\Gamma(G_v)$.

(ii) Assume $N_G(v)$ is fixed by $\Gamma(G_v)$. Let $h \in \Gamma(G_v)$, and let $h^* = h(v)$ be the corresponding permutation on $V(G)$ which fixes v . We show that $h^* \in \Gamma(G)$. Let u_1, u_2 be adjacent vertices of G_v . As $h \in \Gamma(G_v)$, it follows that $u_1^{h^*} \sim u_2^h$. Thus $u_1^{h^*} \sim u_2^{h^*}$. Now let $u \in N_G(v)$, so that $u \sim v$ in G . By hypothesis, $N_G(v)$ is fixed by $\Gamma(G_v)$, so that $u^h = w$ for some $w \in N_G(v)$. It follows that $u^h = w$. Now $v^{h^*} = v$, so that the edge (u, v) of G is mapped onto the edge (w, v) of G by h^* . We

deduce that h^* preserves adjacency in G . As G is a finite graph, it follows immediately that $h^* \in \Gamma(G)$. Consequently, $h \in \Gamma(G)_v$, so that $\Gamma(G_v) \leq \Gamma(G)_v$. By Lemma 1, $\Gamma(G)_v \leq \Gamma(G_v)$. We deduce that $\Gamma(G_v) = \Gamma(G)_v$, and that G is semi-stable at v .

The importance of this characterisation is that to decide whether or not G is semi-stable at v we need only decide whether or not $N_G(v)$ is fixed by $\Gamma(G_v)$. This in many cases is an easier task than deciding whether or not $\Gamma(G_v) = \Gamma(G)_v$. As an example of its power, we prove a generalisation of Holton's result (1973a), that all regular graphs are semi-stable at each of their vertices, that is *completely semi-stable*. We require a definition. The graph G is *congruence-regular* if there exists an integer $n > 1$ and a non-negative integer $m < n$ such that $\deg_G(v) \equiv m \pmod{n}$ for all $v \in V(G)$. We have:

THEOREM 3. *All congruence-regular graphs are completely semi-stable.*

PROOF. Let G be a congruence-regular graph, such that for $n > 1$ and $0 \leq m < n$, $\deg_G(u) \equiv m \pmod{n}$ for all $u \in V(G)$. Let $v \in V(G)$. In G_v , if $w \in N_G(v)$, then $\deg_{G_v}(w) \equiv (m - 1) \pmod{n}$, whereas if $w \notin N_G(v)$, then $\deg_{G_v}(w) \equiv m \pmod{n}$. It follows that $N_G(v)$ is fixed by $\Gamma(G_v)$, so that, by Theorem 2, G is semi-stable at v . The theorem follows.

Immediate is the following corollary, which includes Holton's result.

COROLLARY. (a) *All regular graphs are completely semi-stable.*

(b) *All Eulerian graphs are completely semi-stable.*

PROOF. (a) If G is regular of degree r , we can choose $n = r$, $m = 0$ and use Theorem 3.

(b) If G is Eulerian, then by Theorem 3.1 of Behzad and Chartrand (1971), we can choose $n = 2$, $m = 0$ and apply Theorem 3.

Theorem 3 suggests the following definition. A *semi-stable extension* of a graph G is a graph obtained from G by adding a new vertex v and the edges (v, v_i) , $i = 1, 2, \dots, n$, where the subset $\{v_i\}_{i=1}^n$ of $V(G)$ is fixed by $\Gamma(G)$. Theorem 3 can thus be interpreted as: G is semi-stable at v if and only if G is a semi-stable extension of G_n . We shall use the notion of a semi-stable extension in the next section.

3. Some aspects of stable regular graphs

In Holton (1973b), the following quite simple result was used, without proof. We now supply a short proof.

THEOREM 4. *If G is stable, then either G is K_1 or $\Gamma(G)$ contains a transposition.*

PROOF. K_1 is stable. Assume $|V(G)| = p > 1$ and let $S = \{v_1, v_2, \dots, v_p\}$ be

a stabilising sequence for G . Now $G_{S_{p-2}}$ is either K_2 or \bar{K}_2 , so that $(v_{p-1}v_p) \in \Gamma(G_{S_{p-2}})$. As G is stable, $\Gamma(G)_{S_{p-2}}$ contains $(v_{p-1}v_p)$. It follows that $\Gamma(G)$ contains the transposition $(v_1)(v_2)\cdots(v_{p-2})(v_{p-1}v_p)$.

Holton (1973b) showed that if G is a tree and $\Gamma(G)$ contains a transposition, then G is stable. McAvaney, Grant and Holton (1974) showed that a similar result holds if G is unicyclic. It is not true, however, that if G is regular and $\Gamma(G)$ contains a transposition then G is stable. For example, the graph G of Figure 1 is regular of degree 2, and its automorphism group contains the transposition $(v_1 v_2)$. However, G is $K_3 \cup C_5$, and so by Theorem 5 of Holton (1973) is stable if and only if both K_3 and C_5 are stable. However, $\Gamma(C_5)$ does not contain a transposition, so that C_5 is not stable. It follows that G is not stable. We can clearly construct several other examples of this type. By the above argument,

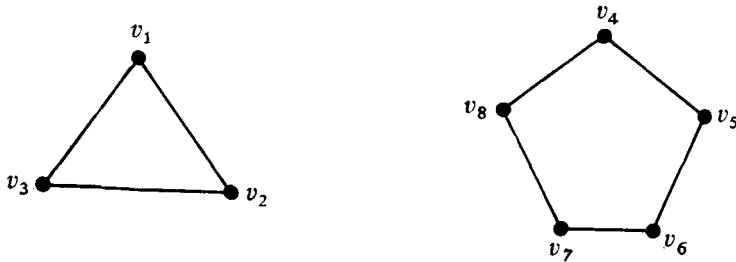


Fig. 1

if G is a stable regular graph, all components of G are stable regular graphs. Thus to find all stable regular graphs, it is sufficient to find all connected stable regular graphs. Moreover, as either a graph or its complement is connected, we need only find all stable regular graphs G which are *complement-connected* (that is both G and \bar{G} are connected).

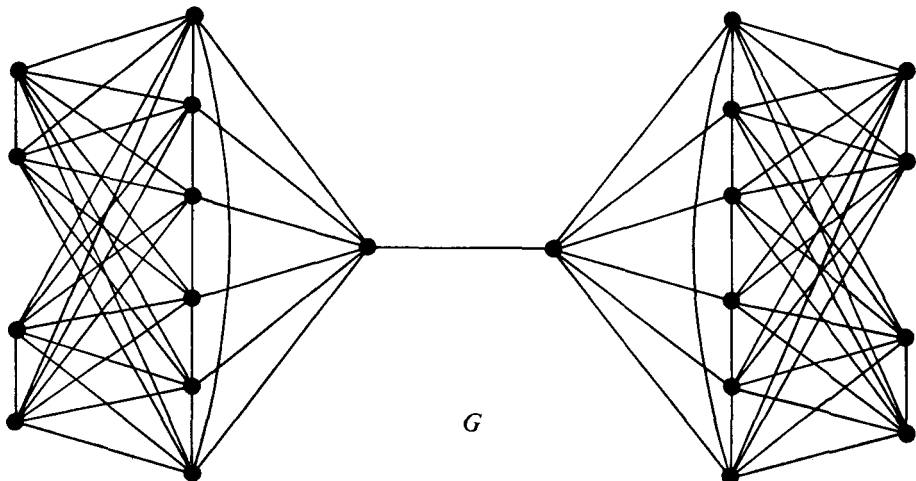


Fig. 2

In Figure 2 we provide an example of a complement-connected regular graph G , such that $\Gamma(G)$ contains a transposition, which is not stable. We can readily construct an infinite number of such graphs. It thus remains a difficult problem to characterise complement-connected stable regular graphs. Below we make a start on the investigation of this problem.

We say that G is *vertex-transitive* if $\Gamma(G)$ is a transitive group. Clearly if G is vertex-transitive, then G is regular. We now proceed to investigate the

CONJECTURE. *If G is vertex-transitive and $\Gamma(G)$ contains a transposition, then G is stable.*

Our first task is to characterise vertex-transitive graphs whose automorphism group contains a transposition. We need three preliminary lemmas, which we state without proof.

LEMMA 5. *Let $u, v \in V(G)$. Then $(uv) \in \Gamma(G)$ if and only if (i) in the case that $u \sim v$, $N_G(u) = N_G(v)$, (ii) in the case that $u \not\sim v$, $N_G(u) = N_G(v)$.*

LEMMA 6 (Wielandt (1964), Theorem 13.3). *If P is a primitive permutation group containing a transposition, then P is a symmetric group.*

LEMMA 7. *If $|V(G)| = p$, then $\Gamma(G) = \mathcal{S}_p$ if and only if G is K_p or \bar{K}_p . (Here \mathcal{S}_p is the symmetric group of degree p).*

Define F to be the family of all graphs G such that (i) G is vertex-transitive and (ii), if $(uv) \in \Gamma(G)$, then $u \sim v$. Recall that if G_1 and G_2 are graphs, then $G_1[G_2]$ denotes the composition of G_1 round G_2 (see Harary (1969), page 22). We have:

THEOREM 8. *If G is vertex-transitive and $\Gamma(G)$ contains a transposition, then G is of the form $H[K_n]$ or $H[\bar{K}_n]$ for some $n > 1$, where $H \in F$. Conversely, if G is such a graph, then G is a vertex-transitive graph whose automorphism group contains a transposition.*

PROOF. (i) Let G be vertex-transitive, and suppose $u, v \in V(G)$ are such that $(uv) \in \Gamma(G)$. By Lemma 6, if $\Gamma(G)$ is primitive, then $\Gamma(G)$ is a symmetric group, and so by Lemma 7, either G or \bar{G} is a complete graph, and the theorem holds with $H = K_1$. Now suppose that $\Gamma(G)$ is imprimitive, and let $H_1 \subset V(G)$ be a minimal non-trivial block of $\Gamma(G)$, of cardinality $n > 1$. Let H_1, H_2, \dots, H_m be the complete block system of $\Gamma(G)$ containing H_1 . Denote by $\langle H_i \rangle$ the subgraph of G induced by H_i . As G is vertex-transitive, the $\langle H_i \rangle$ are all isomorphic. Moreover, if Γ_i denotes the restriction of $\Gamma(G)$ to H_i , then Γ_i is transitive, is a subgroup of $\Gamma(\langle H_i \rangle)$, and, by the minimality of H_1 , is primitive. Now suppose $u \in H_j$ and $v \in H_k$, $k \neq j$. As $(uv) \in \Gamma(G)$, it follows from the fact that H_j and H_k are blocks that $|H_j| = |H_k| = n = 1$, a contradiction. Thus if $u \in H_j$,

then $v \in H_j$. We deduce that $(uv) \in \Gamma_j$. By Lemma 6, Γ_j is symmetric, so that $\Gamma_j = \Gamma(\langle H_j \rangle)$ and by Lemma 7, $\langle H_j \rangle$ is either K_n or \bar{K}_n . It follows that either $\langle H_i \rangle = K_n$ for all i or $\langle H_i \rangle = \bar{K}_n$ for all i .

Let a and c be adjacent vertices of G such that $a \in H_j$ and $c \in H_k$, $k \neq j$. Let $d \in H_j \setminus \{a\}$. As G is vertex-transitive and $\Gamma(G)$ contains a transposition, we may suppose using the above argument, that there is $b \in H_j \setminus \{a\}$ such that $(ab) \in \Gamma(G)$. If $b \neq d$, then $(bd) \in \Gamma_j$. Let g be a permutation in $\Gamma(G)$ whose restriction to Γ_j is (bd) . It follows that $g^{-1}(ab)g = (ad) \in \Gamma(G)$. Thus in all cases $(ad) \in \Gamma(G)$. We deduce by Lemma 5, that $d \sim c$. It follows that each vertex in H_j is adjacent to each vertex in H_k , so that G is in fact $H[K_n]$ or $H[\bar{K}_n]$ for some graph H . If H were not vertex-transitive, then for some p and q there would be no automorphism of G mapping H_p onto H_q , whence G would not be vertex-transitive. Moreover, if H were not in F , then for some r and s , for all $w \in H_r \cup H_s$, $\overline{N_G(w)}$ would be the same, so by Lemma 5, there would exist $x \in H_r$ and $y \in H_s$ such that $(xy) \in \Gamma(G)$, contradicting our previous conclusion. It follows that $H \in F$.

(ii) The converse result follows by noting that if u and v are vertices in the same copy of K_n in $G = H[K_n]$, then $\overline{N_G(u)} = \overline{N_G(v)}$, so that by Lemma 5, $(uv) \in \Gamma(G)$, and that a similar result holds for $H[\bar{K}_n]$.

Thus to prove our conjecture, we need only show that graphs of the type $H[K_n]$ and $H[\bar{K}_n]$ for $H \in F$ are stable. We now indicate that in fact we need not examine all such graphs.

First of all, from the definition of the composition of two graphs we can deduce the following two lemmas.

LEMMA 9. $\overline{G_1[G_2]} = \bar{G}_1[\bar{G}_2]$.

LEMMA 10. If $n = pq$, where p and q are integers greater than 1, then $H[K_n] = (H[K_p])[K_q]$.

In Grant (to appear), the following result was proved.

LEMMA 11. If G is stable, then $G[K_n]$ is stable for all $n > 1$.

Finally, Holton (1973) proved

LEMMA 12. G is stable if and only if \bar{G} is stable.

We deduce

THEOREM 13. If the conjecture is true for all graphs of the form $H[K_p]$ where p is prime and H is a vertex-transitive graph whose automorphism group contains no transpositions, then it is true a conjecture.

PROOF. Assume the conjecture holds for all graphs of the form stated. We shall show by induction that it holds for all graphs given by Theorem 8. It is

readily observed that such graphs with few vertices are stable. (We omit the details). Suppose that $G = H[K_n]$ or $H[\bar{K}_n]$, where $H \in \mathcal{F}$ and $\Gamma(H)$ contains a transposition. In the former case, we may suppose by the inductive hypothesis that H is stable, so that G is stable by Lemma 11. In the latter, \bar{G} is $\bar{H}[K_n]$. As $\Gamma(H) = \Gamma(\bar{H})$, it follows that \bar{H} is vertex-transitive and that $\Gamma(\bar{H})$ contains a transposition. As above, \bar{H} is stable, so that \bar{G} is stable. By Lemma 12, G is stable. Now suppose that $G = H[K_n]$ or $H[\bar{K}_n]$ where H is vertex-transitive, but $\Gamma(H)$ does not contain a transposition. In the latter case, $\bar{G} = \bar{H}[K_n]$, where \bar{H} is vertex-transitive, but $\Gamma(\bar{H})$ does not contain a transposition. By Lemma 12, it follows therefore, that we need once again only consider the former case, as the latter is contained in it. We may suppose that n is composite, and that p is the smallest prime dividing n . Let $q = n/p$. By Lemma 10, $G = (H[K_p])[K_q]$. $\Gamma(H[K_p])$ contains a transposition, and $H[K_p]$ is certainly vertex-transitive, so that by the inductive hypothesis, $H[K_p]$ is stable. It follows from Lemma 11 that G is stable. This completes the proof.

To conclude, we offer the following result which supports our conjecture. We construct its proof using semi-stable extensions, but note that it is just as easy to obtain a proof by directly constructing a stabilising sequence.

THEOREM 14. $C_m[K_n]$ is stable for all $m \geq 3, n \geq 2$.

PROOF. $C_3[K_2] = K_6$, and K_6 is stable. We thus assume $m + n > 5$. In Figure 3, below, the symbol in (a) denotes a copy of K_n and a dot (see vertex $v_{3,1}$) a single vertex. A line joining two symbols and/or dots indicates that each vertex represented by one symbol and/or dot is adjacent to each vertex represented by the other. We denote the copies of K_n by D_1, D_2, \dots, D_m and the vertices in D_i by $v_{i,1}, v_{i,2}, \dots, v_{i,n}$. As K_n is stable, we can clearly start with a copy D_1 of K_n (see (a)). By adding the isolated vertex $v_{3,1}$, as in (b), we produce, by definition, a semi-stable extension of K_n . Since the set of all vertices in this graph is fixed by its automorphism group, we can add vertex $v_{2,1}$, and join it to $v_{3,1}$ and to each vertex in D_1 , so that the graph in (c) is a semi-stable extension of that in (b). By a number of semi-stable extensions, we next build up copy D_2 of K_n , successively adding vertices $v_{2,2}, v_{2,3}, \dots, v_{2,n}$ (see (d) and (e)). After adding vertex $v_{4,1}$, joined to $v_{3,1}$ (see (f)), we proceed as in (d), (e) and (f) until we reach (g). We then build up copy D_{m-1} of K_n (see (h)). Finally, we add successively the vertices $v_{m,1}, v_{m,2}, \dots, v_{m,n}$, joining $v_{m,i}$ to all vertices in D_1 and D_{m-1} , and to $v_{m,j}$ for $j < i$ (see (i) and (j)). We obtain the graph $C_m[K_n]$, which is therefore stable. The stabilising sequence we construct is $\{v_{m,n}, v_{m,n-1}, \dots, v_{m,1}, v_{m-1,n}, \dots, v_{m-1,2}, v_{m-2,n}, \dots, v_{m-2,2}, v_{m-1,1}, v_{m-3,n}, \dots, v_{m-3,2}, v_{m-2,1}, v_{m-4,n}, \dots, v_{4,1}, v_{2,n}, \dots, v_{2,1}, v_{3,1}, v_{1,n}, \dots, v_{1,1}\}$.

COROLLARY. $P_k[K_n]$ is stable for all $k \geq 2, n \geq 2$.

PROOF. $P_2[K_2] = K_4$, and is stable. For $k + n > 4$, we note that at step (h) in Figure 3, putting $m - 1 = k$, we have the graph $P_k[K_n]$, which is therefore stable.

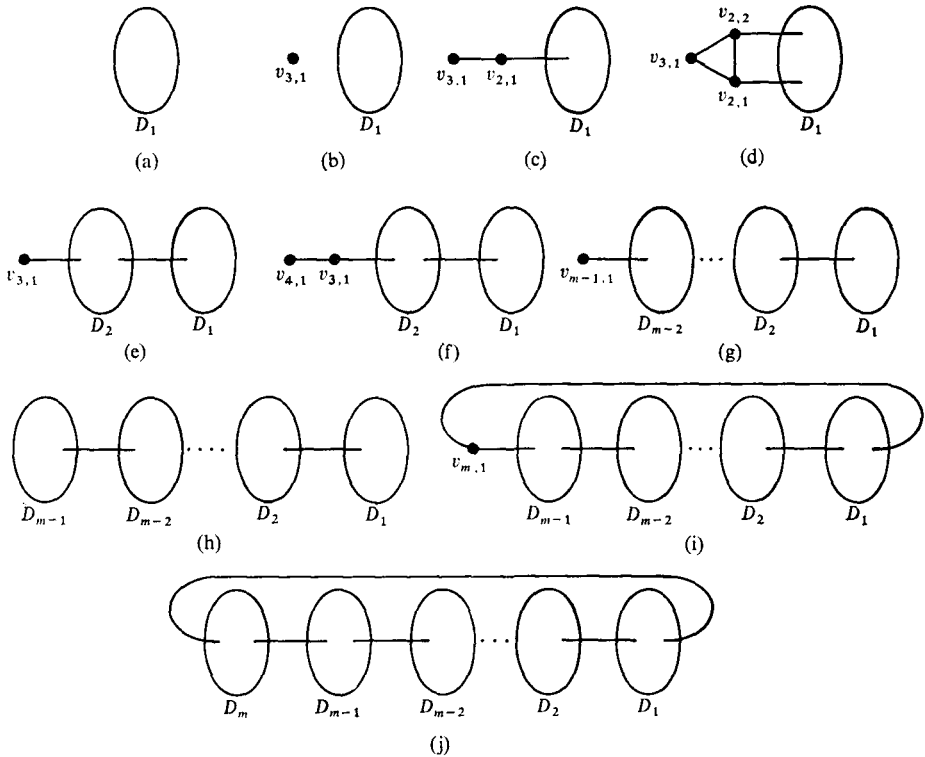


Fig. 3

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